

Continuity, compactness and connectedness

Definition 3.1 (Continuity and sequential continuity).

Let X, Y be topological spaces, $f : X \rightarrow Y$ a map, and $a \in X$

1. We say that f is *sequentially continuous* at a , if for a sequence (x_n) , $\lim_{n \rightarrow \infty} x_n = a$ implies that

$$\lim_{n \rightarrow \infty} f(x_n) = f(a).$$

2. We say that f is *continuous* at a , if

$$\forall U \in \mathcal{U}(f(a)) \exists V \in \mathcal{U}(a) : f(V) \subset U.^1$$

If a function is (sequentially) continuous at all points $a \in X$, then we say that f is *(sequentially) continuous on X* .

Proposition 3.2. *If $f : X \rightarrow Y$ is continuous at $x \in X$, then f is also sequentially continuous at x .*

Proposition 3.3 (ε - δ -continuity in metric spaces). *A function $f : X \rightarrow Y$ between metric spaces X, Y is continuous at $x \in X$, if and only if*

$$\forall \varepsilon > 0 \exists \delta > 0 : f(B_\delta(x)) \subset B_\varepsilon(f(x))$$

Proposition 3.4. *A function $f : X \rightarrow Y$ between metric spaces X, Y is continuous at $a \in X$, if and only if it is sequentially continuous at a .*

Proof. $\Rightarrow$ Proposition 3.2

¹ $\mathcal{U}(x)$ is the set of all neighbourhoods of the point x .

$\Leftarrow$ (by contraposition $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$)

Assume that f is not continuous at a , i.e.

$$\exists \varepsilon > 0 \forall \delta > 0 : f(B_\delta(a)) \not\subset B_\varepsilon(f(a)).$$

For $\delta = \frac{1}{n}$ choose $x_n \in B_\delta(a) \setminus f^{-1}(B_\varepsilon(f(a))) \neq \emptyset$. Then $\lim_{n \rightarrow \infty} x_n = a$, but $f(x_n) \notin B_\varepsilon(f(a)) \forall n \Rightarrow f$ is not sequentially continuous.

□

Theorem 3.5. *Let X, Y be topological spaces. A map $f : X \rightarrow Y$ is continuous (on X), if the preimage $f^{-1}(O) \subset X$ of any open set $O \subset Y$ is open.*

Example 3.6. 1. In a metric space (X, d) the distance function to a point $b \in X$,

$$d_b : X \rightarrow [0, \infty), \quad x \mapsto d_b(x) := d(x, b)$$

is continuous.²

2. In a normed space $(V, \|\cdot\|)$ the norm:

$$\|\cdot\| : V \rightarrow [0, \infty),$$

addition:

$$+ : V \times V \rightarrow V, \quad (x, y) \mapsto x + y,$$

and multiplication by scalars:

$$\cdot : \mathbb{K} \times V \rightarrow V, \quad (\lambda, v) \mapsto \lambda \cdot v$$

are all continuous.

3. The composition of continuous functions is continuous. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then also $g \circ f : X \rightarrow Z$ is continuous.

4. If X is equipped with the discrete topology, then every map $f : X \rightarrow Y$ is continuous. If X is equipped with the trivial topology, then every map $f : Y \rightarrow X$ is continuous.

Remark 3.7. 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Then a metric on $X \times Y$ is for example

$$d((x_1, y_1), (x_2, y_2)) := (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{1/p} \quad 1 \leq p < \infty$$

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ topological space. Then the (product) topology on $X \times Y$ is generated by

$$\{O_1 \times O_2 : O_1 \in \mathcal{T}_X, O_2 \in \mathcal{T}_Y\}$$

also called *bose topology*.

²Also $d : X \times X \rightarrow [0, \infty)$ is continuous using a suitable metric on $X \times X$. For the definition of this metric, see Remark 3.7.

3. Let $(X_i, \mathcal{T}_i)$, $i \in I$, be topological spaces. Then the product topology on $\prod_{i \in I} X_i$ is generated by

$$\left\{ \prod_{i \in I} O_i : O_i \in \mathcal{T}_i \text{ and } O_i \neq X_i \text{ only for finitely many } i \in I \right\}.$$

Definition 3.8 (Lipschitz continuity).

Let X, Y be metric spaces. A function $f : X \rightarrow Y$ is called *Lipschitz-continuous*, if there exists $0 \leq L \leq \infty$ such that

$$\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq L \cdot d_X(x_1, x_2).$$

Then L is called a *Lipschitz-constant* for f . If f has a Lipschitz-constant $L < 1$, then f is called *contraction*.

Example 3.9. 1. $f(x) = ax + b$ is Lipschitz continuous with $L = a$.

2. $f \in C^1(\mathbb{R})$ then $L = \sup_{x \in \mathbb{R}} |f'(x)|$.
3. $f(x) = x^2$ is continuous but not Lipschitz continuous in $\mathbb{R}$.
4. $f(x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous in $\mathbb{R}$, as its derivative around 0 diverges.

Definition 3.10 (Homeomorphic functions, isometries and isometric isomorphisms).

1. Two topological spaces X, Y are *homeomorphic* if there exists a bicontinuous bijection

$$f : X \rightarrow Y \quad \text{a homeomorphism}$$

2. A map $f : X \rightarrow Y$ between metric spaces is an *isometry*, if

$$\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

X and Y are *isometric*, if there exists a bijective isometry $f : X \rightarrow Y$.

3. Two normed spaces V and W are *isometrically isomorphic*, if there exists a linear bijection (isomorphism) $A : V \rightarrow W$ such that

$$\forall v \in V : \|Av\|_W = \|v\|_V.$$

Example 3.11. 1. The interval $(a, b) \subset \mathbb{R}$ is homeomorphic, but not isometric to $\mathbb{R}$. The map

$$f : (a, b) \rightarrow \mathbb{R}, \quad x \mapsto f(x) = \frac{1}{a-x} + \frac{1}{b-x}$$

is an example of a homeomorphism.

2. The isometries of Euclidean space $(\mathbb{R}^n, d_2)$ are translations, rotations and reflections and compositions thereof (euclidean group).
3. $\mathbb{R}^2$ and $\mathbb{C}$ with the standard norms are isometrically isomorphic.

Definition 3.12 (Pointwise and uniform convergence).

Let X be a set, Y a metric space and

$$f_n : X \rightarrow Y, n \in \mathbb{N} \quad \text{and} \quad f : X \rightarrow Y$$

both functions.

1. We say that f_n *converges pointwise* to f , if

$$\forall x \in X : \lim_{n \rightarrow \infty} d_Y(f_n(x), f(x)) = 0. \quad \Leftrightarrow \quad \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

2. We say that f_n *converges uniformly* to f , if

$$\lim_{n \rightarrow \infty} \sup_{x \in X} d_Y(f_n(x), f(x)) = 0$$

If $(Y, \|\cdot\|)$ is a normed space, then $f_n \rightarrow f$ uniformly, if and only if

$$\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0$$

Example 3.13. $f_n : [0, 1] \rightarrow [0, 1]$, $x \mapsto f_n(x) = x^n$, then pointwise

$$f_n(x) \xrightarrow{n \rightarrow \infty} f(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \end{cases}.$$

However, (f_n) does not converge uniformly to f since $\sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1$.

To see this consider $x = 1 - \delta$ for arbitrarily small $\delta > 0$. Then, $f_n(x) = (1 - \delta)^n = 1 - n\delta + O(\delta^2)$, whereas $f(x) = 0$, so after sending $\delta \rightarrow 0$ we get $\sup_{x \in [0, 1]} |f_n(x) - f(x)| \geq 1$.

Proposition 3.14 (Uniform limits of continuous functions are continuous).

Let $(X, \mathcal{T})$ a topological and (Y, d) a metric space. Let $f_n : X \rightarrow Y$ be a sequence of continuous functions and let $f_n \rightarrow f$ uniformly. Then f is continuous.

Corollary 3.15. Let X be a topological space, $(Y, \|\cdot\|_Y)$ a complete normed space and $C_b(X, Y)$ the space of continuous bounded functions, i.e.

$$C_b(X, Y) = \{f : X \rightarrow Y \text{ continuous} \mid \sup_{x \in X} \|f(x)\|_Y < \infty\}.$$

Then the normed space $(C_b(X, Y), \|\cdot\|_\infty)$ is complete.

Definition 3.16 (Open cover and finite subcover).

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. A family $(U_i)_{i \in I}$ of open sets, $U_i \in \mathcal{T} \forall i \in I$, is called an *open cover* of Y , if

$$Y \subset \bigcup_{i \in I} U_i$$

A set $K \subset X$ is called *compact*, if any open cover $(U_i)_{i \in I}$ of K admits a finite subcover, i.e. there exists $i_1, \dots, i_n \in I$ such that:

$$K \subset \bigcup_{i=i_1, \dots, i_n} U_i$$

Example 3.17. 1. Every finite subset $K = \{x_1, \dots, x_n\}$ of a topological space is compact.

2. $(0, 1] \subset \mathbb{R}$ is not a compact set. The open cover $(0, 1] \subset \bigcup_{n=2}^{\infty} (\frac{1}{n}, 2)$ admits no finite subcover.

Theorem 3.18 (Bolzano-Weierstraß). *Let $K \subset X$ be compact. Then any sequence in K has a cluster point in K .*

Remark 3.19. In metric spaces also the converse is true, namely, that if every sequence in a subset has a cluster point, then it is compact.

Proposition 3.20. *Let $f : X \rightarrow Y$ be a continuous function and $K \subset X$ a compact set. Then also $f(K) \subset Y$ is compact.*

Proposition 3.21. 1. *Let X be a topological space and $K \subset X$ compact. Then any close subset $A \subset K$ is also compact.*

2. *If X is a Hausdorff space and K compact, then K is closed.*

Definition 3.22 (Sequential compactness).

Let X be a topological space. Then, $K \subset X$ is called *sequentially compact* if every sequence in K has a convergent subsequence with limit in K .

Proposition 3.23. *A subset $K \subset (X, d)$ of a metric space is compact if and only if it is sequentially compact.*

Definition 3.24 (Bounded sets and the diameter of a set).

Let X be a metric space.

1. A subset $B \subset X$ is *bounded*, if

$$\exists C \in \mathbb{R} \forall x, y \in B : d(x, y) \leq C$$

2. The *diameter* of the set $Y \subset X$ is

$$\text{diam}(Y) = \sup\{d(x, y) \mid x, y \in Y\} \in [0, \infty) \cup \{\infty\}$$

Theorem 3.25.

Let X be a metric space and $K \subset X$ compact. Then K is bounded and closed.

Theorem 3.26 (Heine-Borel). A subset K of a finite-dimensional normed space is compact if it is bounded and closed.

Theorem 3.27 (Weierstraß). Let $f : K \rightarrow \mathbb{R}$ be a continuous function and K compact. Then f is bounded ($f(K) \subset \mathbb{R}$ is bounded) and attains its maximum and its minimum.

Definition 3.28 (Equicontinuity).

Let X, Y be metric spaces and $A \subset C(X, Y)$. Then the set A is called *equicontinuous* at $x \in X$, if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall f \in A : f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

Theorem 3.29 (Arzela-Ascoli). Let X be a compact metric space and consider $C(X, \mathbb{C})$ equipped with the $\|\cdot\|_\infty$ -norm. A subset $K \subset C(X, \mathbb{C})$ is compact, if and only if it is closed, bounded pointwise (i.e. $\forall x \in X$:

$$\sup_{f \in K} |f(x)| < \infty)$$

and equicontinuous.

Definition 3.30 (Connected, disconnected and path connected spaces).

Let X be a topological space. If X is the union of two disjoint, open, non-empty sets, then X is *disconnected*, otherwise *connected*.

X is *path-connected*, if any two points $x_0, x_1 \in X$ can be connected by a continuous path, i.e. there exists

$$\gamma : [0, 1] \rightarrow X$$

continuous, with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Proposition 3.31. If X is path-connected then X is connected.

Proposition 3.32. Let O be an open subset of a normed space. Then O is connected, if and only if it is path connected.

Proposition 3.33. Let $f : X \rightarrow Y$ be continuous and $A \subset X$ (path) connected. Then also $f(A) \subset Y$ is (path) connected.

Definition 3.34 (Bounded functions).

A function $f : X \rightarrow Y$ with X a set and (Y, d) a metric space, is called bounded, if and only if $f(X) \subset Y$ is bounded.

Definition 3.35 (Bounded linear maps and their norms).

A linear map $A : V \rightarrow W$ between normed spaces is called bounded, if $A(B_1(0))$ is *bounded*, i.e.

$$\exists C \in \mathbb{R} \forall x \in V : \|Ax\|_W \leq C\|x\|_V.$$

The smallest such constant C is called the *operator norm* of A , i.e.

$$\|A\|_{op} := \sup\{\|Ax\|_W \mid x \in \overline{B_1(0)}\}$$

The space of bounded linear maps $V \rightarrow W$ is denoted by

$$\mathcal{L}(V, W) \text{ or } \mathcal{B}(V, W)$$

and $\|\cdot\|_{op}$ is a norm on $\mathcal{L}(V, W)$.

Remark 3.36. 1. If $A \in \mathcal{L}(V, W)$ we have for all $x \in V$

$$\|Ax\|_W \leq \|A\|_{op} \cdot \|x\|_V$$

2. $A \in \mathcal{L}(V, W)$ is bounded if and only if it is continuous.
3. If $\dim V < \infty$, then all linear maps $V \rightarrow W$ are bounded.
4. If $(W, \|\cdot\|_W)$ is a Banach space, then $(\mathcal{L}(V, W), \|\cdot\|_{op})$ is also complete.