Foundations of Quantum Mechanics: Assignment 6

Exercise 22: Essay question. (20 points)
Describe what the Heisenberg uncertainty relation asserts.

Exercise 23: Pauli matrices (30 points)
The three *Pauli matrices* are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) For each of σ_1 and σ_2, find an orthonormal basis of eigenvectors in $\mathbb{C}^2$.

(b) Show that for every unit vector $\boldsymbol{n} \in \mathbb{R}^3$, the Pauli matrix in direction $\boldsymbol{n}$, $\sigma_{\boldsymbol{n}} := \boldsymbol{n} \cdot \boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$, has eigenvalues ± 1. (Hint: compute det and trace.)

(c) Show that every self-adjoint complex 2×2 matrix A is of the form $A = cI + \boldsymbol{u} \cdot \boldsymbol{\sigma}$ with $c \in \mathbb{R}$ and $\boldsymbol{u} \in \mathbb{R}^3$.

Exercise 24: Spinors (25 points)
Verify that $|\boldsymbol{\omega}(\phi)| = \|\phi\|_S^2 = \phi^*\phi$. Proceed as follows: By (2.108), $\boldsymbol{\omega}(z\phi) = |z|^2\boldsymbol{\omega}(\phi)$, it suffices to show that unit spinors are associated with unit vectors. By (2.108) again, it suffices to consider ϕ with $\phi_1 \in \mathbb{R}$ (else replace ϕ by $e^{i\theta}\phi$ with appropriate θ). So we can assume, without loss of generality, $\phi = (\cos\alpha, e^{i\beta}\sin\alpha)$ with $\alpha, \beta \in \mathbb{R}$. Evaluate $\phi^*\boldsymbol{\sigma}\phi$ explicitly in terms of α and β, using the explicit formulas (2.105) for $\boldsymbol{\sigma}$. Then check that it is a unit vector.

Exercise 25: Angles in Hilbert space (25 points)
In my book, I said that the angle θ between two vectors ϕ, χ in Hilbert space is $\theta = \arccos \frac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}$. I take it back. In this exercise, we look at the reasons why the natural definition for θ actually is

$$\theta = \arccos \frac{\mathrm{Re}\langle\phi|\chi\rangle}{\|\phi\|\,\|\chi\|}, \tag{2}$$

while the one for the angle α between the 1d subspaces $\mathbb{C}\phi, \mathbb{C}\chi$ is

$$\alpha = \arccos \frac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}. \tag{3}$$

(a) For the most natural identification mapping $J : \mathbb{C}^d \to \mathbb{R}^{2d}$ given by $J(\phi_1, \ldots, \phi_d) = (\mathrm{Re}\,\phi_1, \mathrm{Im}\,\phi_1, \ldots, \mathrm{Re}\,\phi_d, \mathrm{Im}\,\phi_d)$, show that

$$\langle J\phi|J\chi\rangle_{\mathbb{R}^{2d}} = \mathrm{Re}\langle\phi|\chi\rangle_{\mathbb{C}^d} \quad \text{and} \quad \|J\phi\|_{\mathbb{R}^{2d}} = \|\phi\|_{\mathbb{C}^d}, \tag{4}$$

where $\langle x|y\rangle_{\mathbb{R}^n} = \sum_{i=1}^n x_i y_i$ and $\langle\phi|\chi\rangle_{\mathbb{C}^d} = \sum_{j=1}^d \phi_j^*\chi_j$, and the norms are accordingly defined.

(b) Explain by means of an example in $\mathbb{R}^3$ why the intuitive notion of the angle β between two subspaces U, V of $\mathbb{R}^n$ is given by

$$\beta = \inf_{u \in U\setminus\{0\}} \inf_{v \in V\setminus\{0\}} \theta(u, v), \tag{5}$$

where $\theta(u,v)$ means the angle between u and v. We adopt the same definition in $\mathbb{C}^d$.

(c) Show that for $\phi, \chi \in \mathbb{C}^d \setminus \{0\}$,

$$\inf_{u \in \mathbb{C}\phi \setminus \{0\}} \inf_{v \in \mathbb{C}\chi \setminus \{0\}} \arccos \frac{\mathrm{Re}\langle u|v\rangle}{\|u\|\,\|v\|} = \arccos \frac{|\langle \phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}\,. \tag{6}$$

Hand in: By Saturday November 22, 2025, at noon via `urm.math.uni-tuebingen.de`

No reading assignment this week.