

FOUNDATIONS OF QUANTUM MECHANICS: ASSIGNMENT 2

Exercise 5: Essay question: Explain the concept of reductionism. (You can use Newtonian mechanics as an example.)

Exercise 6: Unitary operators.

(a) For any vector $\mathbf{a} \in \mathbb{R}^3$, the *translation operator* $T_{\mathbf{a}}$ is defined on $L^2(\mathbb{R}^{3N})$ by

$$(T_{\mathbf{a}}\psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi(\mathbf{x}_1 - \mathbf{a}, \dots, \mathbf{x}_N - \mathbf{a}). \quad (1)$$

It shifts the wave function by $\mathbf{a}$ in every $\mathbf{x}_i$. Show that $T_{\mathbf{a}}$ is unitary.

(b) For any rotation matrix $R \in SO(3)$, the *rotation operator* U_R is defined on $L^2(\mathbb{R}^{3N})$ by

$$(U_R\psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi(R^{-1}\mathbf{x}_1, \dots, R^{-1}\mathbf{x}_N). \quad (2)$$

It rotates the wave function according to R in every $\mathbf{x}_i$. Show that U_R is unitary.

Exercise 7: Orthonormal system.

For $n \in \mathbb{Z}$, let the function $\varphi_n : [-\pi, \pi] \rightarrow \mathbb{C}$ be defined by

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}. \quad (3)$$

Show that they form an *orthonormal system* in $L^2([-\pi, \pi])$, i.e., that

$$\langle \varphi_n | \varphi_m \rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \quad (4)$$

with

$$\langle \psi | \phi \rangle = \int_{-\pi}^{\pi} \psi(x)^* \phi(x) dx. \quad (5)$$

Exercise 8: Dense subspace.

The space $\ell^2 = \{(x_1, x_2, \dots) : x_n \in \mathbb{C}, \sum |x_n|^2 < \infty\}$ of all square-summable sequences is a Hilbert space with inner product

$$\langle x | y \rangle = \sum_{n=1}^{\infty} x_n^* y_n. \quad (6)$$

A subset S of a Hilbert space $\mathcal{H}$ is called *dense* if for every $\psi \in \mathcal{H}$ and $\varepsilon > 0$ there is $\phi \in S$ with $\|\psi - \phi\| < \varepsilon$. (For example, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.) Let S be the subspace of ℓ^2 consisting of all sequences $(x_1, x_2, \dots)$ with only finitely many nonzero entries.

(a) Show that S is dense in ℓ^2 .

(b) Show that in a finite-dimensional Hilbert space (that is, without loss of generality, in $\mathbb{C}^n$), the only dense subspace is the full space $\mathbb{C}^n$.

Hand in: by Saturday October 25, 2025, at noon via urm.math.uni-tuebingen.de

Reading assignment due Thursday October 30, 2025: First three pages of J. Bell: De Broglie–Bohm, delayed-choice double-slit experiment, and density matrix. *International Journal of Quantum Chemistry* **14**: 155–159 (1980)