Prof. Dr. Roderich Tumulka
Eberhard Karls University Tübingen

Winter semester 2025/26
November 6, 2025

Foundations of Quantum Mechanics: Assignment 5

Exercise 18: Essay question. (30 points)
Describe the delayed-choice double-slit experiment. Why does it seem paradoxical? How does the paradox get resolved in Bohmian mechanics?

Exercise 19: Fourier transform
(a) (20 points) Find the Fourier transform $\widehat{\psi}$ of the function

$$\psi(x) = \begin{cases} 0 & x < -1 \\ 2^{-1/2} & -1 \leq x \leq 1 \\ 0 & x > 1, \end{cases} \tag{1}$$

where x is a 1-dimensional variable.

(b) (optional, 15 extra points) Plot $\widehat{\psi}$ using a computer.

(c) (optional, 15 extra points) Using suitable software (such as Mathematica, Maple, or Matlab), make the computer find the formula for $\widehat{\psi}$. (That is, you need to find the command for symbolically computing Fourier transforms and the one for defining a function like (1), and run them.)

Exercise 20: Spectral theorem (20 points)
In this problem we use without proof the following generalization of the spectral theorem (formulated here in finite dimension): If the self-adjoint $d \times d$ matrices A and B commute, then they can be simultaneously unitarily diagonalized, i.e., there is an orthonormal basis $\{\phi_1, \ldots, \phi_d\}$ such that each ϕ_j is an eigenvector of A and an eigenvector of B.

Show that a $d \times d$ matrix C can be unitarily diagonalized iff C commutes with $C^\dagger$. Such a matrix is called "normal." (Hint: write $C = A + iB$.)

Exercise 21: Potential step (30 points)
Consider a potential step $V(x) = V_0 \, 1_{0 \leq x}$ in 1d with $V_0 > 0$. We want to compute the reflection and transmission probabilities as a function of the incoming momentum $\hbar k_0$, assuming that $E := \hbar^2 k_0^2/2m > V_0$. A recipe for that is to consider a plane wave $e^{ik_0 x}$ coming from $x = -\infty$ and see how much gets reflected and transmitted by constructing an eigenfunction ψ of H, $H\psi = E\psi$, from the ansatz

$$\psi(x) = \begin{cases} Ae^{ik_0 x} + Be^{-ik_0 x} & \text{for } x < 0 \\ Ce^{iK_0 x} & \text{for } x > 0 \end{cases} \tag{2}$$

with complex coefficients A, B, C, representing the incoming wave $e^{ik_0 x}$, the reflected wave $e^{-ik_0 x}$, and the transmitted wave $e^{iK_0 x}$. Regarding V as a limit of continuous functions leads to the further requirement that ψ be continuous and continuously differentiable at 0.

(a) Determine K_0 from $H\psi = E\psi$.

(b) Determine B and C from A.

(c) The recipe says that the three waves are associated with probability currents $j_{\text{in}} = \hbar k_0 |A|^2/m$, $j_R = \hbar k_0 |B|^2/m$, and $j_T = \hbar K_0 |C|^2/m$; and that the reflection probability is $P_R = j_R/j_{\text{in}}$, while the transmission probability is $P_T = j_T/j_{\text{in}}$. Compute P_R and P_T.

(d) To justify the recipe, consider a "plane wave packet" $\psi(x) = \phi(x)\, e^{ik_0 x}$ with a (non-Gaussian) envelope profile $\phi(x)$ that is nearly constant over a region of length L much larger than the wave length $2\pi/k_0$ and then drops to 0 over a length much smaller than L but still much larger than the wave length. Let us take for granted that under the free Schrödinger evolution the envelope function ϕ will approximately maintain its shape (in particular, its length) for a long time and simply move at speed $\hbar k_0/m$. Suppose the packet hits the step at $t = 0$; forget about what happens at the edges of the packet and focus on the bulk. A reflected plane wave packet and a transmitted one are generated; at time $\tau > 0$, the incoming plane wave packet is used up, and the two outgoing packets end. During $0 < t < \tau$, the region around 0 is well approximated by (2); after τ, the outgoing packets keep moving away from the origin. Determine τ and the lengths L_R and L_T of the reflected and transmitted packets ψ_R and ψ_T.

(e) Determine $P_R = \|\psi_R\|^2$ and $P_T = \|\psi_T\|^2$ and verify that they agree with part (c).

(f) Draw a space-time diagram of the Bohmian trajectories in the bulk (that is, ignoring any edge effects).

Hand in: by Saturday November 15, 2025, at noon via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday November 20, 2025: Section II of T. Norsen: The Pilot-Wave Perspective on Quantum Scattering and Tunneling. *American Journal of Physics* **81**: 258 (2013)