Foundations of Quantum Mechanics: Assignment 4

Exercise 14: Essay question. (20 points)
How does Bohmian mechanics explain the double-slit experiment?

Exercise 15: Conserved quantities (20 points)
Recall the equation of motion of Newtonian mechanics:

$$m_i \frac{d^2 \boldsymbol{Q}_i}{dt^2} = -\nabla_i V(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N)\,. \tag{1}$$

Suppose that V is invariant under rotations and translations:

$$V\Big(R\boldsymbol{Q}_1 + \boldsymbol{a}, \dots, R\boldsymbol{Q}_N + \boldsymbol{a}\Big) = V\big(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N\big) \tag{2}$$

for all $\boldsymbol{a} \in \mathbb{R}^3$ and $R \in SO(3)$. Show that

$$\text{the energy } E = \sum_{k=1}^N \frac{m_k}{2}\boldsymbol{v}_k^2 + V(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N) \tag{3}$$

$$\text{the momentum } \boldsymbol{p} = \sum_{k=1}^N m_k \boldsymbol{v}_k \tag{4}$$

$$\text{the angular momentum } \boldsymbol{L} = \sum_{k=1}^N m_k \boldsymbol{Q}_k \times \boldsymbol{v}_k \tag{5}$$

are conserved.

Exercise 16: Conserved operators (30 points)
Show that if the potential $V(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$ is C^1 and translation invariant [as in (2) with $R = I$], then each component of the total momentum operator

$$P_a = -i\hbar \sum_{j=1}^N \frac{\partial}{\partial x_{ja}} \tag{6}$$

($a = 1, 2, 3$) commutes with the Hamiltonian

$$H = -\sum_{j=1}^N \frac{\hbar^2}{2m_j}\nabla_j^2 + V \tag{7}$$

when acting on $\psi \in C^3 \cap L^2$. Show further that if V is C^1 and rotation invariant [as in (2) with $\boldsymbol{a} = \boldsymbol{0}$], then each component of the total angular momentum operator

$$\boldsymbol{L} = -i\hbar \sum_{j=1}^N \boldsymbol{x}_j \times \nabla_j \tag{8}$$

commutes with H when acting on $\psi \in C^3 \cap L^2$.

Exercise 17: Galilean relativity (30 points)
A Galilean change of space-time coordinates ("Galilean boost") is given by

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t\,, \quad t' = t \tag{9}$$

with a constant $\boldsymbol{v} \in \mathbb{R}^3$ called the relative velocity. Suppose the potential V is translation invariant.
(a) Show that Newton's equation of motion (1.58) is invariant under Galilean boosts, i.e., if $t \mapsto (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ is a solution, then so is $t \mapsto (\boldsymbol{Q}'_1, \ldots, \boldsymbol{Q}'_N)$.
(b) Show that if $\psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is a solution of the Schrödinger equation, then so is

$$\psi'(t', \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = \exp\left[\tfrac{i}{\hbar} \sum_{i=1}^{N} m_i (\boldsymbol{x}'_i \cdot \boldsymbol{v} - \tfrac{1}{2}\boldsymbol{v}^2 t')\right] \psi\Big(t', \boldsymbol{x}'_1 - \boldsymbol{v}t', \ldots, \boldsymbol{x}'_N - \boldsymbol{v}t'\Big). \tag{10}$$

(c) Show that if $t \mapsto (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ is a solution of Bohmian mechanics then so is $t \mapsto (\boldsymbol{Q}'_1, \ldots, \boldsymbol{Q}'_N)$.

Hand in: by Saturday November 8, 2025, at noon via `urm.math.uni-tuebingen.de`

No reading assignment this week.