ETH Zürich

Bachelor's thesis

Cusps, Hauptmoduln and Modular Functions for Congruence Subgroups

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Abstract

Modular functions have many applications ranging from number theory and group theory to string theory. We first consider modular functions for $SL(2,\mathbb{Z})$ and show that the j-function generates the field of modular functions. Then we look at the congruence subgroups $\Gamma(N)$ and $\Gamma_0(N)$ and find the equivalence classes of cusps. For $\Gamma_0(p)$, where p is a prime, we calculate the fundamental region. We analyse the behaviour of some Hauptmoduln for $\Gamma_0(N)$ at the cusps. Then we consider two congruence subgroups $\Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$ conjugate to $\Gamma_0(2)$ and describe corresponding Hauptmoduln. Finally, for modular functions for $\Gamma_{(1,1)}$ with only nonnegative real coefficients in the Fourier expansion we show that the pole at infinity already gives a bound for poles at other cusps. This makes it possible to write holomorphic modular functions with nonnegative real coefficients as rational functions of the Hauptmodul just using the first few coefficients of the Fourier expansion.

Contents

1 Introduction

Modular functions are an excellent example for how different mathematical areas are connected to each other. To study modular functions, one needs complex analysis and some elementary algebra. But the theory of modular functions leads to amazing results ranging from group theory, sphere packings to string theory. This diversity of applications shows that modular functions are important even in current research and certainly worth studying.

Modular functions are meromorphic functions which have a certain symmetry property. In conformal field theory, so-called partition functions can have these symmetries and then they are modular functions (Di Francesco et al.; 1997). A very special modular function, the j-function, is related to the monster group, the largest sporadic simple group. The coefficients in the Fourier expansion of j are connected to the dimensions of the irreducible representations of the monster group (Borcherds; 1992).

As it turns out, the j-function generates the field of modular functions, i.e. every modular function can be written as rational function of j. This leads to the useful principle that finitely many coefficients are enough to determine a modular function uniquely. Using this principle, one can prove number-theoretical identities by comparing the first few terms of the Fourier expansions (Bruinier et al.; 2008).

In Section 2 we will introduce the most important concepts, which we will generalise throughout the thesis. Then we will study the j-function in Section 3, where we will see that the j-function generates all modular functions. We then generalise the notion of modular functions in Section 4 by loosening the symmetry conditions and considering functions invariant under genus 0 congruence subgroups. It turns out that these more general modular functions, sometimes also called automorphic functions, can be generated again by one single function. We study some examples of these generating functions, which are also called Hauptmoduln, and determine their zeros and singularities. In Section 5 we look at symmetry groups showing up in conformal field theory. Then again we look at modular functions symmetric under these groups and find generating functions. For one of these groups we find that modular functions with only nonnegative real coefficients in their Fourier expansion have a very nice property. The behaviour of the function at infinity already restricts its behaviour on the whole real axis. If the function is holomorphic on the upper half plane, this allows us to write the function as a rational function of the Hauptmodul, when we only know the behaviour at infinity and the first few coefficients of the Fourier expansion.

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2 Preliminaries

2.1 Modular Functions

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the upper half plane. We can define an action of the special linear group $SL(2,\mathbb{R})$ on $\mathbb{H}$ in the following way: For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, let

$$\gamma(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

By direct calculation one can verify that this is in fact a well-defined group action.

Let Γ be a subgroup of $SL(2,\mathbb{R})$. The group action induces an equivalence relation on $\mathbb{H}$.

Definition 2.1. Two points $z_1, z_2 \in \mathbb{H}$ are Γ-*equivalent* if and only if there is a $\gamma \in \Gamma$ such that $\gamma(z_1) = z_2$.

Definition 2.2. A *fundamental domain* for Γ is an open subset $\mathcal{F} \subset \mathbb{H}$ such that no two points in $\mathcal{F}$ are Γ-equivalent and every point in $\mathbb{H}$ is Γ-equivalent to some point in $\overline{\mathcal{F}}$.

We will mainly consider the subgroup

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \mid ad - bc = 1 \right\}$$

The group of transformations on $\mathbb{H}$ induced by $SL(2,\mathbb{Z})$ is often referred to as the *full modular group.* The full modular group is generated by the transformations $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ (Apostol; 1990).

Lemma 2.3. *(Apostol; 1990) For $SL(2,\mathbb{Z})$ a fundamental domain is given by*

$$\mathcal{F} = \left\{ z \in \mathbb{H} \mid |z| > 1, |\Re(z)| < \frac{1}{2} \right\}$$

Definition 2.4. A function $f : \mathbb{H} \to \mathbb{C}$ is a *modular function* if it satisfies the following three conditions:

(i) f is meromorphic

(ii) f is invariant under $SL(2,\mathbb{Z})$, i.e. $f(\gamma(\tau)) = f(\tau)$ for all $\gamma \in SL(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$

(iii) The Fourier expansion of f is of the form

$$f(\tau) = \sum_{n=-m}^{\infty} a_n e^{2\pi i n \tau}$$

Remark 2.5. Condition (ii) applied for $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ gives that $f(\tau+1) = f(\tau)$. Thus f is a meromorphic function of $q = e^{2\pi i \tau}$. Because τ lies in the upper half-plane, q is a complex number with $0 < |q| < 1$. The Fourier expansion of f is now given by its Laurent expansion in q around 0:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n$$

Condition (iii) asserts that $f(q)$ is meromorphic at $q = 0$.

Remark 2.6. A modular function has finitely many poles in the fundamental domain.

Proof. We prove this by contradiction. Suppose f is a modular function with infinitely many poles in $\boldsymbol{\mathcal{F}}$. Because f is meromorphic, its poles are isolated. Therefore, the set of poles must be unbounded and we can find a sequence of poles $(t_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \Im(t_k) = \infty$. Then $q_k = e^{2\pi i \tau_k}$ are poles of the meromorphic function $f(q)$. Since the q_k converge to 0 for $k \to \infty$, $f(q)$ is not meromophic at $q = 0$. By Remark 2.5 this contradicts condition (iii) of the definition of a modular function. Thus, every modular function can only have finitely many poles in $\boldsymbol{\mathcal{F}}$. $\square$

Lemma 2.7. *(Apostol; 1990) Every bounded modular function is constant.*

2.2 Lattices

Definition 2.8. A set $L \subset \mathbb{C}$ is a *lattice* if it is of the form

$$L = [z_1, z_2] = \{mz_1 + nz_2 \mid m, n \in \mathbb{Z}\}$$

for two $\mathbb{R}$-linearly independent complex numbers z_1 and z_2.

Definition 2.9. Two lattices L and L' are *homothetic* if there is a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $L = \lambda L'$.

Definition 2.10. Let L be a lattice. For $n \geq 3$ the *Eisenstein series* of order n for L is defined as

$$G_n(L) = \sum_{\omega \in L\setminus\{0\}} \frac{1}{\omega^n}.$$

It can be shown that the Eisenstein series converges absolutely for every L and $n \geq 3$ (Cox; 2013). Two important series are

$$g_2(L) = 60G_4(L) = 60 \sum_{\omega \in L\setminus\{0\}} \frac{1}{\omega^4}$$

and

$$g_3(L) = 140G_6(L) = 140 \sum_{\omega \in L\setminus\{0\}} \frac{1}{\omega^6}.$$

Definition 2.11. Let L be a lattice. The *Weierstrass $\wp$-function* is defined by

$$\wp(z, L) = \frac{1}{z^2} + \sum_{\omega \in L\setminus\{0\}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

for all $z \in \mathbb{C}$.

Lemma 2.12. *(Cox; 2013) The set of singularities of $\wp(z, L)$ consists of poles at the lattice points of L.*

Lemma 2.13. *(Cox; 2013) The Laurent expansion of the Weierstrass $\wp$-function at the origin can be written as*

$$\wp(z, L) = \frac{1}{z^2} + \sum_{n=1}^{\infty} p_n(g_2(L), g_3(L)) z^{2n}$$

where p_n are polynomials independent of L.

3 The j-Function

Definition 3.1. For a lattice L the *j-invariant* is the complex number

$$j(L) = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27 g_3(L)^2}$$

From this we define the *j-function* as the map $j : \mathbb{H} \to \mathbb{C}$ given by $j(\tau) := j([1, \tau])$.

First, we have to check that the j-invariant is well defined.

Lemma 3.2. *(Apostol; 1990) For every lattice $\Delta(L) = g_2(L)^3 - 27 g_3(L)^2 \neq 0$.*

Proof sketch. Let $L = [\omega_1, \omega_2]$ be a lattice. Consider the polynomial $p(x) = 4x^3 - g_2(L)x - g_3(L)$. Its discriminant is $16 \cdot \Delta(L)$. In Apostol (1990) it is shown that $p(x)$ has three distinct roots. Thus, the discriminant of $p(x)$ and hence also $\Delta(L)$ are non-zero. $\square$

The aim of this Section is to study different properties of the j-function. One of the main results will be the following theorem:

Theorem 3.3. *(Scherer; 2010) The j-function is a bijection between $\mathbb{H}/SL(2, \mathbb{Z})$ and $\mathbb{C}$.*

The proof of this theorem is split into Lemma 3.5 and Lemma 3.10. First, we need more preparation.

Lemma 3.4. *(Cox; 2013) Two lattices L and L' in $\mathbb{C}$ are homothetic if and only if $j(L) = j(L')$.*

Proof. ($\Rightarrow$) Let L and L' be homothetic, i.e. $L' = \lambda L$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. From the definitions of g_2 and g_3 we see that $g_2(L') = g_2(\lambda L) = \lambda^{-4} g_2(L)$ and $g_3(L') = g_3(\lambda L) = \lambda^{-6} g_3(L)$. Calculating the j-invariant we get

$$j(L') = \frac{1728 g_2(L')^3}{g_2(L')^3 - 27 g_3(L')^2} = \frac{1728 \lambda^{-12} g_2(L)^3}{\lambda^{-12} g_2(L)^3 - 27 \lambda^{-12} g_3(L)^2} = j(L)$$

($\Leftarrow$) Let $j(L) = j(L')$. We begin by proving the following claim:
Claim: *There is a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $g_2(L') = \lambda^{-4} g_2(L)$ and $g_3(L') = \lambda^{-6} g_3(L)$.*
We distinguish two cases.

Case 1: $g_2(L') = 0$
By Lemma 3.2 $\Delta(L') = g_2(L')^3 - 27g_3(L')^2$ is nonzero, hence $g_3(L') \neq 0$. Choose $\lambda \in \mathbb{C}$ such that $\lambda^6 = g_3(L)/g_3(L')$. We have that

$$0 = \frac{1728 g_2(L')^3}{g_2(L')^3 - 27g_3(L')^2} = j(L') = j(L) = \frac{1728 g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2}$$

Therefore, $g_2(L) = 0$ and $\lambda \neq 0$ because otherwise $\Delta(L)$ would be zero. Hence, $g_2(L') = 0 = \lambda^{-4} g_2(L)$.

Case 2: $g_2(L') \neq 0$
Choose $\lambda \in \mathbb{C}$ such that $\lambda^4 = g_2(L)/g_2(L')$. From $j(L') = j(L)$ we get

$$\frac{g_2(L')^3}{g_2(L')^3 - 27g_3(L')^2} = \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = \frac{\lambda^{12} g_2(L')^3}{\lambda^{12} g_2(L')^3 - 27g_3(L)^2}$$

Dividing by $g_2(L')^3$ and multiplying by the denominators we get

$$\lambda^{12} g_2(L')^3 - 27g_3(L)^2 = \lambda^{12} g_2(L')^3 - 27\lambda^{12} g_3(L')^2$$

Therefore, $g_3(L) = \pm\lambda^6 g_3(L')$. If there is a minus, we replace λ by $i\lambda$. If λ was zero, $g_3(L)$, $g_2(L)$ and hence also $\Delta(L)$ would be zero. Thus, $\lambda \neq 0$ and we get $\lambda^{-6} g_3(L) = g_3(L')$ and have proven the claim.

Combining the claim with the definitions of g_2 and g_3 we get $g_2(L') = \lambda^{-4} g_2(L) = g_2(\lambda L)$ and $g_3(L') = \lambda^{-6} g_3(L) = g_3(\lambda L)$. Now we look at the Laurent expansion of the Weierstrass $\wp$-function around 0 (Lemma 2.13):

$$\wp(z, L') = \frac{1}{z^2} + \sum_{n=1}^{\infty} p_n(g_2(L'), g_3(L')) z^{2n} = \frac{1}{z^2} + \sum_{n=1}^{\infty} p_n(g_2(\lambda L), g_3(\lambda L)) z^{2n} = \wp(z, \lambda L)$$

Both functions $\wp(z, L')$ and $\wp(z, \lambda L)$ are holomorphic on $\mathbb{C} \setminus \{L' \cup \lambda L\}$. The Laurent expansion converges on a deleted neighbourhood $U \subset \mathbb{C} \setminus \{L' \cup \lambda L\}$ of 0. The two functions therefore agree on U and by the identity theorem for holomorphic functions they agree on all of $\mathbb{C} \setminus \{L' \cup \lambda L\}$. Thus, $\wp(z, L')$ and $\wp(z, \lambda L)$ have the same poles. By Lemma 2.12 the set of poles is exactly $L' = \lambda L$. Therefore, L and L' are homothetic. $\square$

Lemma 3.5. *(Cox; 2013) Let $\tau, \tau' \in \mathbb{H}$. Then, $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma(\tau)$ for some $\gamma \in SL(2, \mathbb{Z})$.*

Proof. ($\Rightarrow$) Let $\tau, \tau' \in \mathbb{H}$ such that $j(\tau) = j(\tau')$. From the definition of $j(\tau)$ and Lemma 3.4 it follows that $[1, \tau]$ and $[1, \tau']$ are homothetic, i.e. $[1, \tau'] = [\lambda, \lambda\tau]$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, there are $r, s, p, q \in \mathbb{Z}$ such that $\lambda = r\tau' + s$ and

$\lambda\tau = p\tau' + q$ or

$$\begin{pmatrix} \lambda\tau \\ \lambda \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

Dividing the two equations, we get

$$\tau = \frac{p\tau' + q}{r\tau' + s} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} (\tau')$$

We need to show that $\det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = 1$.

Analogously, we find $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} a\lambda\tau + b\lambda \\ c\lambda\tau + d\lambda \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda\tau \\ \lambda \end{pmatrix}$$

It follows that

$$\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

where $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. We have that $a'\tau' + b' = \tau'$ and thus $a' + \frac{b'}{\tau'} = 1$. Since $a', b' \in \mathbb{Z}$ and $\tau' \notin \mathbb{R}$, $b' = 0$ and, therefore, $a' = 1$. Similarly, $c'\tau' + d' = 1$ implies $c' = 0$ and $d' = 1$. Thus,

$$\det \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = 1 = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

The right hand side is the product of two integers. They must be equal to ± 1. We now just have to show that $\det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) > 0$. We write $\tau' = x + yi$ for some $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} 0 < \Im(\tau) &= \Im\left(\frac{p\tau' + q}{r\tau' + s}\right) = \Im\left(\frac{pr(x^2 + y^2) + qr(x - yi) + ps(x + yi) + sq}{|r\tau' + s|^2}\right) \\ &= \frac{psy - qry}{|r\tau' + s|^2} = \frac{\Im(\tau')(ps - qr)}{|r\tau' + s|^2} \end{aligned}$$

Since $\Im(\tau') > 0$, we get $0 < ps - qr = \det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Thus, $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$.

($\Leftarrow$) Let $\tau' = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)(\tau)$ for some $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. If we show that $[1, \tau']$ is homothetic to $[1, \tau]$, the result folllows from Lemma 3.4. Let $\lambda = r\tau + s$. Then,

$$\lambda[1, \tau'] = (r\tau + s)\left[1, \frac{p\tau + q}{r\tau + s}\right] = [r\tau + s, p\tau + q] \subset [1, \tau]$$

We have

$$\begin{aligned} -q(r\tau + s) + s(p\tau + q) &= \tau \\ p(r\tau + s) - r(p\tau + q) &= 1 \end{aligned}$$

Thus, $[1,\tau] \subset \lambda[1,\tau']$. Together we have $[1,\tau] = \lambda[1,\tau']$. $\square$

For $\tau \in \mathbb{H}$ we define $\Delta(\tau) := \Delta([1,\tau])$.

Corollary 3.6. *For all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau)$$

Proof. From the definition of Δ it follows that $\Delta(\lambda L) = \lambda^{-12}\Delta(L)$ for all lattices L and $\lambda \in \mathbb{C}\setminus\{0\}$. Let $\tau' = \frac{a\tau+b}{c\tau+d}$. In the proof of Lemma 3.5 we showed that $[1,\tau] = \lambda[1,\tau']$ for $\lambda = c\tau+d$. Thus, $\Delta(\tau) = \Delta([1,\tau]) = \lambda^{-12}\Delta([1,\tau']) = \lambda^{-12}\Delta(\tau')$. $\square$

We will take the following two results as given.

Lemma 3.7. *(Apostol; 1990; Cox; 2013) The j-function is holomorphic on $\mathbb{H}$.*

Lemma 3.8. *(Apostol; 1990; Cox; 2013) The Fourier expansion of the j-function is*

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n$$

where $q = e^{2\pi i\tau}$ and $c_n \in \mathbb{Z}$.

Remark 3.9. Lemma 3.5 together with Lemma 3.7 and Lemma 3.8 implies that $j(\tau)$ is a modular function. Moreover, we can view j as an injective map from $\mathbb{H}/SL(2,\mathbb{Z})$ to $\mathbb{C}$.

Lemma 3.10. *(Cox; 2013) The j-function is surjective, i.e. $j(\mathbb{H}) = \mathbb{C}$.*

Proof. Since j is an injective map from the fundamental domain $\mathcal{F}$ to $\mathbb{C}$ and $\mathcal{F}$ contains more than one point, j is certainly not constant. Moreover, $j(\tau)$ is holomorphic on $\mathbb{H}$. By the open mapping theorem, the image $j(\mathbb{H})$ is open in $\mathbb{C}$. If we prove that $j(\mathbb{H})$ is closed, it follows that j is surjective, i.e. $j(\mathbb{H}) = \mathbb{C}$ because $\mathbb{C}$ is connected.

Let $(j(t_k))_{k\in\mathbb{N}}$ be a sequence in $j(\mathbb{H})$ converging to some $z \in \mathbb{C}$. Because j is invariant under $SL(2,\mathbb{Z})$, we can assume that the t_k lie in $\mathcal{F}$. Therefore,

$$\forall k \in \mathbb{N} : |\Re(t_k)| < \frac{1}{2} \text{ and } |\Im(t_k)| \geq \frac{\sqrt{3}}{2}$$

Suppose that $\Im(t_k)$ is unbounded. For a subsequence with $\Im(t_{k_i})$ going to infinity, it follows form Lemma 3.8 that $\lim_{i\to\infty} j(t_{k_i}) = \infty$. This contradicts the assumption that $\lim_{k\to\infty} j(t_k) = z$. Hence, $\Im(t_k)$ is bounded and the t_k are contained in a compact set $K \subset \mathbb{H}$. Since K is compact and j is continuous, $j(K)$ is compact and thus closed. Therefore, $\lim_{k\to\infty} j(t_k) = z \in \overline{j(K)} = j(K)$. Because $j(K) \subset j(\mathbb{H})$, it follows that $z \in j(\mathbb{H})$. Thus $j(\mathbb{H})$ is closed. $\square$

Lemma 3.11. *(Scherer; 2010) Every holomorphic modular function for $SL(2,\mathbb{Z})$ can be written as a polynomial in $j(\tau)$.*

Proof. Let $f(\tau)$ be such a function and let

$$f(\tau) = \sum_{n=-m}^{\infty} a_n q^n, \quad m \in \mathbb{Z}_{\geq 0}$$

be its q-expansion. The q-expansion of $j(\tau)$ has only one negative q-power term, that is q^{-1}. Thus, we can find a polynomial p such that $f(\tau) - p(j(\tau))$ has no negative q-powers. (We can do this inductively. Let $p_m = a_{-m}$. Then $f(\tau) - p_m j(\tau)^m = \sum_{n=-m+1}^{\infty} a_n^{(1)} q^n$ for some $a_n^{(1)} \in \mathbb{C}$. Repeat this procedure until p_1 is defined. Then $p(z) = \sum_{n=1}^{m} p_n z^n$.) Then $f(\tau) - p(j(\tau))$ is bounded on $\mathbb{H}$, since f and j are holomorphic. Lemma 2.7 implies that $f(\tau) - p(j(\tau))$ is constant and thus $f(\tau) = c + p(j(\tau))$ for a $c \in \mathbb{C}$. □

Theorem 3.12. *(Apostol; 1990) Every modular function is a rational function of the j-function.*

Proof. Let $f(\tau)$ be a modular function for $SL(2,\mathbb{Z})$. By Remark 2.6, f has a finite number of poles in $\boldsymbol{\mathcal{F}}$. Let $\{\tau_k\}_{1\leq k\leq n}$ be the poles of f with orders m_k. Because j is holomorphic at τ_k, the zero of $j(\tau) - j(\tau_k)$ in t_k has at least order 1. Hence, $f(\tau)(j(\tau) - j(\tau_k))^{m_k}$ is holomorphic at τ_k. Thus,

$$f(\tau) \prod_{k=1}^{n} (j(\tau) - j(\tau_k))^{m_k}$$

is holomorphic on $\mathbb{H}$. Let $q(j(\tau)) := \prod_{k=1}^{n} (j(\tau) - j(\tau_k))^{m_k}$. By Lemma 3.11 we have $q(j(\tau))f(\tau) = p(j(\tau))$ and thus

$$f(\tau) = \frac{p(j(\tau))}{q(j(\tau))}$$

is a rational function in $j(\tau)$. □

Remark 3.13. Let f be a modular function with q-expansion

$$f(\tau) = \sum_{n=-m}^{\infty} a_n q^n.$$

Suppose f is holomorphic on $\boldsymbol{\mathcal{F}}$ except for n poles of order m_k at $\tau_k \in \boldsymbol{\mathcal{F}}$. Then already finitely many a_n determine f uniquely. More precisely, it is sufficient to know the first $m + \sum_{k=1}^{n} m_k + 1$ coefficients.

Proof. First, suppose that f is holomorphic on $\mathbb{H}$. Using the construction given in the proof of Lemma 3.11, we can find a polynomial $p(z) = \sum_{n=1}^{m} p_n z^n$ such that

$$f(\tau) = c + \sum_{n=1}^{m} p_n j(\tau)^n.$$

To determine p, we only used the values of $a_{-m}, a_{-m+1}, ..., a_{-1}$ and c is determined by a_0.

If f is holomorphic except for n poles of order m_k at $t_k \in \mathcal{F}$, let

$$r(\tau) = \prod_{k=1}^{n} (j(\tau) - j(\tau_k))^{m_k}.$$

Since j has a pole of order one in q, $r(\tau) = \sum_{n=-M}^{\infty} b_n q^n$ with $M := \sum_{k=1}^{n} m_k$. Now, $f(\tau)r(\tau) = \sum_{n=-M-m}^{\infty} c_n q^n$ is holomorphic on $\mathbb{H}$. By the first case, we need the coefficients $c_{-M-m}, ..., c_1, c_0$ to determine $f(\tau)r(\tau)$. These coefficients are determined by $a_{-m}, a_{-m+1}, ..., a_M$ and $b_{-M}, .., b_m$. But the b_k only depend on the location and order of the poles, not on f itself. Hence, if we know the first $m+1+M$ coefficients a_k, the product $f(\tau)r(\tau)$ is uniquely determined and so is f. $\square$

4 Congruence Subgroups

We write (a, b) or $\gcd(a, b)$ for the greatest common divisor of a and b and we follow the convention that $\pm 1/0 = \infty$.

Definition 4.1. For a positive integer N we define $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$ as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod N \right\}$$

$$\Gamma_1(N) \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

A *congruence subgroup* of $SL(2, \mathbb{Z})$ is a subgroup which contains $\Gamma(N)$ for some N. In particular, $\Gamma_1(N)$ and $\Gamma_0(N)$ are congruence subgroups for every N.

Definition 4.2. A *cusp* is an element of $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ which is $SL(2, \mathbb{Z})$-equivalent to ∞.

Lemma 4.3. *The set of cusps is exactly* $\mathbb{Q} \cup \{\infty\}$.

Proof. For every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ we have that $\gamma(\infty) = \frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$. Conversely, we can write every $q \in \mathbb{Q}$ as $q = \frac{m}{n}$ for relatively prime integers m and n. There are integers b and d such that $dm - bn = (m, n) = 1$. Then $\gamma := \left(\begin{smallmatrix} m & b \\ n & d \end{smallmatrix}\right)$ lies in $SL(2, \mathbb{Z})$ and $\gamma(\infty) = q$. $\square$

For a subgroup Γ of $SL(2, \mathbb{Z})$ not all cusps need to be Γ-equivalent. In this section we will study the equivalence classes of cusps for $\Gamma(N)$ and $\Gamma_0(N)$. Then we will examine $\Gamma_0(p)$ more precisely for the case that p is prime and look at modular functions for $\Gamma_0(N)$.

4.1 Cusps under $\Gamma(N)$

Lemma 4.4. *(Shimura; 1971) Let a, b, c, d be integers such that $(a, b) = 1$ and (c, d)=1. Then,*

$$\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} \pmod N \Leftrightarrow \exists \gamma \in \Gamma(N) : \begin{pmatrix} a \\ b \end{pmatrix} = \gamma \begin{pmatrix} c \\ d \end{pmatrix}$$

Proof. ($\Leftarrow$) Since $\gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N$,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \gamma \begin{pmatrix} c \\ d \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} \pmod N$$

($\Rightarrow$) First, assume that $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then $a \equiv 1 \pmod N$ and hence $1 - a$ is divisible by N. Since $(a, b) = 1$, we can find integers p' and q' such that $ap' - bq' = 1$. Now let $p = p'(1 - a)/N$ and $q = q'(1 - a)/N$ and $\gamma = \left(\begin{smallmatrix} a & Nq \\ b & 1+Np \end{smallmatrix}\right)$. Then $\det(\gamma) = a + ap'(1 - a) - bq'(1 - a) = a + 1 - a = 1$. Then $\gamma \in \Gamma(N)$ and $\gamma \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$.
In the general case, let r and s be integers such that $cr + ds = 1$ and $\sigma = \left(\begin{smallmatrix} c & -s \\ d & r \end{smallmatrix}\right)$. Since $\sigma \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \pmod N$, we get $\sigma^{-1} \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \pmod N$. By the first case, we can find a $\gamma \in \Gamma(N)$ such that $\gamma \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \sigma^{-1} \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \sigma\gamma \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \sigma\gamma\sigma^{-1} \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$ and $\sigma\gamma\sigma^{-1} \equiv \sigma I_2 \sigma^{-1} \pmod N \equiv I_2 \pmod N$. Thus $\sigma\gamma\sigma^{-1}$ has the desired properties. ☐

Lemma 4.5. *If*

$$\frac{a}{b} = \frac{pc + qd}{rc + sd}$$

for $a, b, c, d, p, q, r, s \in \mathbb{Z}$ with $(a, b) = (c, d) = \det \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = 1$ and $b \neq 0$, then $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$.

Proof. We get that $\lambda \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$ for $\lambda = (rc + sd)/b \in \mathbb{Q}$. Write $\lambda = m/n$ for relatively prime integers m and n. Then,

$$m \begin{pmatrix} a \\ b \end{pmatrix} = n \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \tag{1}$$

We see that $n \mid a, b$ because $(n, m) = 1$. Since $(a, b) = 1$, it follows that $n = \pm 1$. Multiplying Equation 1 from left by $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^{-1}$, we see that $m \mid c, d$ because $(m, n) = 1$. Since $(c, d) = 1$, it follows that $m = \pm 1$ and thus $\lambda = \pm 1$. ☐

Theorem 4.6. *(Shimura; 1971) Let $z = a/b$ and $z' = c/d$ be cusps of $\Gamma(N)$ written as quotients of relatively prime integers (where $\pm 1/0 = \infty$). Then z and z' are $\Gamma(N)$-equivalent if and only if $\pm \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} \pmod N$.*

This gives us all equivalence classes of cusps for $\Gamma(N)$.

Proof. ($\Rightarrow$) Take $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \Gamma(N)$ such that $\gamma(z') = z$. If $bd = 0$, we can assume w.l.o.g. that $d = 0$. Then $c = \pm 1$ and $\gamma(z') = p/r = z$. Since $(p, r) \mid \det(\gamma) = 1$, we have that $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} p \\ r \end{smallmatrix}\right)$ and because $\gamma \in \Gamma(N)$ we get $\pm \left(\begin{smallmatrix} p \\ r \end{smallmatrix}\right) \equiv \pm \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \pmod{N} \equiv \pm \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \pmod{N}$.

If $bd \neq 0$, we get

$$z = \frac{a}{b} = \frac{pc + qd}{rc + sd}$$

By Lemma 4.5 we obtain $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \equiv \pm \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \pmod{N}$.

($\Leftarrow$) By Lemma 4.4 there is a $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \Gamma(N)$ such that $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. If $d = 0$, we have $c = \pm 1$ and thus $\left(\begin{smallmatrix} p \\ r \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. Moreover, $z' = \infty$ and $\gamma(z') = p/r = a/b = z$. If $d \neq 0$, we get

$$\gamma(z') = \frac{pz' + q}{rz' + s} = \frac{pc + qd}{rc + sd} = \frac{a}{b} = z$$

Therefore, z and z' are $\Gamma(N)$-equivalent. □

4.2 Cusps under $\Gamma_0(N)$

Definition 4.7. Let n be a positive integer. Define $\varphi(n)$ as the number of integers k such that $1 \leq k \leq n$ and $(k, n) = 1$. This function φ is called *Euler's totient function.*

Definition 4.8. Let n be a positive integer. A *reduced residue system modulo* n is a set $R \subset \mathbb{Z}$ such that for all r in R we have $(r, n) = 1$ and no two elements of R are congruent modulo n.

A reduced residue system modulo n contains $\varphi(n)$ elements, where φ is Euler's totient function.

Lemma 4.9. *Let N be a positive integer and $c \in \mathbb{Z}$ dividing N. Then there exists a reduced residue system $R_{c,N}$ modulo $(c, N/c)$ with $(c, d) = 1$ for all d in $R_{c,N}$.*

Proof. Suppose d' is relatively prime to $(c, N/c)$. Let $d := d' + (c, N/c) \prod_{p|c, p \nmid d'} p$, where p are prime. Suppose there was a prime number p dividing (c, d). If $p \mid d'$, then also $p \mid (c, N/c)$. Since $(d', (c, N/c)) = 1$, it follows that $p \mid 1$, a contradiction. If $p \nmid d'$, then $p \mid \prod_{p'|c, p' \nmid d'} p'$ and hence also $p \mid d - (c, N/c) \prod_{p'|c, p' \nmid d'} p' = d'$, again a contradiction. Therefore, $(c, d) = 1$ and also $((c, N/c), d) = 1$. Moreover, $d \equiv d' \pmod{(c, N/c)}$. Thus if we take any reduced residue system modulo $(c, N/c)$ and replace all its elements d' by the corresponding d, we obtain $R_{c,N}$. □

Theorem 4.10. *(Wang and Pei; 2012) Let N be a positive integer. For all $c \in \mathbb{Z}_{>0}$ dividing N let $R_{c,N}$ be a reduced residue system modulo $(c, N/c)$ with $(c, d) = 1$ for all d in $R_{c,N}$. The set $M := \{d/c \mid c \in \mathbb{Z}_{>0},\ c|N,\ d \in R_{c,N}\}$ contains one representative of each equivalence class of cusps of $\Gamma_0(N)$. The number of these equivalence classes is equal to*

$$|M| = \sum_{c|N} \varphi((c, N/c))$$

Proof. First, we count the elements in M. Let $M_c := \{d/c \mid d \in R_{c,N}\}$. Then for $c \neq c'$ the sets M_c and $M_{c'}$ are disjoint. Thus, $|M| = \sum_{c|N} |M_c| = \sum_{c|N} \varphi((c, N/c))$.

We have to prove that (i) every cusp is $\Gamma_0(N)$-equivalent to some element of M and that (ii) no two different elements of M are $\Gamma_0(N)$-equivalent. We begin by proving (i). First let d'/c and d/c be two cusps (written as reduced fractions) such that $c \mid N$, and $d \equiv d' \pmod{(c, N/c)}$. Then we can find $a, b, a', b' \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a' & d' \\ b' & c \end{smallmatrix}\right)$ lie in $SL(2, \mathbb{Z})$. Then, $bd \equiv b'd' \equiv -1 \pmod{(c, N/c)}$. Thus $b \equiv b' \pmod{(c, N/c)}$ and there are $m, n \in \mathbb{Z}$ such that $b = b' + mc + nN/c$. Let

$$\gamma = \begin{pmatrix} a - md & d \\ b - mc & c \end{pmatrix} \begin{pmatrix} c & -d' \\ -b' & a' \end{pmatrix}$$

Since γ is the product of two matrices with determinant one, $\det \gamma = 1$. The bottom left entry of γ is $\gamma_{21} = bc - mc^2 - b'c = nN$ and thus $\gamma \in \Gamma_0(N)$. Moreover, direct calculation gives $\gamma(d'/c) = d/c$ and thus d'/c is $\Gamma_0(N)$-equivalent to d/c. By the definition of M, for every cusp d'/c as above, we can find a corresponding $d/c \in M$. Hence, all cusps of this form are equivalent to some element of M. Since $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$, we see that ∞ is equivalent to $1/N$ and hence to some element of M.

Now let n/m with $(n, m) = 1$ be a cusp. Let $c := (m, N)$. Then also $(m, nN) = c$ and hence there are $\alpha, \beta \in \mathbb{Z}$ with

$$\alpha m + \beta n N = c. \tag{2}$$

Define $\alpha' := \alpha + nN/c \prod_{p|N, p\nmid\alpha} p$ and $\beta' := \beta - m/c \prod_{p|N, p\nmid\alpha} p$ where p are prime. Since $\alpha' m/c + \beta' nN/c = 1$, we have that $(\alpha', \beta') = 1$. From Equation (2) it follows that $(\alpha m/c, \beta nN/c) = 1$ and thus also $(\alpha, nN/c) = 1$. We find $(\alpha', N) = 1$ by the same argument as in the proof of Lemma 4.9. Hence also $(\alpha', \beta' N) = 1$ and there exists a $\sigma \in \Gamma_0(N)$ of the form

$$\sigma = \begin{pmatrix} a & b \\ \beta' N & \alpha' \end{pmatrix}$$

Then we have

$$\sigma(n/m) = \frac{an + bm}{\beta' N n + \alpha' m} = \frac{an + bm}{c} = \frac{d}{c'}$$

for some $d, c' \in \mathbb{Z}$ with $(d, c') = 1$ and $c' \mid N$. Since d/c' is $\Gamma_0(N)$-equivalent to some element of M, so is n/m.

To prove statement (ii), we assume that p and q are two $\Gamma_0(N)$-equivalent elements of M. We write them as reduced fractions $p = d/c$ and $q = d'/c'$. Since they are $\Gamma_0(N)$-equivalent, there is a $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma N & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ such that

$$\frac{\alpha d + \beta c}{\gamma N d + \delta c} = \frac{d'}{c'}$$

By Lemma 4.5 and after replacing σ with $-\sigma$ if necessary, we get

$$\alpha d + \beta c = d' \tag{3}$$
$$\gamma N d + \delta c = c' \tag{4}$$

Since $c \mid N$, Equation (4) implies $c \mid c'$. By the same argument with p and q exchanged, we also get $c' \mid c$ and thus $c = c'$. After dividing Equation (4) by c, we get $\delta \equiv 1 \pmod{N/c}$. Because $\det \sigma = 1$, we have $\alpha\delta \equiv 1 \pmod{N} \equiv 1 \pmod{N/c}$ and hence $\alpha \equiv 1 \pmod{N/c}$. Now it follows from Equation (3) that $d \equiv d' \pmod{(c, N/c)}$. By the definition of $R_{c,N}$, d and d' must be equal. Hence, $p = d/c$ and $q = d'/c'$ are the same element of M. □

Corollary 4.11. *If p is prime, there are exactly two equivalence classes of cusps under $\Gamma_0(p)$. They are represented by 1 and ∞.*

4.3 Fundamental Region for $\Gamma_0(p)$

Throughout this section, we assume p to be any prime.

Lemma 4.12. *(Apostol; 1990) Let $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ be the generators of the full modular group $SL(2, \mathbb{Z})$. Then every $V \in SL(2, \mathbb{Z}) - \Gamma_0(p)$ can be written as*

$$V = PST^k$$

for some $P \in \Gamma_0(p)$ and some integer $0 \leq k < p$.

Proof. We have that $V = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for $C \not\equiv 0 \pmod{p}$. We want to find $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \pmod{p}$ and an integer $0 \leq k < p$ so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}$$

Solving this for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} kA - B & A \\ kC - D & C \end{pmatrix}$$

Because $C \not\equiv 0 \pmod{p}$, there is a $0 \leq k < p$ with $kC \equiv D \pmod{p}$. Choose

$$c = kC - D, \quad a = kA - B, \quad b = A, \quad d = C.$$

Then $c \equiv 0 \pmod{N}$ and hence $P \in \Gamma_0(p)$. □

Remark 4.13. The sets $\Gamma_k := \{PA_k \mid P \in \Gamma_0(p)\}$ where $A_k = ST^k$ if $0 \leq k < p$ and $A_p = I_2$ are pairwise disjoint.

Proof. Suppose $PST^k = QST^l$ for some P and Q in $\Gamma_0(p)$ and $0 \le k, l < p$. Then we have that $P^{-1}Q = ST^{k-l}S^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -k+l & 1 \end{smallmatrix}\right)$. Since $P^{-1}Q \in \Gamma_0(p)$ we get $k - l \equiv 0 \pmod{p}$. Because of the bounds on l and k, the only solution is $k = l$. Thus, the sets Γ_k are pairwise disjoint for $0 \le k < p$.

Suppose $P = QST^l$ for some P and Q in $\Gamma_0(p)$ and $0 \le l < p$. Then we get $S = Q^{-1}PT^{-l} \in \Gamma_0(p)$, but since $S \notin \Gamma_0(p)$ this is a contradiction. Thus, Γ_p is disjoint from any Γ_l with $0 \le l < p$. $\square$

Let $\boldsymbol{\mathcal{F}}$ be a fundamental region of $SL(2,\mathbb{Z})$.

Theorem 4.14. *(Apostol; 1990) A fundamental region of $\Gamma_0(p)$ is given by*

$$\boldsymbol{\mathcal{F}}_p = \boldsymbol{\mathcal{F}} \cup \bigcup_{k=0}^{p-1} ST^k(\boldsymbol{\mathcal{F}})$$

In Figure 1 the fundamental region $\boldsymbol{\mathcal{F}}_5$ is shown for the choice of $\boldsymbol{\mathcal{F}}$ as in Lemma 2.3.

Proof. We have to prove that

(i) every $\tau \in \mathbb{H}$ is $\Gamma_0(p)$-equivalent to some point in the closure of $\boldsymbol{\mathcal{F}}_p$ and,

(ii) no two distinct points in $\boldsymbol{\mathcal{F}}_p$ are $\Gamma_0(p)$-equivalent.

We begin by proving (i). Let $\tau \in \mathbb{H}$. Since $\boldsymbol{\mathcal{F}}$ is a fundamental region for $SL(2,\mathbb{Z})$, we can find an $A \in SL(2,\mathbb{Z})$ with $A(\tau) = \tau_1 \in \overline{\boldsymbol{\mathcal{F}}}$. By Lemma 4.12 there are $P \in \Gamma_0(p)$, $0 \le k < p$ and $W = I_2$ or $W = ST^k$ such that $A^{-1} = PW$. Let $V := P^{-1} = WA$. We have that $V \in \Gamma_0(p)$ with $V(\tau) = WA(\tau) = W(\tau_1) \in \overline{\boldsymbol{\mathcal{F}}_p}$. This implies (i).

To prove (ii) suppose τ_1 and τ_2 are in $\boldsymbol{\mathcal{F}}_p$ and there is a $V \in \Gamma_0(p)$ with $V(\tau_1) = \tau_2$. We want to show that $\tau_1 = \tau_2$. We look at three cases:

(a) $\tau_1, \tau_2 \in \boldsymbol{\mathcal{F}}$. Since $V \in SL(2,\mathbb{Z})$ and $\boldsymbol{\mathcal{F}}$ is a fundamental domain we have $\tau_1 = \tau_2$.

(b) $\tau_1 \in \boldsymbol{\mathcal{F}}$, $\tau_2 \in ST^k(\boldsymbol{\mathcal{F}})$. Write $\tau_2 = ST^k(\tau_3)$ for some $\tau_3 \in \boldsymbol{\mathcal{F}}$. Then $\tau_1 = V^{-1}(\tau_2) = V^{-1}ST^k(\tau_3)$. Since τ_1 and τ_3 both lie in $\boldsymbol{\mathcal{F}}$, they must be equal. Let $U := \left(V^{-1}ST^k\right)^{-1}(\boldsymbol{\mathcal{F}})$. Because the map $\tau \mapsto V^{-1}ST^k(\tau)$ is continuous, U is open. On the open and nonempty set $U \cap \boldsymbol{\mathcal{F}}$ the map $\tau \mapsto V^{-1}ST^k(\tau)$ is the identity and by the identity theorem, it is the identity on all of $\mathbb{H}$. Thus $V^{-1}ST^k = \pm I_2$. Hence, $V = \pm ST^k = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & k \end{smallmatrix}\right)$ which contradicts $V \in \Gamma_0(p)$.

(c) $\tau_1 \in ST^{k_1}(\boldsymbol{\mathcal{F}})$, $\tau_2 \in ST^{k_2}(\boldsymbol{\mathcal{F}})$. There are $\tau_1', \tau_2' \in \boldsymbol{\mathcal{F}}$ with $\tau_1 = ST^{k_1}(\tau_1')$ and $\tau_2 = ST^{k_2}(\tau_2')$. Because $V(\tau_1) = \tau_2$, we get $VST^{k_1}(\tau_1') = ST^{k_2}(\tau_2')$ and hence as above $VST^{k_1-k_2}S^{-1} = \pm I_2$. Therefore, $V = \pm ST^{k_2-k_1}S^{-1} = \pm\left(\begin{smallmatrix} 1 & 0 \\ k_1-k_2 & 1 \end{smallmatrix}\right)$. Since $V \in \Gamma_0(p)$, we get $k_2 \equiv k_1 \pmod{p}$. But k_1 and k_2 both lie between 0 and $p-1$, thus they must be equal. We get that $V = \pm ST^0S^{-1} = \pm I_2$ and $\tau_1 = \tau_2$.

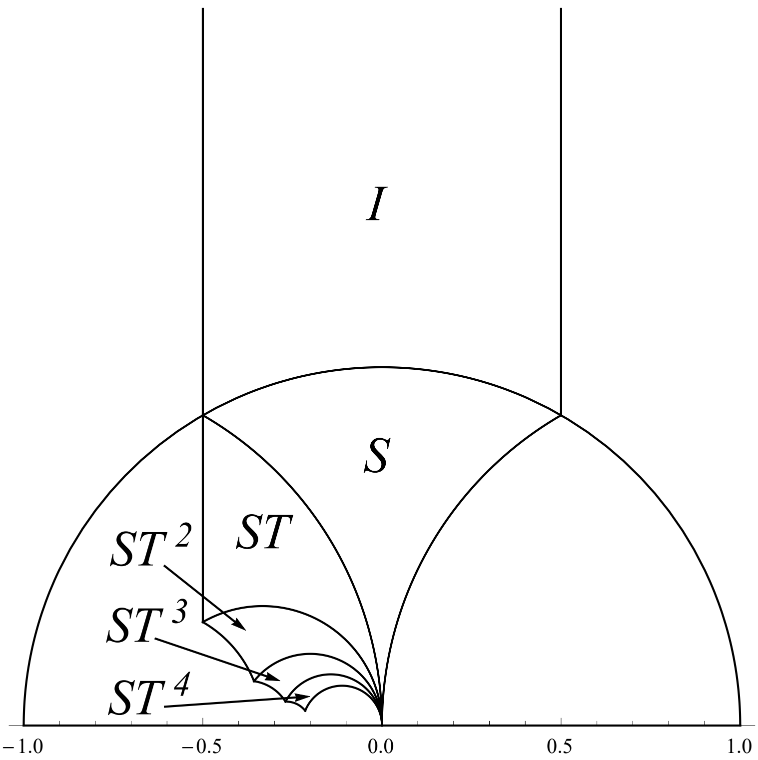

Figure 1: Plot of a fundamental region for $\Gamma_0(5)$.

□

4.4 Automorphic Functions

Let N be a positive integer. Let p be any prime. First, we generalise the definition of modular functions.

Definition 4.15. Let G be a subgroup of $SL(2,\mathbb{Z})$ conjugate to $\Gamma_0(N)$. A function $f : \mathbb{H} \to \mathbb{C}$ is *automorphic under G* or a *modular function for G* if it satisfies the following three conditions:

(i) f is meromorphic on $\mathbb{H}$

(ii) f is invariant under G, i.e. $f(\gamma(\tau)) = f(\tau)$ for all $\gamma \in G$ and $\tau \in \mathbb{H}$

(iii) f is meromorphic at the cusps, i.e. for very $\gamma \in SL(2,\mathbb{Z})$ the Fourier expansion of $f(\gamma(\tau))$ is of the form

$$f(\gamma(\tau)) = \sum_{n=-m}^{\infty} a_n e^{2\pi i n\tau/N}$$

for some $m \in \mathbb{Z}$.

If $f \not\equiv 0$ we can choose m in condition (iii) such that $a_{-m} \neq 0$. If $m > 0$ or $m < 0$, we say that f has a pole or a zero of order m/N at the cusp $\gamma(i\infty)$, respectively.

Remark 4.16. (Cox; 2013) For condition (iii) to make sense, we have to show that $f(\gamma(\tau))$ is invariant under $\tau \mapsto \tau + N$, which corresponds to $U := \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$. Then $f(\gamma(\tau))$ is a meromorphic function of $q^{1/N} = e^{2\pi i n\tau/N}$.

Proof. We have $G = \sigma\Gamma_0(N)\sigma^{-1}$ for some $\sigma \in SL(2, \mathbb{Z})$. For any $\gamma \in SL(2, \mathbb{Z})$ we have that $\gamma U \gamma^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ cd-c^2N-cd & * \end{smallmatrix}\right)$ lies in $\Gamma_0(N)$. Hence, also $(\sigma^{-1}\gamma)U(\sigma^{-1}\gamma)^{-1}$ lies in $\Gamma_0(N)$ and $\sigma(\sigma^{-1}\gamma)U(\sigma^{-1}\gamma)^{-1}\sigma^{-1} = \gamma U \gamma^{-1}$ belongs to G. Thus, for any $\gamma \in SL(2, \mathbb{Z})$ we have $f(\gamma U(\tau)) = f(\gamma U \gamma^{-1}\gamma(\tau)) = f(\gamma(\tau))$. Therefore, $f(\gamma(\tau))$ is invariant under $\tau \mapsto \tau + N$. $\square$

Theorem 4.17. *(Apostol; 1990) Every function f which is automorphic under $\Gamma_0(p)$ and bounded in $\mathbb{H}$, is constant.*

Proof. By Lemma 4.12 we can write

$$SL(2, \mathbb{Z}) = \bigcup_{k=0}^{p} \{PA_k \mid P \in \Gamma_0(p)\}$$

where $A_k = ST^k$ if $k < p$ and $A_p = I_2$. Let $V_k \in \Gamma_k := \{PA_k \mid P \in \Gamma_0(p)\}$ and define

$$f_k(\tau) = f(V_k(\tau)).$$

These functions are well defined since

$$f_k(\tau) = f(V_k(\tau)) = f(PA_k(\tau)) = f(A_k(\tau))$$

which depends only on k and not on the choice of V_k. Note that $f_p(\tau) = f(P(\tau)) = f(\tau)$. Now let $V \in SL(2, \mathbb{Z})$. Then $f_k(V(\tau)) = f(A_k V(\tau))$. Since $A_k V \in SL(2, \mathbb{Z})$, $A_k V = QA_m$ for some $Q \in \Gamma_0(p)$ and an integer $0 \leq m \leq p$. Therefore,

$$f_k(V(\tau)) = f(QA_m(\tau)) = f_m(\tau).$$

If $A_k V = QA_m$ and $A_l V = RA_m$ for some Q and $R \in \Gamma_0(p)$, then $A_l = RQ^{-1}A_k$ and hence $\Gamma_l = \Gamma_k$. Since Γ_l and Γ_k are disjoint for $k \neq l$ by Remark 4.13, we have $l = k$. Thus, there is a permutation σ of $\{0, 1, ..., p\}$ with $f_k(V\tau) = f_{\sigma(k)}(\tau)$ for $0 \leq k \leq p$. Now let $w \in \mathbb{H}$ be fixed and let

$$\phi(\tau) = \prod_{k=0}^{p} (f_k(\tau) - f(w)).$$

Because f and hence each f_k is bounded, ϕ is bounded as well. Therefore, ϕ has no poles in $\mathbb{H} \cup \{\infty\}$. For $V \in SL(2, \mathbb{Z})$

$$\phi(V\tau) = \prod_{k=0}^{p} (f_k(V(\tau)) - f(w)) = \prod_{k=0}^{p} (f_{\sigma(k)}(\tau) - f(w)) = \phi(\tau).$$

So ϕ is a holomorphic modular function holomorphic at ∞. By Lemma 2.7, ϕ is constant and since $\phi(w) = 0$, we have $\phi \equiv 0$. Thus for $\tau = i$ we have

$$0 = \prod_{k=0}^{p} (f_k(i) - f(w))$$

hence one factor must be zero. Since w was arbitrary, f can only take values in $\{f_k(i)\}_{k=0}^{p}$. Because f is continuous, f must be constant. □

4.5 Hauptmoduln for $\Gamma_0(N)$

Definition 4.18. Let G be a subgroup of $SL(2,\mathbb{Z})$ conjugate to $\Gamma_0(N)$ for some N. A *Hauptmodul* for G is a function which generates the field of modular functions for G. A Hauptmodul for $\Gamma_0(N)$ is also called *Hauptmodul of level N*.

Example 4.19. In Section 3 we proved that the j-function is a Hauptmodul of level 1, i.e. it is a Hauptmodul for $SL(2,\mathbb{Z})$.

Table 1: Hauptmoduln of level N (Beneish and Larson; 2014)

N	2	3	4	5	6	7	13
$j_N(\tau)$	$\frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$	$\frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}}$	$\frac{\eta(\tau)^{8}}{\eta(4\tau)^{8}}$	$\frac{\eta(\tau)^{6}}{\eta(5\tau)^{6}}$	$\frac{\eta(2\tau)^{3}\eta(3\tau)^{9}}{\eta(\tau)^{3}\eta(6\tau)^{9}}$	$\frac{\eta(\tau)^{4}}{\eta(7\tau)^{4}}$	$\frac{\eta(\tau)^{2}}{\eta(13\tau)^{2}}$

The *Dedekind eta function* is defined as

$$\eta(\tau) = \left(\frac{\Delta(\tau)}{(2\pi)^{12}}\right)^{1/24} = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q^{1/24} = e^{2\pi i/24}$. The Dedekind eta function is holomorphic and nonzero on $\mathbb{H}$ (Apostol; 1990).

In Table 1 some Hauptmoduln j_N of level N are listed. In this section, we show that the given Hauptmoduln are invariant under $\Gamma_0(N)$ and we examine their behaviour at the cusps. Since the Hauptmoduln are fractions of the Dedekind eta function, we need to know how η transforms under $SL(2,\mathbb{Z})$. Recall Corollary 3.6: For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\Delta(\tau).$$

Taking the 24th root we get that

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a,b,c,d)(c\tau + d)^{1/2}\eta(\tau),$$

for some ϵ with $\epsilon^{24} = 1$ depending on our transformation. In Apostol (1990) a formula is derived for this ϵ:

Theorem 4.20. *(Apostol; 1990) If* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ *with* $c > 0$ *and* $\tau \in \mathbb{H}$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\left(-i(c\tau+d)\right)^{1/2}\eta(\tau),$$

where

$$\epsilon(a,b,c,d) = \exp\left(\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right)$$

and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

Remark 4.21. For a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ with $c \neq 0$ we can fulfill the condition $c > 0$ in Theorem 4.20 by replacing γ with $-\gamma$, which describes the same transformation. If $c = 0$, γ is a power of $\tau \mapsto \tau + 1$ and transforms according to Lemma 4.24.

The function $s(h,k)$ is called *Dedekind sum*. It has the following properties:

Theorem 4.22. *(Apostol; 1990) Suppose (h,k)=1 and k is positive.*

(i) If $a \in \mathbb{Z}$ *with* $ha \equiv \pm 1 \pmod{k}$, *then* $s(a,k) = \pm s(h,k)$.

(ii) If $h^2 + 1 \equiv 0 \pmod{k}$, *then* $s(h,k) = 0$.

Theorem 4.23. *(Apostol; 1990) Let* $N = 3, 5, 7$ *or* 13 *and* $r = 24/(N-1)$. *For integers* a, b, c, d *with* $ab - Ncd = 1$ *and* $c > 0$, *let*

$$\delta = \left(s(a, Nc) - \frac{a+d}{12Nc}\right) - \left(s(a,c) - \frac{a+d}{12c}\right).$$

The product $r\delta$ *then is an even integer.*

Lemma 4.24. *(Apostol; 1990) For the generators* $T : \tau \mapsto \tau + 1$ *and* $S : \tau \mapsto -1/\tau$ *of* $SL(2,\mathbb{Z})$ *we have*

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$$
$$\eta\left(\frac{-1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau)$$

Proof. By definition of $\eta(\tau)$ and with $q = e^{2\pi i\tau}$, we have

$$\eta(\tau+1) = e^{2\pi i(\tau+1)/24}\prod_{n=1}^{\infty}(1 - e^{2\pi i n(\tau+1)}) = e^{\pi i/12}q^{1/24}\prod_{n=1}^{\infty}(1 - q^n e^{2\pi i n}) = e^{\pi i/12}\eta(\tau).$$

To obtain the second equation, we apply Theorem 4.20 for $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$\eta\left(\frac{-1}{\tau}\right) = \epsilon(0,-1,1,0)(-i\tau)^{1/2}\eta(\tau),$$

with $\epsilon(0,-1,1,0) = \exp(\pi i\ s(0,1))$ and $s(0,1) = 0$. $\square$

Lemma 4.25. *If $k \mid N$ and $\gamma = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c > 0$, then*

$$\eta(k\gamma(\tau)) = \epsilon(a, kb, cN/k, d)(-i(Nc\tau + d))^{1/2}\eta(k\tau).$$

Proof. Using Theorem 4.20, we get

$$\begin{aligned}\eta(k\gamma(\tau)) &= \eta\left(\frac{ka\tau + kb}{Nc\tau + d}\right) = \eta\left(\begin{pmatrix} a & kb \\ Nc/k & d \end{pmatrix}(k\tau)\right) \\ &= \epsilon(a, kb, cN/k, d)(-i(Nc\tau + d))^{1/2}\eta(k\tau).\end{aligned}$$

□

Lemma 4.26. *For $N = 2, 3, 4, 5, 7$ or 13, $j_N(\tau)$ is invariant under $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. By Lemma 4.24, we have

$$\eta\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) = e^{\pi i/12}\eta(\tau)$$

and

$$\eta\left(N\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) = \eta\left(\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)(N\tau)\right) = e^{N\pi i/12}\eta(N\tau).$$

Hence,

$$j_N\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) = \left(e^{-(N-1)\pi i/12}\frac{\eta(\tau)}{\eta(N\tau)}\right)^{24/(N-1)} = e^{-2\pi i}j_N(\tau) = j_N(\tau).$$

□

Theorem 4.27. *For $N = 2, 3, 5, 7$ or 13, $j_N(\tau)$ is invariant under $\Gamma_0(N)$.*

Proof. If $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_0(N)$, then $\det(\gamma) = 1$ implies $a = d = \pm 1$. Hence, the transformation induced by γ is the same as the transformation induced by some power of T. From Lemma 4.26 it follows that j_N is invariant under γ.

If γ is not a power of T, we distinguish two cases.

$N = 2$: Let $\gamma = \left(\begin{smallmatrix} a & b \\ 2c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ with $c \neq 0$. If $c < 0$ we replace γ by $-\gamma$ which induces the same transformation. By Lemma 4.25 and using $\epsilon^{24} = 1$ we have

$$j_2(\gamma(\tau)) = \frac{\eta(\gamma(\tau))^{24}}{\eta(2\gamma(\tau))^{24}} = \frac{(2c\tau + d)^{12}\eta(\tau)^{24}}{(2c\tau + d)^{12}\eta(2\tau)^{24}} = j_2(\tau).$$

$N \in \{3, 5, 7, 13\}$: Let $\gamma = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c \neq 0$ and $r = 24/(N-1)$. If $c < 0$ we replace γ by $-\gamma$ which induces the same transformation. With Lemma 4.25 we get

$$\begin{aligned}j_N(\gamma(\tau)) = \left(\frac{\eta(\gamma(\tau))}{\eta(N\gamma(\tau))}\right)^r &= \left(\frac{\epsilon(a, b, cN, d)(-i(Nc\tau + d))^{1/2}\eta(\tau)}{\epsilon(a, Nb, c, d)(-i(Nc\tau + d))^{1/2}\eta(N\tau)}\right)^r \\ &= \left(\frac{\epsilon(a, b, cN, d)}{\epsilon(a, Nb, c, d)}\right)^r j_N(\tau).\end{aligned}$$

Now,

$$\left(\frac{\epsilon(a,b,cN,d)}{\epsilon(a,Nb,c,d)}\right)^r = \exp\left(r\pi i\left(\frac{a+d}{12Nc} + s(-d,Nc) - \frac{a+d}{12c} - s(-d,c)\right)\right).$$

Since $ad - Ncb = 1$, we have $ad \equiv 1 \pmod{Nc}$ and $ad \equiv 1 \pmod{c}$. By Theorem 4.22, we get $s(-d,Nc) = -s(a,Nc)$ and $s(-d,c) = -s(a,c)$. Hence, together with Theorem 4.23, we have

$$\left(\frac{\epsilon(a,b,cN,d)}{\epsilon(a,Nb,c,d)}\right)^r = \exp\left(r\pi i\left(\frac{a+d}{12Nc} - s(a,Nc)\right) - \left(\frac{a+d}{12c} - s(a,c)\right)\right) = 1.$$

□

Lemma 4.28. *(Bruinier et al.; 2008) The group* $\Gamma_0(4)$ *is generated by* $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ *and* $R = \left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$ *and* $-I_2$ *.*

Proof. Let $\gamma = \left(\begin{smallmatrix}a&b\\4c&d\end{smallmatrix}\right) \in \Gamma_0(4)$. Let $T^\pm = \left(\begin{smallmatrix}1&\pm1\\0&1\end{smallmatrix}\right)$ and $R^\pm = \left(\begin{smallmatrix}1&0\\\pm4&1\end{smallmatrix}\right)$. Then $T^\pm$ and $R^\pm$ lie in $\Gamma_0(4)$. The coefficients a and d are odd, since $2 \nmid \det(\gamma) = 1$. Hence, $|a| \neq 2|b|$. If $|a| < 2|b|$, either $|b+a|$ or $|b-a|$ is smaller than $|b|$. Multiplying from right with T^+ or T^-, respectively, we get $\gamma' = \gamma T^\pm = \left(\begin{smallmatrix}a&b\pm a\\4c&d\pm4c\end{smallmatrix}\right)$ with $|b \pm a| < |b|$. Hence, $a^2 + (b\pm a)^2 < a^2 + b^2$. If $|a| > 2|b| \neq 0$, either $|a+4b|$ or $|a-4b| < |a|$. Multiplying from right with R^+ or R^-, respectively, we obtain $\gamma' = \gamma R^\pm = \left(\begin{smallmatrix}a\pm4b&b\\4c\pm4d&d\end{smallmatrix}\right)$ with $|a \pm 4b| < |a|$ and hence $(a\pm4b)^2 + b^2 < a^2 + b^2$. Thus, multiplying γ from right with $R^\pm$ or $T^\pm$ reduces $a^2 + b^2 \in \mathbb{Z}_{\geq 0}$ if $b \neq 0$. Hence, we can do this until $b = 0$. Then we are left with $\gamma' = \left(\begin{smallmatrix}a&0\\4c&d\end{smallmatrix}\right)$. Since $\det\gamma' = 1$, we have $a = d = \pm1$. Therefore, $\pm\gamma'$ is a power of $R^\pm$. Note that $R^- = R^{-1}$ and $T^- = T^{-1}$. Hence, $\Gamma_0(4)$ is generated by R, T and $-I_2$. □

Theorem 4.29. *The function* $j_4(\tau)$ *is invariant under* $\Gamma_0(4)$*.*

Proof. By Lemma 4.28 and since I_2 and $-I_2$ represent the same transformation, we only need to verify that $j_4(\tau)$ is invariant under T and R. We have already proven the invariance under T in Lemma 4.26. With Theorem 4.20 we find

$$\eta\left(\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)(\tau)\right) = \eta\left(\frac{\tau}{4\tau+1}\right) = \epsilon(1,0,4,1)\,(-i(4\tau+1))^{1/2}\,\eta(\tau)$$
$$\eta\left(4\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)(\tau)\right) = \eta\left(\frac{4\tau}{4\tau+1}\right) = \epsilon(1,0,1,1)\,(-i(4\tau+1))^{1/2}\,\eta(4\tau)$$

with

$$\epsilon(1,0,4,1) = \exp\left(\pi i\left(\frac{2}{48} + s(-1,4)\right)\right)$$
$$\epsilon(1,0,1,1) = \exp\left(\pi i\left(\frac{2}{12} + s(-1,1)\right)\right)$$

and

$$s(-1,4) = \frac{1}{4}\left(-\frac{1}{4} - \left\lfloor -\frac{1}{4} \right\rfloor - \frac{1}{2}\right) + \frac{2}{4}\left(-\frac{2}{4} - \left\lfloor -\frac{2}{4} \right\rfloor - \frac{1}{2}\right) + \frac{3}{4}\left(-\frac{3}{4} - \left\lfloor -\frac{3}{4} \right\rfloor - \frac{1}{2}\right)$$
$$= -\frac{1}{8} = -\frac{6}{48}$$

$s(-1,1) = 0$ by Theorem 4.22.

Thus, we get

$$j_4\left(\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)(\tau)\right) = \left(\frac{\exp(-\pi i/12)}{\exp(2\pi i/12)}\right)^8 j_4(\tau) = j_4(\tau).$$

□

Theorem 4.30. *The function $j_6(\tau)$ is invariant under $\Gamma_0(6)$.*

Proof. Using the mathematical software SageMath (The Sage Developers; 2017) one finds that $\Gamma_0(6)$ is generated by the set $\left\{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 7 & -3 \\ 12 & -5 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}$. We have to verify that $j_6(\tau)$ is invariant under these generators. For $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ this is clear, because the transformation induced is the identity. For the other generators we proceed as in the proof of Theorem 4.29. The calculations can be found in the Appendix. □

Now we want to examine the behaviour of $j_N(\tau)$ at the cusps. We begin with the cusp at ∞. For our purpose we use the following result proven in Apostol (1990):

Theorem 4.31. *(Apostol; 1990) The Fourier expansion of $\Delta(\tau)$ is of the form*

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} a_n q^n$$

with $a_1 = 1$ and $a_2 = -24$.

Theorem 4.32. *(Apostol; 1990) For $N = 2, 3, 4, 5, 7$ and 13 the Hauptmodul $j_N(\tau)$ has a pole of order 1 at infinity.*

Proof. For these N, we have

$$j_N(\tau) = \left(\frac{\eta(\tau)}{\eta(N\tau)}\right)^{24/(N-1)}.$$

Moreover, by Theorem 4.31 we have $\eta(\tau)^{24} = q(1 + I(q))$, where $I(q)$ denotes some power series in q. Hence,

$$j_N(\tau)^{N-1} = \frac{\eta(\tau)^{24}}{\eta(N\tau)^{24}} = \frac{q(1 + I(q))}{q^N(1 + I(q^N))}$$

has a pole of order $N - 1$ at $q = 0$. Since $j_N(\tau)$ is meromorphic, it has a pole of order 1 at $q = 0$. □

Theorem 4.33. *For $N = 2, 3, 4, 5, 7$ and 13 the Hauptmodul $j_N(\tau)$ has a zero of order $1/N$ at $\tau = 0$.*

Proof. For $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ we have $S(i\infty) = 0$. As described in Definition 4.15 we look at the q-expansion of

$$j_N(S\tau) = \left(\frac{\eta(S(\tau))}{\eta(NS(\tau))}\right)^{24/(N-1)}$$

where $\eta(S(\tau)) = (-i\tau)^{1/2}\eta(\tau)$ and $\eta(NS(\tau)) = (-i\tau/N)^{1/2}\eta(\tau/N)$. Therefore,

$$\begin{aligned} j_N(S\tau) = N^{12/(N-1)} \left(\frac{\eta(\tau)}{\eta(\tau/N)}\right)^{24/(N-1)} &= N^{12/(N-1)} \frac{1}{q^{-1/N} + \sum_{n=0}^{\infty} a_n q^{n/N}} \\ &= N^{12/(N-1)} \frac{q^{1/N}}{1 + \sum_{n=0}^{\infty} a_n q^{(n+1)/N}}, \end{aligned}$$

where we used that by Theorem 4.32 we can write $j_N(\tau) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$. Therefore, j_N has a zero of order $1/N$ at zero. $\square$

Recall that the equivalence classes of cusps for $\Gamma_0(N)$ are described in Theorem 4.10. For prime numbers, there are only two equivalence classes of cusps. Thus, apart from $N = 4$ and 6, we have already described the behaviour of the Hauptmoduln at all cusps. For $N = 4$ we still need to study the pole at $1/2$.

Theorem 4.34. *For the Hauptmodul $j_4(\tau)$ we have*

$$\lim_{\tau \to 1/2} j_4(\tau) = -16.$$

Proof. For $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ we have $\gamma(i\infty) = 1/2$. By Theorem 4.20 we have

$$\eta(\gamma(z)) = \epsilon(1, 0, 2, 1)(-i(2z + 1))^{1/2}\eta(z),$$

with $\epsilon(1, 0, 2, 1) = \exp(\pi i(2/24 + s(-1, 2)))$ and $s(-1, 2) = 0$ by Theorem 4.22. Moreover, we have $\eta(4\gamma(z)) = \eta\left(\left(\begin{smallmatrix} 4 & 0 \\ 2 & 1 \end{smallmatrix}\right)(z)\right)$. Now with $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ we get

$$\eta\left(\alpha^{-1}\alpha\left(\begin{smallmatrix} 4 & 0 \\ 2 & 1 \end{smallmatrix}\right)(z)\right) = \eta\left(\alpha^{-1}\left(\frac{2z+1}{2}\right)\right) = \epsilon(2, -1, 1, 0)\left(-i\frac{2z+1}{2}\right)^{1/2}\eta\left(z + \frac{1}{2}\right),$$

with $\epsilon(2, -1, 1, 0) = \exp(\pi i(2/12 + s(0, 1)))$ and $s(0, 1) = 0$. Thus we have

$$\frac{\eta(\gamma(z))}{\eta(4\gamma(z))} = \frac{e^{\pi i/12}(-i(2z + 1))^{1/2}\eta(z)}{e^{2\pi i/12}\left(-\frac{1}{2}i(2z + 1)\right)^{1/2}\eta\left(z + \frac{1}{2}\right)}. \tag{5}$$

Now, $\eta(z) = q^{1/24}I(q)$ and $\eta(z + 1/2) = e^{\pi i/24}q^{1/24}I(-q)$ for $q = e^{2\pi iz}$ and some $I(q)$ with $\lim_{q \to 0} I(q) = 1$. Together with Equation (5) we have

$$\lim_{\tau \to 1/2} j_4(\tau) = \lim_{z \to i\infty} \frac{\eta(\gamma(z))^8}{\eta(4\gamma(z))^8} = \lim_{q \to 0} e^{-8\pi i/12} \cdot 2^4 \cdot e^{-8\pi i/24} \frac{q^{1/24}I(q)}{q^{1/24}I(-q)} = -2^4.$$

$\square$

Table 2: The values of $j_6(\tau)$ at the cusps.

Cusp	0	1/6	1/3	1/2
Value	8	∞	0	$e^{3\pi i/4}$

Theorem 4.35. *The values of $j_6(\tau)$ at the cusps are those listed in Table 2. At infinity, $j_6(\tau)$ has a pole of order one and the zero at $1/3$ is of order $1/2$.*

The proof of Theorem 4.35 can be found in the Appendix.

5 Modular Functions for $\Gamma_{(1,1)}$

Let $i, j \in \mathbb{Z}/n\mathbb{Z} =: \mathbb{Z}_n$. We define

$$\Gamma_{(i,j)} := \{\gamma \in SL(2,\mathbb{Z}) | (i,j)\gamma = (i,j)\}.$$

For every positive integer n, the sets $\Gamma_{(i,j)}$ are subgroups of $SL(2,\mathbb{Z})$ containing $\Gamma(n)$.

Theorem 5.1. *Let n be a positive integer and $i, j \in \mathbb{Z}_n$. Then*

$$\Gamma_{(i,j)} \cong \Gamma_1\left(\frac{n}{\gcd(n,i,j)}\right).$$

Proof. For $i = j = 0$ the statement is clear since $\Gamma_{(0,0)} = SL(2,\mathbb{Z}) = \Gamma_1(1)$. Now let $j \neq 0$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{(0,j)}$. The definition of $\Gamma_{(0,j)}$ implies that $cj \equiv 0 \pmod{n}$ and $dj \equiv j \pmod{n}$. Hence, $n \mid cj$ and $n \mid (d-1)j$. For $r := n/\gcd(n,j)$ we thus have $c \equiv 0 \pmod{r}$ and $d \equiv 1 \pmod{r}$. Therefore, $\gamma \equiv \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \pmod{r}$. Since $\det(\gamma) = 1$, we get $a \equiv 1 \pmod{r}$ and thus γ lies in $\Gamma_1(r)$. Conversely, for any $\gamma = \left(\begin{smallmatrix} ra+1 & b \\ rc & rd+1 \end{smallmatrix}\right) \in \Gamma_1(r)$ we have

$$(0,j)\left(\begin{smallmatrix} ra+1 & b \\ rc & rd+1 \end{smallmatrix}\right) = (jrc, jrd+j) \equiv (0,j) \pmod{n},$$

because $n = \gcd(j,n) \cdot r$ divides $j \cdot r$. Therefore,

$$\Gamma_{(0,j)} = \Gamma_1(r) = \Gamma_1\left(\frac{n}{\gcd(n,j)}\right). \tag{6}$$

Now for (i,j) with $\gcd(i,j) = k$ we can find integers b and d such that $dj + bi = k$. Define $A_{(i,j)} := \left(\begin{smallmatrix} j/k & b \\ -i/k & d \end{smallmatrix}\right)$. Note that $A_{(i,j)} \in SL(2,\mathbb{Z})$ and

$$(i,j)A_{(i,j)} = (0, bi + dj) = (0,k).$$

Hence, a matrix γ belongs to $\Gamma_{(i,j)}$ if and only if $A_{(i,j)}^{-1}\gamma A_{(i,j)}$ lies in $\Gamma_{(0,k)}$. Thus,

$$\Gamma_{(i,j)} = A_{(i,j)}\Gamma_{(0,k)}A_{(i,j)}^{-1} \cong \Gamma_{(0,k)} = \Gamma_1\left(\frac{n}{\gcd(n,i,j)}\right),$$

by Equation (6) using $\gcd(n, \gcd(i, j)) = \gcd(n, i, j)$. $\square$

Lemma 5.2. *Let N be a positive integer and let H be conjugate to $G := \Gamma_0(N)$, i.e. $G = \sigma H \sigma^{-1}$ for some $\sigma \in SL(2, \mathbb{Z})$. Then $f(\tau)$ is a modular function for G if and only if $f(\sigma\tau)$ is a modular function for H.*

Proof. ($\Rightarrow$) Let $\gamma \in H$. Then $\sigma\gamma\sigma^{-1} \in G$ and

$$f(\sigma\gamma\tau) = f(\sigma\gamma\sigma^{-1}\sigma\tau) = f(\sigma\tau),$$

because f is invariant under G. Hence, $f(\sigma\tau)$ is invariant under H. Furthermore, $f(\sigma\tau)$ is meromorphic on $\mathbb{H}$ since $f(\tau)$ is a meromorphic and $\sigma(\tau)$ is holomorphic on $\mathbb{H}$. Moreover, $f(\sigma\tau)$ is meromorphic at the cusps, because at the cusp $\gamma(i\infty)$ it has the same q-expansion as $f(\tau)$ at the cusp $\sigma\gamma(i\infty)$. Thus, $f(\sigma\tau)$ is a modular function for H.

($\Leftarrow$) If $f(\sigma\tau)$ is a modular function for H, we can apply the above argument for σ^{-1} instead of σ and with G and H interchanged and get that $f(\tau)$ is a modular function for G. $\square$

We write $[\tau]_G$ for the G-equivalence class of a cusp τ.

Remark 5.3. Let G and H be conjugate subgroups of $SL(2, \mathbb{Z})$, i.e. $G = \sigma H \sigma^{-1}$ for some $\sigma \in SL(2, \mathbb{Z})$. Then, $[\tau]_H = \sigma^{-1}[\sigma\tau]_G$. In particular, G and H have the same number of equivalence classes of cusps.

Proof. We have that $z \in [\tau]_H \Leftrightarrow z = \alpha\tau$ for some $\alpha \in H \Leftrightarrow \sigma z = (\sigma\alpha\sigma^{-1})\sigma\tau$ for some $\alpha \in H \Leftrightarrow \sigma z \in [\sigma\tau]_G$, where we used that $\sigma H \sigma^{-1} = G$. Hence, $z \in [\tau]_H \Leftrightarrow z \in \sigma^{-1}[\sigma\tau]_G$. $\square$

We will now focus on the case $n = 2$. We consider the groups $\Gamma_{(0,1)}$, $\Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$ and want to find corresponding Hauptmoduln. Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and let Γ_θ be the group generated by S and T^2.

Lemma 5.4. *We have $\Gamma_1(2) = \Gamma_0(2)$ and $\Gamma_1(2) = (ST)\Gamma_\theta(ST)^{-1}$.*

Proof. Since $ad \equiv 1 \pmod 2$ implies $a \equiv d \equiv 1 \pmod 2$, we have

$$\begin{aligned} \Gamma_0(2) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod 2 \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod 2, a \equiv d \equiv 1 \pmod 2 \right\} = \Gamma_1(2). \end{aligned}$$

For the second statement, we calculate

$$A := (ST)S(ST)^{-1} = \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right) \in \Gamma_1(2)$$

and

$$B := (ST)T^2(ST)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right) \in \Gamma_1(2).$$

Therefore, $(ST)\Gamma_\theta(ST)^{-1} < \Gamma_1(2)$. We have $B^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $(AB)^{-1} = T$, which generate $\Gamma_1(2)$. (This can be proved analogously to Lemma 4.28 using $(TB)^{-2} = -I$.) Thus we get $(ST)\Gamma_\theta(ST)^{-1} = \Gamma_1(2)$. □

Kim and Koo (2004) give a list of Hauptmoduln $j_{1,N}$ for some $\Gamma_1(N)$ in the Appendix of their paper. For $N = 2$ they have

$$j_{1,2}(\tau) = \frac{\theta_2(\tau)^8}{\theta_4(2\tau)^8},$$

where $\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}$ and $\theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}$ for $\tau \in \mathbb{H}$. Since $\Gamma_0(2) = \Gamma_1(2)$, we expect the Hauptmodul $j_2(\tau) = (\eta(\tau)/\eta(2\tau))^{24}$ for $\Gamma_0(2)$ given in Beneish and Larson (2014) to be compatible with $j_{1,2}(\tau)$, meaning that we can express $j_{1,2}$ as a rational function of j_2 and vice versa. To check this, we use the Jacobi triple product which is proven in the book by Apostol (1976).

Theorem 5.5 (Jacobi triple product)**.** *(Apostol; 1976) For $x, z \in \mathbb{C}$ with $|x| < 1$ and $z \neq 0$ we have the following identity:*

$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1}z^2)(1 + x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}.$$

Corollary 5.6. *(Conway and Sloane; 1999) We can express θ_2 and θ_4 as the following η-quotients*

$$\theta_2(\tau) = \frac{2\eta(2\tau)^2}{\eta(\tau)}$$
$$\theta_4(\tau) = \frac{\eta(\tau/2)^2}{\eta(\tau)}$$

Proof. Applying the Jacobi triple product with $x = q^{1/2}$ and $z = q^{1/4}$ we have

$$\begin{aligned}
\theta_2(\tau) &= \sum_{m\in\mathbb{Z}} q^{(m^2+m+1/4)/2} = q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)(1 + q^{n-1}) \\
&= q^{1/8}(1 + q^0) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)(1 + q^n) = 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^n) \\
&= \frac{2q^{1/6} \prod_{n=1}^{\infty} (1 - q^{2n})^2}{q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)} = \frac{2\eta(2\tau)^2}{\eta(\tau)}.
\end{aligned}$$

Applying the Jacobi triple product with $x = q^{1/2}$ and $z = i$ leads to

$$\begin{aligned}
\theta_4(\tau) &= \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2/2} = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{-1/2+n})^2 \\
&= \prod_{n=1}^{\infty} \frac{(1 - q^n)^2 (1 - q^{-1/2+n})^2}{(1 - q^n)} = \frac{q^{1/24} \prod_{l=1}^{\infty} (1 - q^{l/2})^2}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)} = \frac{\eta(\tau/2)^2}{\eta(\tau)}.
\end{aligned}$$

□

With Corollary 5.6 it is now easy to see that the two Hauptmoduln $j_{1,2}$ and j_2 are compatible, since

$$j_{1,2}(\tau) = \frac{\theta_2(\tau)^8}{\theta_4(2\tau)^8} = \frac{2^8\eta(2\tau)^{16}\eta(2\tau)^8}{\eta(\tau)^8\eta(\tau)^{16}} = \frac{2^8\eta(2\tau)^{24}}{\eta(\tau)^{24}} = \frac{2^8}{j_2(\tau)}.$$

By Theorem 5.1 and Lemma 5.4 we have

$$\begin{aligned}
\Gamma_{(0,1)} &= \Gamma_1(2) = \Gamma_0(2)\\
\Gamma_{(1,0)} &= S^{-1}\Gamma_1(2)S\\
\Gamma_{(1,1)} &= (ST)^{-1}\Gamma_{(0,1)}ST = T^{-1}\Gamma_{(1,0)}T = \Gamma_\theta.
\end{aligned}$$

Theorem 5.7. *The following functions $j_{(0,1)}$, $j_{(1,0)}$ and $j_{(1,1)}$ are Hauptmoduln for $\Gamma_{(0,1)}$, $\Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$, respectively.*

$$\begin{aligned}
j_{(0,1)}(\tau) &= j_2(\tau) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n\\
j_{(1,0)}(\tau) &= \frac{2^{12}}{j_2(S\tau)} = q^{-1/2} + \sum_{\substack{n=0\\ n\in\frac{1}{2}\mathbb{Z}}}^{\infty} b_n q^n\\
j_{(1,1)}(\tau) &= -\frac{2^{12}}{j_2(ST\tau)} = q^{-1/2} + \sum_{\substack{n=0\\ n\in\frac{1}{2}\mathbb{Z}}}^{\infty} c_n q^n,
\end{aligned}$$

where j_2 is the Hauptmodul for $\Gamma_0(2)$ defined in Section 4.5, $q = e^{2\pi i\tau}$ and $a_n, b_n \in \mathbb{Z}$ and $c_n \in \mathbb{Z}_{\geq 0}$. There are two equivalence classes of cusps for $\Gamma_{(0,1)}, \Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$. The Hauptmoduln $j_{(1,0)}$ and $j_{(1,1)}$ have zeros of order 1 at the cusps inequivalent to ∞, whereas $j_{(0,1)}$ has zeros of order $1/2$.

Proof. For $j_{(0,1)}$ we have proven most of the properties in Section 4.5 and we only need to show that the q-expansion has integer coefficients. We have

$$j_{(0,1)}(\tau) = \frac{\Delta(\tau)}{\Delta(2\tau)} = \frac{q\prod_{m=1}^{\infty}(1-q^m)^{24}}{q^2\prod_{n=1}^{\infty}(1-q^{2n})^{24}} = \frac{1}{q}\prod_{k=1}^{\infty}(1-q^{2k-1})^{24}.$$

Hence, $j_{(0,1)}$ has a pole of order one at infinity and comparing coefficients, we see that its q-expansion has integer coefficients.

Let $f(\tau)$ be a modular function for $\Gamma_{(1,0)}$. By Lemma 5.2 then $f(S^{-1}\tau)$ is a modular function for $\Gamma_0(2)$. Since j_2 is a Hauptmodul for $\Gamma_0(2)$, we can write $f(S^{-1}\tau) = r(j_2(\tau))$ for some rational function r. Substituting $S^{-1}\tau$ with τ, we get $f(\tau) = r(j_2(S\tau))$. Hence, $j_2(S\tau)$ is a Hauptmodul for $\Gamma_{(1,0)}$. Therefore, also $j_{(1,0)}(\tau) = 2^{12}/j_2(S\tau)$ is a Hauptmodul for $\Gamma_{(1,0)}$. Analogously, $j_{(1,1)}(\tau) = -2^{12}/j_2(ST\tau)$ is a Hauptmodul for $\Gamma_{(1,1)}$.

Now we look at the q-expansions. We have

$$\begin{aligned} j_{(1,0)}(\tau) &= 2^{12}\frac{\Delta(2S(\tau))}{\Delta(S(\tau))} = 2^{12}\frac{\Delta(S(\tau/2))}{\Delta(S(\tau))} = 2^{12}\frac{(\tau/2)^{12}\Delta(\tau/2)}{\tau^{12}\Delta(\tau)} = \frac{\Delta(\tau/2)}{\Delta(\tau)} \\ &= \frac{q^{1/2}\prod_{m=1}^{\infty}(1-q^{m/2})^{24}}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = \frac{1}{q^{1/2}}\prod_{l=1}^{\infty}(1-q^{-1/2+l})^{24}. \end{aligned}$$

Comparing coefficients, we see that the q-expansion starts with $q^{-1/2} - 24 + ...$ and that it has integer coefficients. For $\Gamma_{(1,1)}$ we have

$$j_{(1,1)}(\tau) = -2^{12}\frac{\Delta(2ST(\tau))}{\Delta(ST(\tau))}.$$

Since $ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, we have $\Delta(ST(\tau)) = (\tau+1)^{12}\Delta(\tau)$. Moreover,

$$\Delta(2ST(\tau)) = \Delta\left(-\frac{2}{\tau+1}\right) = \Delta\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\frac{\tau+1}{2}\right)\right) = \left(\frac{\tau+1}{2}\right)^{12}\Delta\left(\frac{\tau+1}{2}\right).$$

Therefore,

$$j_{(1,1)}(\tau) = -\frac{\Delta\left(\frac{\tau+1}{2}\right)}{\Delta(\tau)} = -\frac{-q^{1/2}\prod_{m=1}^{\infty}(1-(-1)^m q^{m/2})^{24}}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = \frac{1}{q^{1/2}}\prod_{l=1}^{\infty}(1+q^{-1/2+l})^{24}.$$

Comparing coefficients, we have $j_{(1,1)}(\tau) = q^{-1/2} + \sum_{\substack{n=0 \\ n\in\frac{1}{2}\mathbb{Z}}}^{\infty} c_n q^n$ for positive integers c_n.

By Remark 5.3, $\Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$ have the same number of equivalence classes of cusps as $\Gamma_{(0,1)}$ which is two by Corollary 4.11. Representatives are given by $\{0,\infty\}$ and $\{0,-1\}$ for $\Gamma_{(1,0)}$ and $\Gamma_{(1,1)}$, respectively. For the latter, 0 is equivalent to ∞ because $S \in \Gamma_{(1,1)}$ and $S(\infty) = 0$. Now we calculate the q-expansions at 0 and -1 as described in Definition 4.15, respectively.

$$\begin{aligned} j_{(1,0)}(S^{-1}\tau) &= \frac{2^{12}}{j_2(SS^{-1}\tau)} = \frac{2^{12}}{q^{-1}+\sum_{n=0}^{\infty}a_n q^n} = q\frac{2^{12}}{1+\sum_{n=0}^{\infty}a_n q^{n+1}}, \\ j_{(1,1)}((ST)^{-1}\tau) &= -\frac{2^{12}}{j_2(ST(ST)^{-1}\tau)} = -q\frac{2^{12}}{1+\sum_{n=0}^{\infty}a_n q^{n+1}}, \end{aligned}$$

where we used the q-expansion of $j_2(\tau)$. Thus $j_{(1,0)}$ and $j_{(1,1)}$ have a zero of order one at 0 and -1, respectively. □

Remark 5.8. Let $Z_{(1,1)}$ be a holomorphic modular function for $\Gamma_{(1,1)}$ with a pole of order $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ at $\tau = \infty$ and a pole of order $m \leq k$ at $\tau = 1$, where m and k are nonnegative integers. Then since $j_{(1,1)}(\tau)$ has a simple zero at $\tau = 1$, $Z_{(1,1)}(\tau)(j_{(1,1)}(\tau))^k$ is finite at $\tau = 1$ and has a pole of order $n+k/2$ at infinity. Using the same construction as in Lemma 3.11, we can find a polynomial p of degree $2n+k$ such that $f(\tau) := Z_{(1,1)}(\tau)(j_{(1,1)}(\tau))^k - p(j_{(1,1)}(\tau))$ is holomorphic and bounded on $\mathbb{H}$ and has a zero at infinity. For this we need the first $2n+k+1$ coefficients a_l

of the q-expansion of $Z_{(1,1)}$ at infinity. Since $f(\tau)$ is a bounded modular function for $\Gamma_{(1,1)}$, by Lemma 5.2 we have that $f((ST)^{-1}(\tau))$ is a bounded modular function for $\Gamma_0(2)$. But then $f((ST)^{-1}(\tau))$ is constant by Theorem 4.17. Hence, also $f(\tau)$ is constant and since it has a zero at infinity, we have $f \equiv 0$ and

$$Z_{(1,1)}(\tau) = \frac{p(j_{(1,1)}(\tau))}{(j_{(1,1)}(\tau))^k}.$$

Hence, in order to write $Z_{(1,1)}$ as a rational function of $j_{(1,1)}(\tau)$ we need to know the first $2n+k+1$ coefficients of the q-expansion of $Z_{(1,1)}$ at infinity. Note that if m is smaller than k, the coefficients p_l of the polynomial p are zero for $l < k - m$.

Theorem 5.9. *Let $Z_{(1,1)}(\tau)$ be a modular function for $\Gamma_{(1,1)}$ with only nonnegative real coefficients in the q-expansion at infinity, i.e.*

$$Z_{(1,1)}(\tau) = \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k q^k,$$

for some $n \in \frac{1}{2}\mathbb{Z}$ and $a_k \in \mathbb{R}_{\geq 0}$. At the cusp $\tau = -1$ the q-expansion then starts with $b_{-m}q^{-m} + \ldots$ for some $b_{-m} \in \mathbb{C}$ and $m \in \mathbb{Z}$ with $m \leq n$. In particular, if $Z_{(1,1)}$ has a pole of order n at infinity, then it has at most a pole of order n at the inequivalent cusps.

Proof. Let $Z_{(1,1)}((ST)^{-1}\tau) = \sum_{l=-m}^{\infty} b_l q^l$ be the q-expansion at -1. Then we have for all $y \in \mathbb{R}_{>0}$ that

$$Z_{(1,1)}((ST)^{-1}(iy)) = \sum_{l=-m}^{\infty} b_l e^{-2\pi l y}.$$

Since $(ST)^{-1}(iy) = \frac{i}{y} - 1$, we have that

$$Z_{(1,1)}((ST)^{-1}(iy)) = Z_{(1,1)}\left(\frac{i}{y} - 1\right) = \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k (-1)^k e^{-2\pi k/y}.$$

Because the a_k are positive, we have

$$\left| \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k (-1)^k e^{-2\pi k/y} \right| \leq \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k e^{-2\pi k/y} = Z_{(1,1)}\left(\frac{i}{y}\right) = Z_{(1,1)}(iy) = \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k e^{-2\pi k y},$$

where we used that $Z_{(1,1)}$ is invariant under $\tau \mapsto -1/\tau$. Combining the above equations we get

$$\left| \sum_{l=-m}^{\infty} b_l e^{-2\pi l y} \right| \leq \sum_{\substack{k=-n \\ k \in \frac{1}{2}\mathbb{Z}}}^{\infty} a_k e^{-2\pi k y}$$

for all $y \in \mathbb{R}_{>0}$. If we divide this by $e^{2\pi n y}$ and take the limit for $y \to \infty$, the right hand side converges to $a_{-n} < \infty$, whereas the left hand side is finite if and only if $m \leq n$. $\square$

Remark 5.10. Let $Z_{(1,1)}(\tau)$ be a modular function for $\Gamma_{(1,1)}$ holomorphic on $\mathbb{H}$ with only nonnegative real coefficients in the q-expansion at infinity. Then Theorem 5.9 and Remark 5.8 give us the following results. If $Z_{(1,1)}$ is holomorphic at infinity, $Z_{(1,1)}$ must be constant. If $Z_{(1,1)}$ has a pole of order $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ at infinity, we need to know the first $2n + \lfloor n \rfloor + 1$ coefficients to write $Z_{(1,1)}$ as a rational function of $j_{(1,1)}$.

6 Appendix

Proof of Theorem 4.30. We begin with the invariance under $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. With Lemma 4.24 we have

$$\begin{aligned}
\eta\left(2\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)(2\tau)\right) = e^{2\pi i/12}\eta(2\tau) \\
\eta\left(3\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)(3\tau)\right) = e^{3\pi i/12}\eta(3\tau) \\
\eta\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) &= e^{\pi i/12}\eta(\tau) \\
\eta\left(6\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right)(6\tau)\right) = e^{6\pi i/12}\eta(6\tau)
\end{aligned}$$

Thus since $j_6(\tau) = \frac{\eta(2\tau)^3\eta(3\tau)^9}{\eta(\tau)^3\eta(6\tau)^9}$ we have

$$j_6\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(\tau)\right) = \frac{e^{6\pi i/12}e^{27\pi i/12}}{e^{3\pi i/12}e^{54\pi i/12}} j_6(\tau) = j_6(\tau).$$

Now we continue with the invariance under $\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)$. With Theorem 4.20 we get

$$\begin{aligned}
\eta\left(2\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 5 & -2 \\ 3 & -1 \end{smallmatrix}\right)(2\tau)\right) = \epsilon(5,-2,3,-1)(-i(6z-1))^{1/2}\eta(2\tau) \\
\eta\left(3\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 5 & -3 \\ 2 & -1 \end{smallmatrix}\right)(3\tau)\right) = \epsilon(5,-3,2,-1)(-i(6z-1))^{1/2}\eta(3\tau) \\
\eta\left(\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)(\tau)\right) &= \epsilon(5,-1,6,-1)(-i(6z-1))^{1/2}\eta(\tau) \\
\eta\left(6\left(\begin{smallmatrix} 5 & -1 \\ 6 & -1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 5 & -6 \\ 1 & -1 \end{smallmatrix}\right)(6\tau)\right) = \epsilon(5,-6,1,-1)(-i(6z-1))^{1/2}\eta(6\tau)
\end{aligned}$$

with

$$\begin{aligned}
\epsilon(5,-2,3,-1) &= \exp(\pi i(4/36 + s(1,3))) = \exp(\pi i(2/18 + 1/18)) = \exp(\pi i/6) \\
\epsilon(5,-3,2,-1) &= \exp(\pi i(4/24 + s(1,2))) = \exp(\pi i/6) \\
\epsilon(5,-1,6,-1) &= \exp(\pi i(4/72 + s(1,6))) = \exp(\pi i(1/18 + 5/18)) = \exp(\pi i/3) \\
\epsilon(5,-6,1,-1) &= \exp(\pi i(4/12 + s(1,1))) = \exp(\pi i/3)
\end{aligned}$$

Combining everything we get

$$j_6\left(\left(\begin{smallmatrix}5 & -1\\ 6 & -1\end{smallmatrix}\right)(\tau)\right) = \frac{e^{3\pi i/6}e^{9\pi i/6}}{e^{3\pi i/3}e^{9\pi i/3}} j_6(\tau) = j_6(\tau).$$

Now we are only left with the invariance under $\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)$. By Theorem 4.20 we have

$$\begin{aligned}
\eta\left(2\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix}7 & -6\\ 6 & -5\end{smallmatrix}\right)(2\tau)\right) = \epsilon(7,-6,6,-5)(-i(12z-5))^{1/2}\eta(2\tau)\\
\eta\left(3\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix}7 & -9\\ 4 & -5\end{smallmatrix}\right)(3\tau)\right) = \epsilon(7,-9,4,-5)(-i(12z-5))^{1/2}\eta(3\tau)\\
\eta\left(\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)(\tau)\right) &= \epsilon(7,-3,12,-5)(-i(12z-5))^{1/2}\eta(\tau)\\
\eta\left(6\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix}7 & -18\\ 2 & -5\end{smallmatrix}\right)(6\tau)\right) = \epsilon(7,-18,2,-5)(-i(12z-5))^{1/2}\eta(6\tau)
\end{aligned}$$

with

$$\begin{aligned}
\epsilon(7,-6,6,-5) &= \exp(\pi i(2/72 + s(5,6))) = \exp(\pi i(1/36 - 10/36)) = \exp(-\pi i/4)\\
\epsilon(7,-9,4,-5) &= \exp(\pi i(2/48 + s(5,4))) = \exp(\pi i(1/24 + 3/24)) = \exp(\pi i/6)\\
\epsilon(7,-3,12,-5) &= \exp(\pi i(2/144 + s(5,12))) = \exp(\pi i(1/72 - 1/72)) = 1\\
\epsilon(7,-18,2,-5) &= \exp(\pi i(2/24 + s(5,2))) = \exp(\pi i/12)
\end{aligned}$$

Thus, we get

$$j_6\left(\left(\begin{smallmatrix}7 & -3\\ 12 & -5\end{smallmatrix}\right)(\tau)\right) = \frac{e^{-3\pi i/4}e^{9\pi i/6}}{e^{9\pi i/12}} j_6(\tau) = j_6(\tau).$$

Hence, $j_6(\tau)$ is invariant under $\Gamma_0(6)$. □

Proof of Theorem 4.35. By Theorem 4.10 the set $\{1, 1/2, 1/3, 1/6\}$ is a set of representatives of the equivalence classes of cusps under $\Gamma_0(6)$. Because $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 6 & 1\end{smallmatrix}\right)$ lie in $\Gamma_0(6)$, the cusp at zero is equivalent to the cusp at one and the cusp at infinity is equivalent to the cusp at 1/6.

Let us first consider the cusp at infinity. We have

$$j_6(\tau)^8 = \frac{\eta(2\tau)^{24}\eta(3\tau)^{3\cdot 24}}{\eta(\tau)^{24}\eta(6\tau)^{3\cdot 24}}.$$

From Theorem 4.32 we know that $\eta(\tau)^{24}/\eta(2\tau)^{24}$ has a pole of order one at infinity. Therefore, $\eta(2\tau)^{24}/\eta(\tau)^{24}$ has a zero of order one and $\eta(3\tau)^{3\cdot 24}/\eta(6\tau)^{3\cdot 24}$ has a pole of order nine at infinity. In total, we get that $j_6(\tau)^8$ has a pole of order eight and hence $j_6(\tau)$ has a pole of order one at infinity.

For the cusp at zero, we have that $\lim_{z\to 0} j_6(z) = \lim_{\tau\to\infty} j_6(S(\tau))$, with $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. For an integer k we calculate

$$\eta(kS(\tau)) = \eta(-k/\tau) = (-i\tau/k)^{1/2}\eta(\tau/k),$$

where we used Lemma 4.24. Thus, we get

$$j_6(S(\tau)) = \frac{(-i\tau/2)^{3/2}\eta(\tau/2)^3(-i\tau/3)^{9/2}\eta(\tau/3)^9}{(-i\tau)^{3/2}\eta(\tau)^3(-i\tau/6)^{9/2}\eta(\tau/6)^9} = 2^3\frac{\eta(\tau/2)^3\eta(\tau/3)^9}{\eta(\tau)^3\eta(\tau/6)^9}.$$

Since we can write $\eta(\tau) = q^{1/24}I(q)$ with $\lim_{q\to 0} I(q) = 1$, we get

$$\lim_{\tau\to\infty} j_6(S(\tau)) = 2^3 \lim_{q\to 0} \frac{q^{3/48}q^{3/24}}{q^{3/24}q^{3/48}} = 2^3.$$

For the cusp at 1/3, we use that $\lim_{z\to 1/3} j_6(z) = \lim_{\tau\to\infty} j_6\left(\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right)$. With $\alpha_2 = \left(\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right)$ and $\alpha_6 = \left(\begin{smallmatrix} 2 & -3 \\ 1 & -1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -1 & 3 \\ -1 & 2 \end{smallmatrix}\right)$ we get by Theorem 4.20

$$\begin{aligned}
\eta\left(2\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 2 & -2 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) = \eta\left(\alpha_2^{-1}\alpha_2\left(\begin{smallmatrix} 2 & -2 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) = \eta\left(\alpha_2^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)(\tau)\right) \\
&= \epsilon_2(-i(3\tau/2-1))^{1/2}\eta(\tau/2) \\
\eta\left(3\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)(3\tau)\right) = \epsilon_3(-i(3\tau-2))^{1/2}\eta(3\tau) \\
\eta\left(\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) &= \epsilon_1(-i(3\tau-2))^{1/2}\eta(\tau) \\
\eta\left(6\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 2 & -6 \\ 1 & -2 \end{smallmatrix}\right)(3\tau)\right) = \eta\left(\alpha_6^{-1}\alpha_6\left(\begin{smallmatrix} 2 & -6 \\ 1 & -2 \end{smallmatrix}\right)(3\tau)\right) = \eta\left(\alpha_6^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)(3\tau)\right) \\
&= \epsilon_6(-i(3\tau/2-1))^{1/2}\eta(3\tau/2)
\end{aligned}$$

for some constants $\epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_6. Therefore, for some constant c we get

$$\begin{aligned}
j_6\left(\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)(\tau)\right) &= c\frac{(-i(3\tau/2-1))^{3/2}\eta(\tau/2)^3(-i(3\tau-2))^{9/2}\eta(3\tau)^9}{(-i(3\tau-2))^{3/2}\eta(\tau)^3(-i(3\tau/2-1))^{9/2}\eta(3\tau/2)^9} \\
&= 2^3c\frac{\eta(\tau/2)^3\eta(3\tau)^9}{\eta(\tau)^3\eta(3\tau/2)^9} = \frac{2^3c}{j_6(\tau/2)}.
\end{aligned}$$

Now since $j_6(\tau)$ has a simple pole at infinity, $1/j_6(\tau/2)$ has a zero of order 1/2 at infinity. Hence, j_6 has a zero of order 1/2 at the cusp 1/3.

To calculate the value of $j_6(\tau)$ at the cusp 1/2, we use that $\lim_{z\to 1/2} j_6(z) = \lim_{\tau\to\infty} j_6\left(\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right)$. With $\alpha_3 = \left(\begin{smallmatrix} 1 & -1 \\ -2 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)^{-1}$ and $\alpha_6 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1}$ we have by Theorem 4.20

$$\begin{aligned}
\eta\left(2\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)(2\tau)\right) = \epsilon(1,0,1,1)(-i(2\tau+1))^{1/2}\eta(2\tau) \\
\eta\left(3\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\alpha_3^{-1}\alpha_3\left(\begin{smallmatrix} 3 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right) = \eta\left(\alpha_3^{-1}\left(\begin{smallmatrix} 1 & -1 \\ 0 & 3 \end{smallmatrix}\right)(\tau)\right) \\
&= \epsilon(3,1,2,1)(-i(2\tau/3-2/3+1))^{1/2}\eta(\tau/3-1/3) \\
\eta\left(\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right) &= \epsilon(1,0,2,1)(-i(2\tau+1))^{1/2}\eta(\tau) \\
\eta\left(6\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)(\tau)\right) &= \eta\left(\left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right)(2\tau)\right) = \eta\left(\alpha_6^{-1}\alpha_6\left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right)(2\tau)\right) = \eta\left(\alpha_6^{-1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right)(2\tau)\right) \\
&= \epsilon(3,-1,1,0)(-i(2\tau/3+1/3))^{1/2}\eta(2\tau/3+1/3)
\end{aligned}$$

with

$$\begin{aligned}
\epsilon(1,0,1,1) &= \exp(\pi i(2/12 + s(-1,1))) = \exp(\pi i/6)\\
\epsilon(3,1,2,1) &= \exp(\pi i(4/24 + s(-1,2))) = \exp(\pi i/6)\\
\epsilon(1,0,2,1) &= \exp(\pi i(2/12 + s(-1,2))) = \exp(\pi i/6)\\
\epsilon(3,-1,1,0) &= \exp(\pi i(3/12 + s(0,1))) = \exp(\pi i/4)
\end{aligned}$$

Hence, we get

$$\begin{aligned}
j_6\left(\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)(\tau)\right) &= \frac{e^{3\pi i/6}(-i(2\tau+1))^{3/2}\eta(2\tau)^3 e^{9\pi i/6}(-i(2\tau/3+1/3))^{9/2}\eta(\tau/3-1/3)^9}{e^{3\pi i/6}(-i(2\tau+1))^{3/2}\eta(\tau)^3 e^{9\pi i/4}(-i(2\tau/3+1/3))^{9/2}\eta(2\tau/3+1/3)^9}\\
&= e^{-3\pi i/4}\frac{\eta(2\tau)^3\eta(\tau/3-1/3)^9}{\eta(\tau)^3\eta(2\tau/3+1/3)^9}.
\end{aligned}$$

We can write $\eta(\tau) = q^{1/24}I(q)$ with $\lim_{q\to 0} I(q) = 1$. Then, $\eta(\tau/3 - 1/3) = e^{-\pi i/36}q^{1/72}I(e^{-2\pi i/3}q^{1/3})$ and $\eta(2\tau/3+1/3) = e^{\pi i/36}q^{1/36}I(e^{2\pi i/3}q^{2/3})$. Therefore,

$$\lim_{\tau\to\infty} j_6\left(\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)(\tau)\right) = e^{-3\pi i/4}e^{-\pi i/2}\lim_{q\to 0}\frac{q^{1/4}q^{1/8}}{q^{1/8}q^{1/4}} = e^{3\pi i/4}.$$

□

7 References

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