Prof. Dr. Roderich Tumulka
Eberhard Karls University Tübingen

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Foundations of Quantum Mechanics: Assignment 8

Exercise 30: Essay question (20 points)
Explain why the interference pattern in the double-slit experiment disappears if a detector measures through which slit the electron went, even if we ignore the outcome of the detection. Derive this fact from the projection postulate, using the observable P_B defined by $P_B\psi(\boldsymbol{x}) = 1_B(\boldsymbol{x})\psi(\boldsymbol{x})$ as a model of a detector, where B is a suitable neighborhood of one slit. Explain why the pattern on the screen must be the same as if one slit were closed in each run, each slit in 50% of the runs. (Use formulas where appropriate.)

Exercise 31: Symmetrizer and anti-symmetrizer (30 points)
Let S_N be the group of permutations of $\{1, \ldots, N\}$. A function $\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is called

$$\textit{symmetric} \text{ iff } \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) = \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1}$$

$$\textit{anti-symmetric} \text{ iff } \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) = (-)^\pi \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \tag{2}$$

for every $\pi \in S_N$, where $(-)^\pi$ denotes the sign of π (i.e., $+1$ if π is even, -1 if odd). Let $\mathscr{S}_+$ denote the subspace of all symmetric functions in $L^2(\mathbb{R}^{3N})$ and $\mathscr{S}_-$ that of anti-symmetric functions. Show that the operators $P_\pm$ defined by

$$P_+\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \frac{1}{N!} \sum_{\pi \in S_N} \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) \tag{3}$$

$$P_-\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \frac{1}{N!} \sum_{\pi \in S_N} (-)^\pi \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) \tag{4}$$

are the projections to $\mathscr{S}_\pm$. Use without proof that $(-)^\pi (-)^\rho = (-)^{\pi \circ \rho}$ for $\pi, \rho \in S_N$.

Exercise 32: Can't Distinguish Non-Orthogonal State Vectors (20 points)
(a) Alice gives to Bob a single particle whose spin state ψ is either $(1,0)$ or $(0,1)$ or $\frac{1}{\sqrt{2}}(1,1)$. Bob can carry out a quantum measurement of an arbitrary self-adjoint operator. Show that it is impossible for Bob to decide with certainty which of the three states ψ is.

(b) The same with only $(1,0)$ and $\frac{1}{\sqrt{2}}(1,1)$.

Exercise 33: Lie algebras of $SO(3)$ and $SU(2)$ (30 points. Level: difficult)
A *Lie group* G, named after Sophus Lie (1842–1899), is a group that is also a manifold (a curved surface) such that the group multiplication and inversion are smooth mappings. Examples of Lie groups include $GL(n), SO(n), U(n), SU(n)$. The elements infinitesimally close to 1 in G form the *Lie algebra* g of G; more precisely, g is the tangent space of 1, which is here the set

$$\left\{ \frac{dA}{dt}(t=0) \middle| A : (-1,1) \to G \text{ smooth}, A(0) = 1 \right\}.$$

(a) Determine the Lie algebras $so(3)$ and $su(2)$ as subspaces of the space of all real 3×3 (complex 2×2) matrices.

(b) The *exponential mapping* $\exp : g \to G$ can be heuristically understood as follows: For $X \in g$, a corresponding group element infinitesimally close to 1 can be written as $1 + X/n$ with n a large natural number (so $1/n$ serves as an infinitesimal dt). Hence, roughly speaking, $(1 + X/n) \in G$, hence $(1 + X/n)^n \in G$; take the limit $n \to \infty$ to obtain $\exp(X) =: e^X$. Verify that the matrix exponential (defined by the exponential series) actually maps $so(3)$ to $SO(3)$ and $su(2)$ to $SU(2)$. (Hint: diagonalize $X \in g$.)

(c) We now consider the question what the group multiplication of e^X and e^Y looks like for $X, Y \in g$. We know that the solution Z of $e^Z = e^X e^Y$ is $Z = X + Y$ if X and Y commute, but not in general. A version of the *Baker–Campbell–Hausdorff formula* says that

$$\text{the solution of } e^Z = e^{-tX} e^{-tY} e^{t(X+Y)} \text{ is } Z = \tfrac{1}{2} t^2 [X, Y] + \mathcal{O}(t^3)$$

as $t \to 0$, with $[X, Y] = XY - YX$ the *commutator* or *Lie bracket*. The Lie bracket is an operation on g that encodes how the group multiplication deviates from addition in g. Thus, one defines a *Lie algebra* in general as a vector space together with a bracket $[\cdot, \cdot] : g \times g \to g$ that is anti-symmetric, bilinear, and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,.$$

Verify that $so(3)$ and $su(2)$ (with commutators as Lie brackets) are isomorphic Lie algebras. (Hint: The Pauli matrices have something to do with rotations about the x-, y-, and z-axis.)

Hand in: by Saturday December 6, 2025, at noon via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday December 11, 2025: A. Einstein, *Reply to Criticisms*, pages 665–688 in P. Schilpp (editor): *Albert Einstein, Philosopher–Scientist* (1949). Read pages 665–672 and the first quarter of 673.