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Winter semester 2025/26
December 11, 2025

Foundations of Quantum Mechanics: Assignment 10

Exercise 37: Essay question (15 points)
Describe the Einstein–Podolsky–Rosen argument (either in terms of position and momentum or in terms of spin matrices).

Exercise 38: Marginal and conditional distribution (18 points)
Consider two random variables X, Y that assume only values ± 1. Their joint distribution can be described by a 2×2 table of probabilities. **(a)** Give a generic example of such a table (i.e., one without symmetries). For your table, compute **(b)** the marginal distribution of X and **(c)** that of Y, as well as **(d)** the conditional distribution of X, given that $Y = +1$, **(e)** the expectation value $\mathbb{E}(X)$, and **(f)** $\mathbb{E}(XY)$.

Exercise 39: Spin singlet state (17 points)
An element of $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be represented by a complex 2×2 matrix $\psi_{s_1 s_2}$. One calls those elements *anti-symmetric* for which $\psi_{s_2 s_1} = -\psi_{s_1 s_2}$. Show that they form a 1-dimensional subspace. Explain why it follows from this fact that

$$\begin{aligned} &|\boldsymbol{n}\text{-up}\rangle|\boldsymbol{n}\text{-down}\rangle - |\boldsymbol{n}\text{-down}\rangle|\boldsymbol{n}\text{-up}\rangle \\ &= |z\text{-up}\rangle|z\text{-down}\rangle - |z\text{-down}\rangle|z\text{-up}\rangle \end{aligned} \tag{1}$$

up to a phase factor for any direction given by $\boldsymbol{n} \in \mathbb{R}^3$ with $|\boldsymbol{n}| = 1$ and any ONB $\{|\boldsymbol{n}\text{-up}\rangle, |\boldsymbol{n}\text{-down}\rangle\}$ of eigenvectors of $\boldsymbol{n} \cdot \boldsymbol{\sigma}$ with eigenvalues $+1$ and -1.

Exercise 40: Quantum Zeno effect (25 points)
Zeno of Elea (c. 490–c. 430 BCE) was a Greek philosopher who claimed that motion and time cannot exist because they are inherently paradoxical notions, a claim which he tried to support by formulating various paradoxes, including one involving Achilles and a turtle. In modern times, Alan Turing (of computer science fame, lived 1912–1954) reportedly first discovered the following effect, which was later named after Zeno because of its paradoxical flavor: Suppose a quantum particle moves in 1d, and its initial wave function $\psi_0(x)$ is concentrated in the negative half axis $(-\infty, 0)$. We want to model, as a kind of time measurement, a detector, located at the origin, that clicks when the particle arrives. To this end, we imagine that the detector performs, at times $n\tau$ with $n \in \mathbb{N}$ and time resolution $\tau > 0$, a quantum measurement of $1_{x \geq 0}$, i.e., of whether the particle is in the right half axis. The ideal detector would seem to correspond to the limit $\tau \to 0$; however, in this limit, the probability that the detector *ever* clicks is 0. "A watched pot never boils," wrote Misra und Sudarshan.[1]

Prove the following simplified version: In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1, 0)$ evolve with Hamiltonian $H = \sigma_1$, interrupted by a quantum measurement of σ_3 at times $n\tau$ for all $n \in \mathbb{N}$. For any fixed $T > 0$, the probability that any of the $\approx T/\tau$ measurements in the time interval $[0, T]$ yields the result -1 tends to 0 as $\tau \to 0$.

Please turn over.

[1] B. Misra and E.C.G. Sudarshan: The Zeno's paradox in quantum theory. *Journal of Mathematical Physics* **18**: 756–763 (1977)

Exercise 41: No-cloning theorem (25 points)
We show that it is impossible to duplicate the quantum state of an object without destroying the original quantum state. Let $\mathbb{S}(\mathscr{H}) = \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ denote the unit sphere in $\mathscr{H}$. A *cloning mechanism* for the Hilbert space $\mathscr{H}_{\mathrm{obj}}$ would consist of a Hilbert space $\mathscr{H}_{\mathrm{app}}$, a ready state $\phi_0 \in \mathbb{S}(\mathscr{H}_{\mathrm{app}})$ of the apparatus, a ready state $\psi_0 \in \mathbb{S}(\mathscr{H}_{\mathrm{obj}})$ of the copy, and a unitary time evolution U on $\mathscr{H}_{\mathrm{obj}} \otimes \mathscr{H}_{\mathrm{obj}} \otimes \mathscr{H}_{\mathrm{app}}$ such that, for all $\psi \in \mathbb{S}(\mathscr{H}_{\mathrm{obj}})$,

$$U(\psi \otimes \psi_0 \otimes \phi_0) = \psi \otimes \psi \otimes \phi_\psi \tag{2}$$

with some $\phi_\psi \in \mathscr{H}_{\mathrm{app}}$ that may depend on ψ. Prove that if $\dim \mathscr{H}_{\mathrm{obj}} \geq 2$, then no cloning mechanism exists. (*Hint*: Consider $\psi_1 \perp \psi_2$ and $\psi_3 = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$.)

Hand in: by Saturday January 10, 2026, at noon via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday January 8, 2026:
A. Einstein, B. Podolsky, N. Rosen: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review* **47**: 777–780 (1935)