ETH Zürich

Master's thesis

Elementary Classification of Topological Insulators in Low Dimensions

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March 29, 2019

Abstract

Topological insulators can be classified depending on the symmetries and dimension of the physical system. The classification often involves advanced mathematical tools. The goal of this thesis is to understand the classification in low dimensions using elementary tools from topology and differential geometry. Our direct approach without relying on the bulk-boundary correspondence makes the classification more accessible to students who are new to the subject.

We classify the topological insulators via homotopy theory. For each symmetry class in dimension 0, 1 and 2, we either define an index in terms of equivariant vector bundles to distinguish between different homotopy classes or we show that there is only one homotopy class. For the $\mathbb{Z}_2$-indices, we discuss how they are related to the higher dimensional $\mathbb{Z}$ or $\mathbb{Z}_2$-index. Moreover, we provide examples to show that the indices defined are indeed non-trivial.

Contents

1 Introduction

In 2016, David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz were awarded the Nobel Prize in Physics "for theoretical discoveries of topological phase transitions and topological phases of matter" [14]. These topological phases occur in different physical systems such as topological insulators, superconductors and superfluids and can be classified depending on the symmetries and dimension of the physical system [12]. Up to now, this classification has been confirmed experimentally to some extent [4]. For example, the Haldane model [3] and the Kane-Mele model [7] describe interesting topological insulators that have been observed experimentally [6, 9].

There are several approaches to classify topological insulators [8], e.g. via K-theory, via isomorphy classes or via homotopy theory. The classification often involves advanced mathematical tools and is not fully explained. Thus, for many students and scientists it is difficult to understand the classification in detail. The goal of this thesis is to explain the classification of topological insulators in low dimensions using elementary tools from topology and differential geometry. We choose to classify the topological insulators via homotopy theory, because it is the strongest classification method taking into account all of the structure. Our direct approach without making use of the bulk-boundary correspondence makes the classification more accessible to students who are new to the subject.

For ten distinct symmetry classes [1] we consider topological insulators in dimensions 0, 1 and 2. In some cases, all topological insulators can be deformed into one another and we verify that there is only a trivial homotopy class. When there are distinct homotopy classes, we want to distinguish them. In this case, we first define equivariant vector bundles that carry all the relevant information about the topological insulator. In a second step, we define indices allowing us to distinguish between the vector bundles. For class A the 2D-index is known to be given by a Chern number of such a vector bundle. Moreover, for class AII an index is defined in [2]. For symmetry classes C and D it turns out that the index is again given by a Chern number. For class C the index can only take even values. Furthermore, for class D we show that the parity of the 2D-index is related to the 1D-index. For class DIII the 2D-index can be defined as for AII. Also here we examine the relationship of 1D- and 2D-indices.

Our work could be extended in several ways. First, in order to fully explain the classification, one would also have to check that topological insulators with the same index are equivalent. Here we only show this in a few cases. Second, the work could be extended to higher dimensions. Third, one could have a closer look at physical models with non-zero indices.

The thesis is structured as follows. In Section 2, we establish the general setting. First, we define different symmetry classes of lattice Hamiltonians. Second, for periodic Hamiltonians, we explain how the symmetries emerge after Bloch decomposition. Third, we describe the ideas behind the classification via indices. In Section 3, we go through the different symmetry classes and explain the classification in dimensions 0, 1 and 2. Finally, we provide examples of non-trivial Hamiltonians in dimension 2 in Section 4.

I would like to thank Prof. Dr. Gian Michele Graf for taking the time to supervise my work, for providing new insights and for our enlightening discussions.

1.1 Notation

- Let $GL(n) := GL(n, \mathbb{C})$.
- For an operator H the spectrum of H is denoted by $\sigma(H)$.
- For the d-dimensional torus we write $\mathbb{T}^d = (S^1)^d$.
- Identify $\mathbb{T}^d$ with $(\mathbb{R}/2\pi\mathbb{Z})^d$. Let $\tau : \mathbb{T}^d \to \mathbb{T}^d$ be the map $k \mapsto -k$.
- Let $\gamma_1 := \{0\} \times S^1, \gamma_2 := S^1 \times \{\pi\}, \gamma_3 := \{\pi\} \times S^1, \gamma_4 := S^1 \times \{0\}$.

2 Preliminaries

For $m, n \in \mathbb{Z}^d$ let

$$\delta_{m,n} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{else.} \end{cases}$$

The sequences $\delta(m) := (\delta_{m,n})_{n\in\mathbb{Z}^d}$ form an orthonormal basis of the space $\ell^2(\mathbb{Z}^d, \mathbb{C})$.

Definition 2.0.1. A *lattice Hamiltonian on a d-dimensional lattice* is a self-adjoint operator $H : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ which is *local*, i.e. there is a constant $C > 0$ such that for all $m, n \in \mathbb{Z}^d$ with $\|m - n\| > C$:

$$(\delta(m) \otimes v_m, \sum_{n\in\mathbb{Z}^d} H(\delta(n) \otimes v_n)) = 0$$

for any $\{v_n\}_{n\in\mathbb{Z}^d} \subset \mathbb{C}^N$. The number N is the number of *internal degrees of freedom.*

Example 2.0.2. A nearest neighbour hopping Hamiltonian is local with $C = 1$.

Remark 2.0.3. Let $L := (\mathbb{C}^N)^{\mathbb{Z}^d}$ be the space of all $\mathbb{C}^N$-valued functions on $\mathbb{Z}^d$. A Hamiltonian $H : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ can be extended to a map $L \to L$ by the locality condition.

2.1 Symmetries

Definition 2.1.1. Let H be a self-adjoint operator. We say that H has

(i) *chiral symmetry* if there is a unitary operator Π such that $\{H, \Pi\} = H\Pi + \Pi H = 0$ and $\Pi^2 = c1$ for some $c \in \mathbb{C}$.

(ii) *even* or *odd particle hole symmetry (PHS)* if there is an antiunitary operator Σ such that $\{H, \Sigma\} = H\Sigma + \Sigma H = 0$ and $\Sigma^2 = 1$ or $\Sigma^2 = -1$, respectively.

(iii) *even* or *odd time-reversal symmetry (TRS)* if there is an antiunitary operator Θ such that $[H, \Theta] = H\Theta - \Theta H = 0$ and $\Theta^2 = 1$ or $\Theta^2 = -1$, respectively.

In order to obtain a topological insulator, we want the Hamiltonian to have a spectral gap. We observe the following.

Proposition 2.1.2. *For a self-adjoint operator H with chiral symmetry or PHS the spectrum is preserved under the map $\epsilon : \mathbb{R} \to \mathbb{R} : \lambda \mapsto -\lambda$.*

Symmetry				Dimension d							
Class	θ	Σ	Π	1	2	3	4	5	6	7	8 or 0
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Table 1: The periodic table of topological matter. [12]

Proof. Let λ be an eigenvalue of H and $v \neq 0$ a corresponding eigenvector. In the chiral case, since Π is unitary it is injective and thus $\Pi v \neq 0$. Observe that $H\Pi v = -\Pi H v = -\lambda \Pi v$. Thus Πv is an eigenvector of H with eigenvalue $-\lambda$. For the PHS case, since Σ is antiunitary it is injective and thus $\Sigma v \neq 0$. Because λ is real we have again $H\Sigma v = -\Sigma H v = -\bar{\lambda}\Sigma v = -\lambda \Sigma v$. So Σv is an eigenvector of H with eigenvalue $-\lambda$. □

The only fixed point of ϵ is 0. The value 0 is thus distinguished among the possible eigenvalues of H. This motivates us to choose the spectral gap at 0 for chiral and PHS.

Definition 2.1.3. In Table 1 different classes of Hamiltonians are defined. A certain symmetry is present if the corresponding entry is nonzero. Moreover, for the operators Θ and Σ the entry ± 1 indicates whether the symmetry is even or odd. Additionally, the Hamiltonian has a spectral gap $\mu \notin \sigma(H)$. For symmetry classes A, AI and AII the gap can be at any value $\mu \in \mathbb{R}$. Otherwise we assume that $\mu = 0$. The operators Σ and Θ are assumed to commute if H enjoys both PHS and TRS.

Remark 2.1.4. Note that if two different symmetries are present, this implies that the third one is present as well. This is why there are only 10 classes.

Assumption 2.1.5. For lattice Hamiltonians we assume that the symmetries act only on the internal degrees of freedom. Formally, let $H : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ and let $(v_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$. Then the symmetry operators act like $\Pi(v_n)_{n\in\mathbb{Z}^d} = (\widehat{\Pi} v_n)_{n\in\mathbb{Z}^d}$, $\Sigma(v_n)_{n\in\mathbb{Z}^d} = (\widehat{\Sigma} v_n)_{n\in\mathbb{Z}^d}$ and $\Theta(v_n)_{n\in\mathbb{Z}^d} = (\widehat{\Theta} v_n)_{n\in\mathbb{Z}^d}$ for some unitary operator $\widehat{\Pi}$ and antiunitary operators $\widehat{\Sigma}$ and $\widehat{\Theta}$ with $\widehat{\Sigma}^2 = \pm 1$ and $\widehat{\Theta}^2 = \pm 1$.

2.2 Periodic Hamiltonian

For $\psi \in L = (\mathbb{C}^N)^{\mathbb{Z}^d}$ we write $(\psi)_m$ or ψ_m for the entry at $m \in \mathbb{Z}^d$. We define translation operators on the vector space L as follows:

Definition 2.2.1. Let $n \in \mathbb{Z}^d$. The translation by n, denoted by T_n is defined as

$$(T_n \psi)_m := \psi_{m-n}$$

for every $\psi \in L$.

Remark 2.2.2. Note that for all $m, n \in \mathbb{Z}^d$: $T_{m+n} = T_m T_n = T_n T_m$.

Definition 2.2.3. A lattice Hamiltonian on a d-dimensional lattice is *periodic* if for all $n \in \mathbb{Z}^d$

$$T_n H = H T_n.$$

For a local periodic lattice Hamiltonian H we can do a Bloch decomposition. Following Section 8.2. in [13] we obtain the linear isometric bijection

$$\begin{aligned}
\ell^2(\mathbb{Z}^d, \mathbb{C}^N) &\to \int_{\mathbb{T}^d}^{\oplus} \mathbb{H}(k) \mathrm{d}k \\
\psi = (\psi_n)_n &\mapsto \left(k \mapsto \frac{1}{(2\pi)^{d/2}} \sum_{n \in \mathbb{Z}^d} e^{ikn} T_n \psi \right) \\
\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} \psi(k) \mathrm{d}k &\leftarrow\!\shortmid (k \mapsto \psi(k))
\end{aligned}$$

where $\mathbb{H}(k) = \{\psi \in L | \forall n \in \mathbb{Z}^d : T_n \psi = e^{-ikn}\psi\}$. There is an isomorphism $\iota_k : \mathbb{H}(k) \cong \mathbb{C}^N$ given through $(\psi_n)_n \mapsto \psi_0$ with inverse $\psi_0 \mapsto (e^{ikn}\psi_0)_n$. So we obtain another linear isometric bijection

$$\begin{aligned}
\ell^2(\mathbb{Z}^d, \mathbb{C}^N) &\to \int_{\mathbb{T}^d}^{\oplus} \mathbb{C}^N \mathrm{d}k \\
\psi = (\psi_n)_n &\mapsto \left(k \mapsto \frac{1}{(2\pi)^{d/2}} \sum_{n \in \mathbb{Z}^d} e^{ikn} \psi_{-n} \right) \\
\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} \psi_0(k) e^{ikn} \mathrm{d}k \right)_n &\leftarrow\!\shortmid (k \mapsto \psi_0(k))
\end{aligned}$$

Note that $H(\mathbb{H}(k)) \subset \mathbb{H}(k)$ since for all $k \in \mathbb{T}^d, v \in \mathbb{C}^N$ and $m \in \mathbb{Z}^d$ we have

$$T_m H(e^{ikn}v)_n = H T_m (e^{ikn}v)_n = e^{-ikm} H(e^{ikn}v)_n.$$

This means that for every $k \in \mathbb{T}^d, v \in \mathbb{C}^N$ there is a vector $w \in \mathbb{C}^N$ such that

$$(e^{ikn}w)_n = H(e^{ikn}v)_n. \tag{1}$$

For the Hamiltonian H the Bloch decomposition gives $H = \int_{\mathbb{T}^d}^{\oplus} H(k) \mathrm{d}k$ with

$$H(k)v = w(k, v) = \Big(H(e^{ikn}v)_n \Big)_0$$

for any $k \in \mathbb{T}^d$ and $v \in \mathbb{C}^N$. For every $k \in \mathbb{T}^d$ the operator $H(k)$ is self-adjoint and for a fixed basis of $\mathbb{C}^N$ the matrix elements of $H(k)$ depend smoothly on k. Moreover, for the spectrum we have

$$\sigma(H) = \bigcup_{k \in \mathbb{T}^d} \sigma(H(k)).$$

For details see Section XIII.16. in [11]. In particular, if the spectrum of H is gapped, i.e. $\mu \notin \sigma(H)$, then also every $H(k)$ has a spectral gap at μ.

Identify $\mathbb{T}^d$ with $(\mathbb{R}/2\pi\mathbb{Z})^d$. Let $\tau : \mathbb{T}^d \to \mathbb{T}^d$ be the map $k \mapsto -k$.

Proposition 2.2.4. *The symmetries Σ, Θ and Π induce the following symmetries in the Bloch decomposition:*

(i) $\widehat{\Sigma} = \iota_k \circ \Sigma \circ \iota_k^{-1}$ *with* $\widehat{\Sigma} H(k) = -H(\tau k)\widehat{\Sigma}$,

(ii) $\widehat{\Theta} = \iota_k \circ \Theta \circ \iota_k^{-1}$ *with* $\widehat{\Theta} H(k) = H(\tau k)\widehat{\Theta}$,

(iii) $\widehat{\Pi} = \iota_k \circ \Pi \circ \iota_k^{-1}$ *with* $\widehat{\Pi} H(k) = -H(k)\widehat{\Pi}$.

Proof. (i) Let $v \in \mathbb{C}^n$. Then $\iota_k \circ \Sigma \circ \iota_k^{-1} v = (\Sigma(e^{ikn}v)_n)_0 = ((e^{-ikn}\widehat{\Sigma} v)_n)_0 = \widehat{\Sigma} v$. Moreover, $\widehat{\Sigma} H(k)v = (\Sigma(e^{ikn}H(k)v)_n)_0 = (\Sigma H(e^{ikn}v)_n)_0$ by Eq. (1). Since by assumption $\Sigma H = -H\Sigma$, we obtain $\widehat{\Sigma} H(k)v = (-H\Sigma(e^{ikn}v)_n)_0 = -(H(\widehat{\Sigma} e^{ikn}v)_n)_0$. Using the antilinearity of Σ and the definition of $H(k)$ we further get $\widehat{\Sigma} H(k)v = -(H(e^{-ikn}\widehat{\Sigma} v)_n)_0 = -H(\tau k)\widehat{\Sigma} v$. The proof of (ii) and (iii) works analogously. $\square$

Let $E = \mathbb{T}^d \times \mathbb{C}^N$ be the trivial bundle over the torus. Let $H : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ be a periodic local Hamiltonian.

Proposition 2.2.5 (Cf. 8.2.25 and 8.2.26 in [13])**.** *Let P be a projection associated to an isolated part of $\sigma(H)$. Then we may decompose P as*

$$P = \int_{\mathbb{T}^d}^{\oplus} P(k)\mathrm{d}k.$$

Moreover, $\dim(\mathrm{im}(P(k))$ *is constant in k. Thus there is a subbundle of E with fibres* $\mathrm{im}(P(k)) \subset E_k$.

Definition 2.2.6. Let H be a Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ and let $\mu \notin \sigma(H)$. Then the *Fermi projection* P_μ is the projection associated to $\sigma(H) \cap (-\infty, \mu)$.

2.3 Classification

Let H be a local periodic Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ belonging to a certain symmetry class defined in Table 1. Through Bloch decomposition we obtain a smooth family H on $\mathbb{T}^d$ of self-adjoint operators on $\mathbb{C}^N$ with symmetry properties as described in Proposition 2.2.4. Motivated by this, we want to classify continuous families H on $\mathbb{T}^d$ of self-adjoint operators on $\mathbb{C}^N$ for each symmetry class.

Two families are equivalent if they may be deformed continuously into one another while keeping the spectral gap and the symmetry property intact. Formulated in more mathematical terms this is the following:

Definition 2.3.1. Fix the symmetry operators Θ, Σ and Π. Given these operators, let $H_1, H_2 : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ belong to the same symmetry class. Then H_1 *is equivalent to* H_2 if there is a homotopy $F : \mathbb{T}^d \times [0,1] \to \mathbb{C}^{N\times N}$ between H_1 and H_2 and a continuous map $\mu : [0,1] \to \mathbb{R}$ such that for all $t \in [0,1]$ the map $F(\cdot, t)$ belongs to the same symmetry class and has spectral gap at $\mu(t)$.

How many equivalence classes are there for each symmetry class? To answer this question it is useful to label the different equivalence classes with an index, which assigns a different number to every equivalence class. It is known how many equivalence classes there are. For a sufficiently large number of internal degrees of freedom, the equivalence classes can be labelled by $\mathbb{Z}$, $\mathbb{Z}_2$ or 0. The results are listed in Table 1 for different dimensions of the lattice and different symmetries. The list repeats periodically, i.e. the entries are equal for dimensions d and $d+8$. There are some rules how to read the entries of Table 1. First of all, the number of internal degrees of freedom lying above and below the spectral gap has to be sufficiently large. In Section 4.1 we will see that for class A in dimension $d = 2$, there has to be at least one internal degree of freedom above and below the spectral gap to admit a non-trivial index. Secondly, an entry in dimension d counts the equivalence classes of Hamiltonians that have the same lower dimensional $\mathbb{Z}$-indices. Note that the all $\mathbb{Z}_2$-indices are part of a sequence $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}$ when increasing the dimension d. The $\mathbb{Z}_2$-indices are related to the $\mathbb{Z}$-index in this sequence.

In order to show that certain indices vanish, the following results will be useful.

Proposition 2.3.2. *Let X be a topological space and let $f : X \to X$ be a continuous involution. Let $I \in A := \{x \in X | f(x) = x\}$. Let A_I denote the connected component of A containing I. Let $G : S^1 \to X$ be a continuous map satisfying $G(\tau k) = f(G(k))$ and $G(k^*) \in A_I$ for $k^* \in \{0, \pi\}$.*

(i) Then there is a homotopy $F_1 : S^1 \times [0, 1] \to X$ satisfying $F_1(\tau k, t) = f(F_1(k, t))$ and $F_1(\cdot, 0) = G$ and $F_1(k^*, 1) = I$.

(ii) Let $\widehat{G} := F_1(\cdot, 1)$. If the map $\widehat{G}|_{[0,\pi]}$ is homotopic to the constant map I relative to the endpoints, then there is a homotopy $F_2 : S^1 \times [0, 1] \to X$ between G and I such that $F_2(\tau k, t) = f(F_2(k, t))$.

Proof. (i) Let $\alpha : \{0, \pi\} \times [0, 1] \to A_I$ be a continuous map with $\alpha(k^*, 0) = G(k^*)$ and $\alpha(k^*, 1) = I$. There is a map $F_1 : [0, \pi] \times [0, 1] \to X$ with $F_1(\cdot, 0) = G(\cdot)$ extending α, because $\{0, \pi\} \times [0, 1] \cup [0, \pi] \times \{0\}$ is a retract of $[0, \pi] \times [0, 1]$. Extend F_1 to S^1 by $F_1(\tau k, t) = f(F_1(k, t))$.

(ii) Let F be the homotopy between $F_1(\cdot, 1)|_{[0,\pi]}$ and I relative to $\{0, \pi\}$. Then let $F(\tau k, t) := f(F(k, t))$ and $F_2(k, t) = \begin{cases} F_1(k, 2t) & \text{for } t \in [0, 1/2] \\ F(k, 2t-1) & \text{for } t \in [1/2, 1]. \end{cases}$

□

Let $\gamma_1 := \{0\} \times S^1, \gamma_2 := S^1 \times \{\pi\}, \gamma_3 := \{\pi\} \times S^1, \gamma_4 := S^1 \times \{0\} \subset \mathbb{T}$.

Proposition 2.3.3. *Let X be a topological space and $f : X \to X$ a continuous involution. Let $I \in A := \{x \in X | f(x) = x\}$. Let A_I denote the connected component of A containing I. Let $G : \mathbb{T} \to X$ be a continuous map satisfying $G(\tau k) = f(G(k))$ and $G(k^*) \in A_I$ for all fixed points $\tau k^* = k^*$.*

(i) Then there is a homotopy $F_1 : \mathbb{T} \times [0, 1] \to X$ satisfying $F_1(\tau k, t) = f(F_1(k, t))$ and $F_1(\cdot, 0) = G$ and $F_1(k^*, 1) = I$.

(ii) Let $\widehat{G} := F_1(\cdot, 1)$. If $\pi_2(X) = 0$ and for all γ_i the map $\widehat{G}|_{\gamma_i|_{[0,\pi]}}$ is homotopic to the constant map I relative to the endpoints, then $G : \mathbb{T} \to X$ is homotopic to the constant map I via a homotopy F_2 satisfying $F_2(\tau k, t) = f(F_2(k, t))$.

Proof. (i) By Proposition 2.3.2 (i), there is a continuous map $F_1 : (\gamma_1 \cup \gamma_3) \times [0, 1]$ satisfying $F_1(\tau k, t) = f(F_1(k, t))$ and $F_1(k, 0) = G(k)$ for $k \in \gamma_1 \cup \gamma_3$ and $F_1(k^*, 1) = I$ for all fixed points of $\mathbb{T}$. One can extend F_1 to a map $F_1 : [0, \pi] \times S^1 \times [0, 1] \to X$ such that $F_1(k, 0) = G(k)$, because $[0, \pi] \times S^1 \times \{0\} \cup \{0, \pi\} \times S^1 \times [0, 1]$ is a retract of $[0, \pi] \times S^1 \times [0, 1]$. Then extend F_1 to $\mathbb{T} \times [0, 1]$ through $F_1(\tau k, t) = f(F_1(k, t))$.

(ii) For all γ_i let F^i be the homotopy between $\widehat{G}|_{\gamma_i|_{[0,\pi]}}$ and I relative to the endpoints. Then we can extend F^i through $F^i(\tau k, t) = f(F^i(k, t))$ to give a homotopy between $\widehat{G}|_{\gamma_i}$ and I. Let $F_3 : (\gamma_1 \cup \gamma_3 \cup \gamma_2|_{[0,\pi]}) \times [0, 1] \to X$ be given by $F(k, t) = F^i(k, t)$ if $k \in \gamma_i$. There is an extension $F_3 : [0, \pi] \times S^1 \times [0, 1] \to X$ of F_3 satisfying $F_3(k, 0) = \widehat{G}(k)$ because $(\gamma_1 \cup \gamma_3 \cup \gamma_2|_{[0,\pi]}) \times [0, 1] \cup [0, \pi] \times S^1 \times \{0\}$ is a retract of $[0, \pi] \times S^1 \times [0, 1]$. Then $\widetilde{G} := F_3(\cdot, 1) : [0, \pi] \times S^1 \to X$ is constantly equal to I on $\gamma_1 \cup \gamma_3 \cup \gamma_2|_{[0,\pi]}$. So we can view $\widetilde{G}$ as a map from $[0, \pi] \times S^1/(\gamma_1 \cup \gamma_3 \cup \gamma_2|_{[0,\pi]}) \cong S^2$ to X. Since $\pi_2(X) = 0$, there is a homotopy $F_4 : [0, \pi] \times S^1 \to X$ between $\widetilde{G}$ and I such that $F_4(k, t) = I$ for $k \in \gamma_1 \cup \gamma_3 \cup \gamma_2|_{[0,\pi]}$. We can further extend F_3 and F_4 to $\mathbb{T} \times [0, 1]$ via $F_i(\tau k, t) = f(F_i(k, t))$ for $i = 3, 4$. Finally, let

$$F_2(k, t) = \begin{cases} F_1(k, 3t) & \text{for } t \in [0, 1/3] \\ F_3(k, 3t - 1) & \text{for } t \in [1/3, 2/3] \\ F_4(k, 3t - 2) & \text{for } t \in [2/3, 1]. \end{cases}$$

□

3 Topological indices in low dimensions

For every symmetry class we begin with a continuous family H of self-adjoint operators on $\mathbb{C}^N$ with the corresponding symmetry properties. In certain cases we define a corresponding vector bundle carrying all the important information. The vector bundles are said to be equivalent if and only if the corresponding Hamiltonians are equivalent. Equivalent vector bundles are isomorphic, but the converse is not true in general. We focus on dimensions $d \in \{0, 1, 2\}$ and explain the entries in Table 1. If the entry is 0, the aim is to show that indeed, no non-trivial index can be defined. If the entry is non-trivial, we want to formulate the index in terms of the vector bundle corresponding to H. We also want to understand the relationship of the $\mathbb{Z}_2$- and $\mathbb{Z}$-indices.

3.1 A

Any continuous family $H = \{H(k) : k \in \mathbb{T}^d\}$ of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $\mu \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$ belongs to symmetry class A. Let $P_\mu^-(k)$ denote the Fermi-projection and $P_\mu^+(k)$ the projection associated to $\sigma(H) \cap (\mu, \infty)$. From H we obtain the vector bundles $E^\pm(k) = \{(k, P_\mu^\pm(k)(\mathbb{C}^N))\}$ over $\mathbb{T}^d$ with $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$.

Definition 3.1.1. A d-dimensional bundle in symmetry class A is a complex vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$, where the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$.

Remark 3.1.2. Even though the bundle E is trivial, the subbundles E^+ and E^- can be non-trivial.

Given a vector bundle $E = E^+ \oplus E^-$ in class A, we can define $\widehat{H}(k) = 2P_\mu^+(k) - 1$. Then $\widehat{H}$ forms again a continuous family of self-adjoint operators with spectral gap at 0.

Remark 3.1.3. Let H be in class A. Then H is equivalent to the corresponding $\widehat{H}$.

Proof sketch. Let $H : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be in class A with spectral gap at μ. For every $k \in \mathbb{T}^d$ we can write $H(k) = \sum_i \lambda_i(k)P_i(k)$, where $P_i(k)$ is the orthogonal projection onto the eigenspace to eigenvalue $\lambda_i(k)$. Note that P_i and λ_i may not be continuous since the number of distinct eigenvalues and the dimensions of the eigenspaces may vary. For $t \in [0, 1]$ and $\lambda \in \mathbb{R}$ let $f(\lambda, t) = \lambda(1 - t) + t \operatorname{sgn}(\lambda - \mu)$. Now let $F : \mathbb{T}^d \times [0, 1] \to \mathbb{C}^{N\times N}$ be given by

$$F(k, t) = \sum_i f(\lambda_i(k), t)P_i(k).$$

Then for every $t \in [0, 1]$ the map $F(\cdot, t)$ is continuous and in class A with spectral gap at $\mu(1-t)$. Moreover, F is a homotopy between H and $\widehat{H}$. □

For dimension $d \in \{0, 1, 2\}$ we want to define an index $\mathcal{I}_A^d$ for vector bundles in class A. Then the index of H will be $\mathcal{I}_A^d(H) := \mathcal{I}_A^d(E)$.

Definition 3.1.4. In dimension $d = 0$, the index of a vector bundle $E = E^+ \oplus E^-$ is

$$\mathcal{I}_A^0(E) := \operatorname{rank}(E^+) - \operatorname{rank}(E^-).$$

Proposition 3.1.5. *In dimension $d = 0$, for fixed N any two Hamiltonians H_1 and H_2 with $\mathcal{I}_A^0(H_1) = \mathcal{I}_A^0(H_2)$ are equivalent.*

Proof. Let $\mathcal{I}_A^0(H_1) = \mathcal{I}_A^0(H_2) = m$. We know that H_i is equivalent to $\widehat{H}_i$ for $i = 1, 2$ by Remark 3.1.3. Pick unitary frames $v_i^\pm$ of $E_i^\pm$. Then the frames $v_i = (v_i^+, v_i^-)$ lie in $U(N)$. Since $U(N)$ is path connected, there is a path $v : [0, 1] \to U(N)$ with $v(0) = v_1$ and $v(1) = v_2$. Then

$$F(t) = v(t) \begin{pmatrix} \mathbb{I}_{(N+m)/2} & 0 \\ 0 & -\mathbb{I}_{(N-m)/2} \end{pmatrix} v(t)^*$$

is a homotopy between $\widehat{H}_1$ and $\widehat{H}_2$ which belongs to class A for every $t \in [0, 1]$. So $\widehat{H}_1$ and $\widehat{H}_2$ and thus also H_1 and H_2 are equivalent. □

For a given number N of internal degrees of freedom, the index $\mathcal{I}_A^0$ can take any value between $-N$ and N. Thus for $N \to \infty$, the index $\mathcal{I}_A^0$ can take any integer value. This justifies the entry $\mathbb{Z}$ for A in $d = 0$ in Table 1.

Proposition 3.1.6. *The frame bundle of any complex vector bundle over S^1 admits a global section.*

Proof. Let E be a N-dimensional complex vector bundle over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then E induces a complex vector bundle $\dot{E}$ over $[-\pi, \pi]$. Since $[-\pi, \pi]$ is contractible, there is a frame $v : [-\pi, \pi] \to$

$F(\dot{E})$. Moreover, there is a matrix $G \in GL(N)$ such that $v(-\pi) = v(\pi)G$. Since $GL(N)$ is path-connected, there is a path $\gamma : [-\pi, \pi] \to GL(N)$ connecting $\gamma(-\pi) = I_N$ to $\gamma(\pi) = G$. Then $w(k) := v(k)\gamma(k)$ is again a section of $F(\dot{E})$ with $w(-\pi) = w(\pi)$. So w actually defines a global section of $F(E)$. □

Fact 3.1.7. Let $N \in \mathbb{Z}_{>0}$. Two continuous maps $S^1 \to U(N)$ are homotopic if and only if their determinant has the same winding number.

Proposition 3.1.8. *In dimension $d = 1$, for fixed N any two Hamiltonians H_1 and H_2 with $\boldsymbol{\mathcal{I}}_A^0(H_1) = \boldsymbol{\mathcal{I}}_A^0(H_2)$ are equivalent.*

Proof. Let $\boldsymbol{\mathcal{I}}_A^0(H_1) = \boldsymbol{\mathcal{I}}_A^0(H_2) = m$. We know that H_j is equivalent to $\widehat{H}_j$ for $j = 1, 2$ by Remark 3.1.3. By Proposition 3.1.6 we can pick unitary frames $v_j^\pm : S^1 \to E_j^\pm$. Then the frames $v_j = (v_j^+, v_j^-) : S^1 \to U(N)$ have winding numbers $l_j = \boldsymbol{\mathcal{W}}(\det v_j)$. We obtain unitary frames $\widetilde{v}_j$ of $E_j^+ \oplus E_j^-$ with vanishing winding number by setting $\widetilde{v}_j(k) = v_j(k) \begin{pmatrix} e^{-ikl_j} & 0 \\ 0 & \mathbb{I}_{N-1} \end{pmatrix}$. By Fact 3.1.7, there is a homotopy $F : S^1 \times [0, 1] \to U(N)$ between $\widetilde{v}_1$ and $\widetilde{v}_2$. Then

$$F(k,t) \begin{pmatrix} \mathbb{I}_{(N+m)/2} & 0 \\ 0 & -\mathbb{I}_{(N-m)/2} \end{pmatrix} F(k,t)^*$$

is a homotopy between $\widehat{H}_1$ and $\widehat{H}_2$, which lies in class A for every $t \in [0, 1]$. So $\widehat{H}_1$ and $\widehat{H}_2$ and thus also H_1 and H_2 are equivalent. □

Thus we have verified that the entry in Table 1 for A in $d = 1$ must be 0. Now we move on to $d = 2$. Let $\dot{\mathbb{T}} = [-\pi, \pi] \times S^1$ denote the cut torus. For a vector bundle E over $\mathbb{T}$ let $\dot{E}$ denote the induced bundle on $\dot{\mathbb{T}}$.

Proposition 3.1.9. *Let $\dot{E}$ be a complex vector bundle over $\dot{\mathbb{T}}$. There is a global frame $v : \dot{\mathbb{T}} \to F(\dot{E})$.*

Proof. By Proposition 3.1.6, there is a section $v : \{0\} \times S^1 \to F(\dot{E})$. Since $\{0\} \times S^1$ is a deformation retract of $\dot{\mathbb{T}}$, we can extend v to a frame $v : \dot{\mathbb{T}} \to F(\dot{E})$ by Theorem 9.1 in [10]. □

Definition 3.1.10. Let $E = E^+ \oplus E^-$ be in class A, let $n = \mathrm{rank}(E^-)$. Let $v : \dot{\mathbb{T}} \to F(\dot{E}^-)$ be a global frame. There is a map $T : S^1 \to GL(n)$ satisfying

$$v(-\pi, k_2)T(k_2) = v(\pi, k_2)$$

for all $k_2 \in S^1$. Let $\boldsymbol{\mathcal{I}}_A^2(E) := \boldsymbol{\mathcal{W}}(\det T)$ be the winding number of $\det T$.

Lemma 3.1.11. *The index $\boldsymbol{\mathcal{I}}_A^2$ is well-defined, i.e. independent of the choice of the frame v.*

Proof. Consider two frames $v, w : \dot{\mathbb{T}} \to F(\dot{E}^-)$. They are related by $v(k) = w(k)G(k)$ for some $G : \dot{\mathbb{T}} \to GL(n)$. Let $v(-\pi, k_2)T_1(k_2) = v(\pi, k_2)$ and $w(-\pi, k_2)T_2(k_2) = w(\pi, k_2)$. We have

$$\begin{aligned} w(-\pi, k_2)G(-\pi, k_2)T_1(k_2) &= v(-\pi, k_2)T_1(k_2) = v(\pi, k_2) \\ &= w(\pi, k_2)G(\pi, k_2) = w(-\pi, k_2)T_2(k_2)G(\pi, k_2). \end{aligned}$$

Thus,

$$G(-\pi, k_2)T_1(k_2) = T_2(k_2)G(\pi, k_2).$$

Note that the winding numbers $\boldsymbol{\mathcal{W}}(\det G(-\pi,\cdot)) = \boldsymbol{\mathcal{W}}(\det G(\pi,\cdot))$ because $G : [-\pi,\pi] \times S^1 \to GL(n)$ is a homotopy between the two maps. So $\boldsymbol{\mathcal{W}}(\det G(-\pi,\cdot)) + \boldsymbol{\mathcal{W}}(\det T_1) = \boldsymbol{\mathcal{W}}(\det T_2) + \boldsymbol{\mathcal{W}}(\det G(\pi,\cdot))$ implies $\boldsymbol{\mathcal{W}}(\det T_1) = \boldsymbol{\mathcal{W}}(\det T_2)$. Hence, the index is independent of the frame. □

Remark 3.1.12. The number $\boldsymbol{\mathcal{I}}_A^2(E)$ is precisely the first Chern number $Ch_1(E^-)$. This is explained in [13] in Section 8.4.

Remark 3.1.13. Since $E = E^+ \oplus E^- = \mathbb{T} \times \mathbb{C}^N$ is trivial, we have

$$0 = Ch_1(E) = Ch_1(E^+) + Ch_1(E^-).$$

Thus, choosing $\boldsymbol{\mathcal{I}}_A^2$ to be the first Chern number of E^+ instead of E^- would amount to a change of sign.

3.2 AIII

Definition 3.2.1. Let $H = \{H(k) : k \in \mathbb{T}^d\}$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H has chiral symmetry if there is an operator $\Pi : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Π is linear and unitary,

(ii) $\Pi^2 = c1$ for some $c \in \mathbb{C}$,

(iii) for all $k \in \mathbb{T}$,
$$H(k)\Pi = -\Pi H(k).$$

Proposition 3.2.2. *Let $H = \{H(k) : k \in \mathbb{T}^d\}$ have chiral symmetry. Then there is a basis B of $\mathbb{C}^N$ for which $\Pi = \lambda\mathbb{I}_n \oplus -\lambda\mathbb{I}_n$ and*

$$H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}$$

for some continuous $h : \mathbb{T}^d \to GL(N)$.

Proof. Since Π is unitary it is diagonalisable and $0 \notin \sigma(\Pi)$. Moreover if v is an eigenvector of Π with eigenvalue λ, then $\Pi H(k)v = -H(k)\Pi v = -\lambda H(k)v$ for any $k \in \mathbb{T}^d$. Thus $-\lambda$ is an eigenvalue of Π. Let $E_{\Pi,\lambda}$ be the eigenspace of Π to eigenvalue λ. Observe that $H(E_{\Pi,\lambda}) \subset E_{\Pi,-\lambda}$ and $H(E_{\Pi,-\lambda}) \subset E_{\Pi,\lambda}$. Because $0 \notin \sigma(H)$, the operator H is injective. Hence, $\dim(E_{\Pi,\lambda}) \leq \dim(E_{\Pi,-\lambda}) \leq \dim(E_{\Pi,\lambda})$. Thus equality must hold. Since $\Pi^2 = c1$, every eigenvalue λ of Π satisfies $\lambda^2 = c$. So Π has precisely the two eigenvalues $\pm\sqrt{c}$. Hence, there is a basis B of $\mathbb{C}^N$ consisting of eigenvectors of Π for which $\Pi = \lambda\mathbb{I}_n \oplus -\lambda\mathbb{I}_n$ with $N = 2n$.

In the basis B the condition $H(k)\Pi = -\Pi H(k)$ then implies

$$H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}$$

for some $h : \mathbb{T}^d \to GL(n)$. The map h is continuous by the continuity of H. $\square$

Remark 3.2.3. Note that for any continuous map $h : \mathbb{T}^d \to GL(n, \mathbb{C})$ the conditions $0 \notin \sigma(H(k))$ and $\{H(k), \Pi\} = 0$ are automatically satisfied. So two chiral Hamiltonians $H(k)$ and $\widetilde{H}(k)$ are equivalent if and only if the corresponding $h(k)$ and $\widetilde{h}(k)$ are homotopic as maps from $\mathbb{T}^d$ to $GL(n, \mathbb{C})$. Thus, the indices for the symmetry class AIII label the homotopy classes of maps $h : \mathbb{T}^d \to GL(n, \mathbb{C})$.

Remark 3.2.4. Since $GL(n, \mathbb{C})$ is path connected, the 0D-index vanishes for AIII.

Fact 3.2.5. The 1D-index is given through $\boldsymbol{\mathcal{I}}^1_{AIII}(H) = \boldsymbol{\mathcal{W}}(\det(h))$. Moreover, if for two continuous maps $S^1 \to GL(n)$ the determinant has the same winding number, they are homotopic.

Example 3.2.6. For $d = 1$ we can choose $h : S^1 \to GL(n)$ to be given by

$$h(k) = \begin{pmatrix} e^{ikl} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$$

for some $l \in \mathbb{Z}$. Then for the corresponding Hamiltonian we have $\boldsymbol{\mathcal{I}}^1_{AIII}(H) = l$.

This gives the entry $\mathbb{Z}$ in Table 1 for AIII in $d = 1$. In 2D the situation is as follows.

Proposition 3.2.7. *Two maps $h, \widetilde{h} : \mathbb{T} \to GL(n)$ are homotopic if and only if for every path $\gamma : S^1 \to \mathbb{T}$ the maps $h \circ \gamma$ and $\widetilde{h} \circ \gamma$ are homotopic.*

Combining Fact 3.2.5 with Proposition 3.2.7 we obtain:

Corollary 3.2.8. *In 2D, if two families of Hamiltonians $H(k)$ and $\widetilde{H}(k)$ in class AIII cannot be distinguished by the 1D-index, i.e. for all $\gamma : S^1 \to \mathbb{T}$ it holds that $\boldsymbol{\mathcal{I}}^1_{AIII}(H \circ \gamma) = \boldsymbol{\mathcal{I}}^1_{AIII}(\widetilde{H} \circ \gamma)$, then $H(k)$ and $\widetilde{H}(k)$ are homotopic. This justifies the entry 0 in Table 1.*

Before we prove Proposition 3.2.7, we need the following results.

Proposition 3.2.9. *The second homotopy groups $\pi_2(GL(n, \mathbb{C})) = \pi_2(U(n)) = 0$.*

Proof. The space $GL(n, \mathbb{C})$ is homotopy equivalent to $U(n)$ [5]. For $n = 1$ we have $U(1) = S^1$, so $\pi_2(U(1)) = \pi_2(S^1) = 0$. Consider the fibre bundle

$$U(n-1) \to U(n) \to U(n)/U(n-1),$$

where $\iota : U(n-1) \to U(n)$ is given by

$$\iota(A) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

and $\pi : U(n) \to U(n)/U(n-1)$ is the quotient map.

By the long exact sequence for fibre bundles (see [5], Section 4.2.) we obtain the long exact sequence

$$... \to \pi_3(U(n)/U(n-1)) \to \pi_2(U(n-1)) \to \pi_2(U(n)) \to \pi_2(U(n)/U(n-1)) \to ...$$

Note that $U(n)/U(n-1)$ is homeomorphic to S^{2n-1} through the map

$$[(u_1, u_2, ..., u_n)] \mapsto u_1,$$

where u_i denotes the ith column. This map is well defined because the right action of $U(n-1)$ on $U(n)$ leaves the first column invariant. The map is clearly surjective and injective.

We obtain the exact sequence

$$... \to \pi_3(S^{2n-1}) \to \pi_2(U(n-1)) \to \pi_2(U(n)) \to \pi_2(S^{2n-1}) \to ...$$

For $n \geq 3$, the two outer terms vanish and thus $\pi_2(U(n)) \cong \pi_2(U(n-1))$. For $n = 2$, the second and the last term vanish, thus also the third term $\pi_2(U(2))$ must vanish. So inductively, we obtain $\pi_2(U(n)) = 0$ for $n \geq 1$. □

Proof of Proposition 3.2.7. By Fact 3.2.5 a map $f : S^1 \to GL(n)$ is homotopic to a constant map if and only if the winding number $\mathcal{W}(\det f)$ vanishes. Identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Let $\alpha : S^1 \to S^1 \times S^1 : x \mapsto (x, 0)$ and $\beta : S^1 \to S^1 \times S^1 : x \mapsto (0, x)$.

Case 1: Assume $h : \mathbb{T} \to GL(n)$ is trivial in 1D, i.e. for all $\gamma : S^1 \to \mathbb{T}$ the map $h \circ \gamma$ is homotopic to a constant map. Then $\mathcal{W}(\det(h \circ \alpha)) = \mathcal{W}(\det(h \circ \beta)) = 0$. Thus there is a homotopy f between $S|_{S^1 \times \{0\} \cup \{0\} \times S^1}$ and the constant map $\mathbb{I}_n$. Note that $\mathbb{T} \times \{0\} \cup (S^1 \times \{0\} \cup \{0\} \times S^1) \times [0, 1]$ is a retract of $\mathbb{T} \times [0, 1]$. Thus, by the homotopy extension property [5], we can obtain a homotopy $F : \mathbb{T} \times [0, 1] \to GL(n)$ starting at h and extending f. The map $\widetilde{h} := F(\cdot, 1)$ is constant on $S^1 \times \{0\} \cup \{0\} \times S^1$. Thus, it can be viewed as a map $\widetilde{h} : \mathbb{T}/(S^1 \times \{0\} \cup \{0\} \times S^1) \to GL(n)$. But $\mathbb{T}/(S^1 \times \{0\} \cup \{0\} \times S^1)$ is homeomorphic to S^2. Therefore, $\widetilde{h}$ induces a map $S^2 \to GL(n)$ which by Proposition 3.2.9 is homotopic to the constant map $\mathbb{I}_n$. This homotopy induces a homotopy from $\widetilde{h}$ to I_n on $\mathbb{T}$. In total we see that thus h is homotopic to the constant map $\mathbb{I}_n$.

Case 2: Suppose $h_1, h_2 : \mathbb{T} \to GL(n)$ are continuous and the winding numbers $\mathcal{W}(\det(h_1 \circ \alpha)) = \mathcal{W}(\det(h_2 \circ \alpha))$ and $\mathcal{W}(\det(h_1 \circ \beta)) = \mathcal{W}(\det(h_2 \circ \beta))$. Consider $h := h_1 h_2^{-1} : \mathbb{T} \to GL(n)$. Then $\mathcal{W}(\det(h \circ \alpha)) = \mathcal{W}(\det(h_1 \circ \alpha) \det(h_2 \circ \alpha)^{-1}) = \mathcal{W}(\det(h_1 \circ \alpha)) - \mathcal{W}(\det(h_2 \circ \alpha)) = 0$ and analogously $\mathcal{W}(\det h \circ \beta) = 0$. By Case 1, there is a homotopy $F : \mathbb{T} \times [0, 1] \to GL(n)$ from h to $\mathbb{I}_n$. Then $\widetilde{F} : \mathbb{T} \times [0, 1] \to GL(n)$ given through $\widetilde{F}(x, t) := F(x, t)h_2(x)$ is a homotopy from $h_1 = hh_2$ to $h_2 = \mathbb{I}_n h_2$. □

3.3 AI

Definition 3.3.1. Let $H(k)$ for $k \in \mathbb{T}^d$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $\mu \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H has even time-reversal symmetry if there is an operator $\Theta : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ is antiunitary,

(ii) $\Theta^2 = 1$,

(iii) for all $k \in \mathbb{T}$,

$$\Theta H(k) = H(\tau k)\Theta.$$

Remark 3.3.2. We can write $\Theta = UC$ for some $U \in U(N)$ and the complex conjugation C. The condition $\Theta^2 = 1$ implies that U is symmetric. Thus by Autonne-Takagi factorisation, there is $Q \in U(N)$ such that $QUQ^T = \mathbb{I}_N$. So after changing the basis by Q, we may assume that $\Theta = C$. In this basis, the condition $\Theta H(k) = H(\tau k)\Theta$ then becomes

$$H(\tau k) = \overline{H(k)}. \tag{2}$$

Definition 3.3.3. A bundle in class AI is a vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$, with an antiunitary map $\Theta : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$,

(ii) $\Theta^2 = 1$,

(iii) the orthogonal projections $P^\pm(k)$ onto $E^\pm(k)$ satisfy $P^\pm(\tau k)\Theta = \Theta P^\pm(k)$.

Remark 3.3.4. Let H be in class AI. Let P^- be the Fermi projection and P^+ the projection associated to $\sigma(H) \cap (0, \infty)$. Let $E^\pm(k) = \{(k, P^\pm(k)(\mathbb{C}^N))\}$. Then we obtain a bundle in class AI. Conversely, $\widehat{H}(k) := 2P^+(k) - 1$ is self-adjoint, unitary and squares to 1 and belongs to class AI.

Remark 3.3.5. Let H be in class AI. Then H is equivalent to the corresponding $\widehat{H}$.

Proof. Note that the homotopy F constructed in the proof of Remark 3.1.3 satisfies $F(\tau k, t) = \overline{F(k, t)}$. □

Let us first analyse the situation in 0D. As for class A, the number of negative eigenvalues of H is homotopy invariant. So we can define an index in the same way.

Definition 3.3.6. Let H be a in class AI in 0D. Then $\mathcal{I}^0_{AI}(H) := \mathrm{rank}(E^+) - \mathrm{rank}(E^-)$.

Proposition 3.3.7. *Let H_1 and H_2 be in class AI in 0D on $\mathbb{C}^N$. If $\mathcal{I}^0_{AI}(H_1) = \mathcal{I}^0_{AI}(H_2)$ then H_1 can be deformed into H_2 while keeping the symmetry and the spectral gap intact.*

Proof. By Eq. (2) and Hermiticity, $\widehat{H}_1$ and $\widehat{H}_2$ are real and symmetric. Thus they are diagonalisable, i.e. $\widehat{H}_i = Q_i \begin{pmatrix} \mathbb{I}_{N-m} & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix} Q_i^{-1}$ for $Q_i \in O(N)$ for $i = 1, 2$. We can choose the Q_i such that $\det Q_i = 1$ by multiplying with $\begin{pmatrix} -1 & 0 \\ 0 & \mathbb{I}_{N-1} \end{pmatrix}$ from the right if necessary. Then both Q_i lie in $SO(N)$, which is path connected. Let $Q(t)$ be a path in $SO(N)$ from Q_1 to Q_2. Then $F(t) := Q(t) \begin{pmatrix} \mathbb{I}_{N-m} & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix} Q^T(t)$ is a homotopy between $\widehat{H}_1$ and $\widehat{H}_2$ satisfying the conditions we need. By Remark 3.3.5 thus H_1 and H_2 are equivalent. □

Now let us consider the situation in 1D.

Proposition 3.3.8. *Let $E = E^+ \oplus E^-$ be a bundle in AI over S^1. There exist unitary frames $v^\pm : S^1 \to F(E^\pm)$ such that $v^\pm(\tau k) = \overline{v^\pm(k)}$.*

Proof. Let $m = \operatorname{rank}(E^+)$. For $k^* \in \{0, \pi\}$ we may pick real orthonormal frames $v^+(k^*) \in F(E^+_{k^*})$. Extend $v^+(0)$ to a frame $\widetilde{v}^+$ over $[0,\pi]$. Then $v^+(\pi) = \widetilde{v}^+(\pi)G^+(\pi)$ for some $G^+(\pi) \in U(m)$. Since $U(m)$ is path-connected, there is a path $G^+ : [0,\pi] \to U(m)$ from $\mathbb{I}_m$ to $G^+(\pi)$. Then $v^+(k) := \widetilde{v}^+(k)G^+(k)$ extends the $v^+(k^*)$ we chose at the beginning. Now we extend v^+ to S^1 by $v^+(\tau k) = \overline{v^+(k)}$. Analogously, we can obtain a frame $v^- : S^1 \to F(E^-)$ with $v^-(\tau k) = \overline{v^-(k)}$. $\square$

Proposition 3.3.9. *The index for class AI in 1D must be trivial.*

Proof. Let H be in class AI. Consider the corresponding bundle $E = E^+ \oplus E^-$ over S^1, let $m = \operatorname{rank}(E^+)$. Let $v^\pm$ be frames as in Proposition 3.3.8. Let us write $v = (v^+, v^-)$ and let $l := \boldsymbol{\mathcal{W}}(\det v)$. Then $w(k) := v(k)\begin{pmatrix} \operatorname{sgn}(\det v(0))e^{-ikl} & 0 \\ 0 & \mathbb{I}_{N-1}\end{pmatrix}$ is also a section of $F(E^+) \oplus F(E^-)$ satisfying $w(\tau k) = \overline{w(k)}$. Since $0 = \boldsymbol{\mathcal{W}}(\det w) = 2\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det w)$, also $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det w) = 0$. Applying Proposition 2.3.2 with $X = U(N)$, $I = \mathbb{I}_N$, $G = w$ and f being the complex conjugation and noting that $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det \widehat{G}) = 0$ by the reality condition at the endpoints, we obtain a homotopy $F : \mathbb{T} \times S^1 \to U(N)$ between w and $\mathbb{I}_N$ such that $F(\tau k, t) = \overline{F(k,t)}$. Then

$$F(k,t)\begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_{N-m}\end{pmatrix} F(k,t)^*$$

is a homotopy between $\widehat{H}$ and $\begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_{N-m}\end{pmatrix}$ which preserves the spectral gap and the symmetry property. By Remark 3.3.5 H is equivalent to $\widehat{H}$, showing that the index must vanish. $\square$

Now let us consider the situation in 2D.

Proposition 3.3.10. *Let H be in class AI in 2D and $E = E^+ \oplus E^-$ the corresponding bundle. There are unitary frames $v^\pm : \dot{\mathbb{T}} \to F(\dot{E}^\pm)$ such that $v^\pm(\tau k) = \overline{v^\pm(k)}$ and $v^\pm(-\pi, k_2^*) = v^\pm(\pi, k_2^*)$ for $k_2^* \in \{0, \pi\}$.*

Proof. Let $m = \operatorname{rank}(E^+)$. By Proposition 3.3.8 there is a section $v^+ : \{0\} \times S^1 \to F(\dot{E}^+)$ satisfying $v^+(\tau k) = \overline{v^+(k)}$. Extend this section to $v : [0,\pi] \times S^1 \to F(\dot{E}^+)$. Then for $k_2^* \in \{0,\pi\}$ there are $G(\pi, k_2^*) \in U(m)$ such that $v^+(\pi, k_2^*)G(\pi, k_2^*)$ is real, because $H(\pi, k_2^*)$ is real. Pick a loop $G(\pi, \cdot) : S^1 \to U(m)$ extending $G(\pi, 0)$ and $G(\pi,\pi)$. Let $l = \boldsymbol{\mathcal{W}}(\det G(\pi, \cdot))$, then there exists a homotopy $G : [0,\pi] \times S^1 \to U(m)$ between $G(0,k) = \begin{pmatrix} e^{ikl} & 0 \\ 0 & \mathbb{I}_{m-1}\end{pmatrix}$ and $G(\pi, \cdot)$. Now let $w^+ : [0,\pi] \times S^1 \to F(\dot{E}^+)$ be given by $w^+(k) = v^+(k)G(k)$. Note that $w^+(0, \tau k_2) = \overline{w^+(0, k_2)}$. Then choose $w^+(k) := \overline{w^+(\tau k)}$ for $k \in [-\pi, 0] \times S^1$. This gives a frame with all the properties we wanted. The construction of v^- is analogous. $\square$

Proposition 3.3.11. *Let H be in class AI in 2D and $E = E^+ \oplus E^-$ the corresponding bundle. There are global unitary frames $v^\pm : \mathbb{T} \to F(E^\pm)$ such that $v^\pm(\tau k) = \overline{v^\pm(k)}$.*

Proof. Let $m^\pm = \operatorname{rank}(E^\pm)$. Let $v^\pm : \dot{\mathbb{T}} \to F(\dot{E}^\pm)$ be frames as in Proposition 3.3.10. Let $v = (v^+, v^-)$. We need to find a map $G : \dot{\mathbb{T}} \to U(m^+) \oplus U(m^-)$ such that $w = vG$ is a global frame satisfying the symmetry property, i.e. $v(-\pi, \cdot)G(-\pi, \cdot) = v(\pi, \cdot)G(\pi, \cdot)$ and $v(\tau k)G(\tau k) = \overline{v(k)G(k)}$. So G has to satisfy $G(\tau k) = \overline{G(k)}$ and $v(\pi, k_2)G(\pi, k_2) = v(-\pi, k_2)G(-\pi, k_2) = \overline{v(\pi, \tau k_2)G(\pi, \tau k_2)}$.

For $k_2 \in [0, \pi]$ let us choose $G(\pi, k_2) = \mathbb{I}_N$. Then for $k_2 \in [-\pi, 0]$ the condition

$$v(\pi, k_2)G(\pi, k_2) = \overline{v(\pi, \tau k_2)G(\pi, \tau k_2)} = \overline{v(\pi, \tau k_2)} = v(-\pi, k_2)$$

determines $G(\pi, k_2) = G^+(\pi, k_2) \oplus G^-(\pi, k_2) \in U(m^+) \oplus U(m^-)$ uniquely, since

$$F(\dot{E}^\pm)|_{\{-\pi\}\times S^1} = F(\dot{E}^\pm)|_{\{\pi\}\times S^1}.$$

Let $l^\pm = \boldsymbol{\mathcal{W}}(\det G^\pm(\pi, \cdot))$. There are homotopies $f^\pm : [0, \pi]\times S^1 \to U(m^\pm)$ between $\begin{pmatrix} e^{ik_2 l^\pm} & 0 \\ 0 & \mathbb{I}_{m^\pm} \end{pmatrix}$ and $G^\pm(\pi, \cdot)$. For $k \in [0, \pi] \times S^1$ choose $G(k) = \begin{pmatrix} f^+(k) & 0 \\ 0 & f^-(k) \end{pmatrix}$. Then G satisfies $G(0, \tau k_2) = \overline{G(0, k_2)}$. Extend G to $\dot{\mathbb{T}}$ through $G(\tau k) = \overline{G(k)}$. Then G has all the properties we needed. □

Theorem 3.3.12. *The index for class AI in 2D must be trivial.*

Proof. Let H be in class AI. By Remark 3.3.5 we may assume that H only has eigenvalues ± 1. Let $E = E^+ \oplus E^-$ be the corresponding bundle and let $m = \operatorname{rank}(E^+)$. Let $v = (v^+, v^-)$ be a global section as in Proposition 3.3.11. By continuity, $\boldsymbol{\mathcal{W}}_{\gamma_1}(\det v) = \boldsymbol{\mathcal{W}}_{\gamma_3}(\det v) = l_2$ and $\boldsymbol{\mathcal{W}}_{\gamma_2}(\det v) = \boldsymbol{\mathcal{W}}_{\gamma_4}(\det v) = l_1$. Let

$$w(k) := v(k) \begin{pmatrix} \operatorname{sgn}(\det v(0,0)) e^{-ik_1 l_1} e^{-ik_2 l_2} & 0 \\ 0 & \mathbb{I}_{N-1} \end{pmatrix}.$$

Then w is again a frame as in Proposition 3.3.11. Note that

$$H(k) = w(k) \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_{N-m} \end{pmatrix} w(k)^*.$$

Moreover, the frame w satisfies $0 = \boldsymbol{\mathcal{W}}_{\gamma_i}(\det w) = 2\boldsymbol{\mathcal{W}}_{\gamma_i|_{[0,\pi]}}(\det w)$. Thus at all fixed points of the torus, $\det w$ has the same sign. Since at $k = (0, 0)$ the sign is positive, we have that at all fixed points $w(k^*) \in SO(N)$. Note that w satisfies the hypotheses of Proposition 2.3.3 (i) with f being the complex conjugation, $I = \mathbb{I}_N$, $w = G : \mathbb{T} \to U(N)$. So we obtain $\widehat{w}$ such that $\widehat{w}(k^*) = \mathbb{I}_N$. By the reality condition at the fixed points, the winding number of $\det \widehat{w}$ on any $\gamma_i|_{[0,\pi]}$ is still zero. Thus (ii) of Proposition 2.3.3 is satisfied and we obtain a homotopy $F : \mathbb{T} \times [0, 1] \to U(N)$ between w and $\mathbb{I}_N$ such that $F(\tau k, t) = \overline{F(k, t)}$. Then

$$F(k, t) \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_{N-m} \end{pmatrix} F(k, t)^*$$

is a homotopy between H and $\begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_{N-m} \end{pmatrix}$ which preserves the spectral gap and the symmetry property. □

3.4 BDI

Definition 3.4.1. Let $H = \{H(k) : k \in \mathbb{T}^d\}$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H belongs to symmetry class BDI if there are operators $\Theta, \Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ, Σ are antiunitary,

(ii) $\Theta^2 = \Sigma^2 = 1$,

(iii) for all $k \in \mathbb{T}$,

$$H(\tau k)\Sigma = -\Sigma H(k) \quad \text{and} \quad H(\tau k)\Theta = \Theta H(k),$$

(iv) $\Theta\Sigma = \Sigma\Theta = \Pi$.

Note that Π is unitary, squares to the identity and $H(k)\Pi = -\Pi H(k)$.

Remark 3.4.2. Let H be in symmetry class BDI. By Proposition 3.2.2 we can choose a basis such that $\Pi = \mathbb{I}_n \oplus -\mathbb{I}_n$, hence $N = 2n$. Write $\Theta = UC$ and $\Sigma = SC$ for unitary matrices U and S and the complex conjugation C. Since $\Theta\Pi = \Pi\Theta$, we have

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are $n \times n$-matrices. The blocks A and B are unitary because Θ is antiunitary. Moreover, $\Theta^2 = 1$ implies that A and B are symmetric. By Autonne-Takagi factorisation, there exist unitary matrices Q_1 and Q_2 such that $Q_1 A Q_1^T = \mathbb{I}_n$ and $Q_2 B Q_2^T = \mathbb{I}_n$. Let

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Changing the basis by Q^{-1}, we obtain $\Pi = Q\Pi_{old}Q^{-1} = \mathbb{I}_n \oplus -\mathbb{I}_n$ and $U = QU_{old}Q^T = \mathbb{I}_n \oplus \mathbb{I}_n$. Consequently, $S = U\overline{\Pi} = \mathbb{I}_n \oplus -\mathbb{I}_n$. Moreover, $\{\Pi, H(k)\} = 0$ implies

$$H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}, \tag{3}$$

for some continuous $h : \mathbb{T}^d \to GL(n)$. The condition $\Theta H(k) = H(\tau k)\Theta$ gives

$$h(\tau k) = \overline{h(k)}. \tag{4}$$

Note that for given symmetries Θ, Σ, Π we can choose a basis such that the family H belongs to class BDI if and only if H satisfies Eq. (3) and (4).

We want to classify the continuous families H in dimensions 0,1, and 2 satisfying Eq. (3) and (4). In 0D, the condition in Eq. (4) becomes $h \in GL(n,\mathbb{R})$. Since $GL(n,\mathbb{R})$ has two connected components distinguished by the sign of the determinant, we can define the index as follows:

Definition 3.4.3.

$$\boldsymbol{\mathcal{I}}^0_{BDI}(H) := \operatorname{sgn}(\det h).$$

This explains the $\mathbb{Z}_2$ entry in Table 1 for BDI in 0D. In 1D, an index can be defined through

Definition 3.4.4.

$$\boldsymbol{\mathcal{I}}^1_{BDI}(H) := \boldsymbol{\mathcal{W}}(\det h).$$

Example 3.4.5. Note that

$$h(k) = \begin{pmatrix} e^{ikl} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$$

satisfies Eq. (4) for any $l \in \mathbb{Z}$. The family H given by Eq. (3) then has index $\boldsymbol{\mathcal{I}}^1_{BDI}(H) = l$. Thus, the index $\boldsymbol{\mathcal{I}}^1_{BDI}$ can attain all values in $\mathbb{Z}$.

Remark 3.4.6. Note that there is a relation between the indices $\boldsymbol{\mathcal{I}}^0_{BDI}$ and $\boldsymbol{\mathcal{I}}^1_{BDI}$. The 1D-index is even if and only if the 0D-indices a the fixed points are equal.

Proposition 3.4.7. *The 2D-index for BDI vanishes, explaining the entry 0 in Table 1*

Proof. Suppose H_1 and H_2 lie in the class BDI in 2D with N internal degrees of freedom and that the lower dimensional indices agree, i.e. the 0D-index agrees at all fixed points and the 1D-index agrees on γ_i for $i \in \{1,2,3,4\}$. There is a map $G : \mathbb{T} \to GL(n)$ such that $G(k)h_2(k) = h_1(k)$.

Eq. (4) implies $G(\tau k) = \overline{G(k)}$. Since the 0D-index agrees at all fixed points k^*, we have $\operatorname{sgn}\det G(k^*) = 1$. Eq. (4) also implies $\boldsymbol{\mathcal{W}}|_{[-\pi,0]}(\det h_1|_{\gamma_i}) = \boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_1|_{\gamma_i})$. Thus,

$$2\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_1|_{\gamma_i}) = \boldsymbol{\mathcal{I}}^1_{BDI}(h_1|_{\gamma_i}) = \boldsymbol{\mathcal{I}}^1_{BDI}(h_2|_{\gamma_i}) = 2\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_2|_{\gamma_i}),$$

and we conclude that $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det G|_{\gamma_i}) = 0$.

Thus G satisfies the assumptions of Proposition 2.3.3 with $X = GL(n)$, $I = \mathbb{I}_n$ and f being the complex conjugation. So we obtain a homotopy F_1, which by the reality condition at the fixed points has winding number $\boldsymbol{\mathcal{W}}_{[-\pi,0]}(\det F_1(\cdot,t)|_{\gamma_i}) = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det F_1(\cdot,t)|_{\gamma_i}) = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det G|_{\gamma_i}) = 0$ for every t. Thus $\widehat{G} = F_1(\cdot,1)$ satisfies the assumption of part (ii) of Proposition 2.3.3 and since $\pi_2(GL(n,\mathbb{C})) = 0$ by Proposition 3.2.9, we obtain a homotopy F_2 between G and $\mathbb{I}_n$ satisfying $F_2(\tau k,t) = \overline{F_2(k,t)}$. Then we can define $h(k,t) := F_2(k,t)h_2(k)$, which defines a homotopy between h_1 and h_2 satisfying $h(\tau k,t) = \overline{h(k,t)}$. Thus H_1 can be deformed into H_2 while keeping the symmetry and the spectral gap intact. Hence, the 2D-index for BDI vanishes. □

3.5 D

Definition 3.5.1. Let $H(k)$ for $k \in \mathbb{T}^d$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H has even particle-hole symmetry if there is an operator $\Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Σ is antiunitary,

(ii) $\Sigma^2 = 1$,

(iii) for all $k \in \mathbb{T}$,

$$\Sigma H(k) = -H(\tau k)\Sigma.$$

Let $P^-(k)$ be the Fermi projection and $P^+(k)$ the projection associated to $\sigma(H) \cap (0, \infty)$. Let $E^\pm(k) = \{(k, P^\pm(k)(\mathbb{C}^N))\}$.

Proposition 3.5.2. *The projections satisfy $P^-(\tau k)\Sigma = \Sigma P^+(k)$.*

Proof. Let $\widetilde{H} = \Sigma H \Sigma^{-1}$. Denoting by $H = \sum_\lambda \lambda P_\lambda$ the (unique) spectral decomposition of H, that of $\widetilde{H}$ is $\widetilde{H} = \sum_\lambda \lambda \widetilde{P}_\lambda$, where $\widetilde{P}_\lambda = \Sigma P_\lambda \Sigma^{-1}$, since $\lambda \in \mathbb{R}$ and the $\widetilde{P}_\lambda$ remain orthogonal projections. Thus $\widetilde{P}^+ = \Sigma P^+ \Sigma^{-1}$ for $P^+ = \sum_{\lambda > 0} P_\lambda$. Apply this to $H = H(k)$, $\widetilde{H} = -H(\tau k)$. □

Thus a family of Hamiltonians with even PHS induces a bundle satisfying the following definition.

Definition 3.5.3. A bundle with even particle hole symmetry is a vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$ with an antiunitary map $\Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$,

(ii) $\Sigma^2 = 1$,

(iii) the orthogonal projections $P^\pm(k)$ onto $E^\pm(k)$ satisfy $P^-(\tau k)\Sigma = \Sigma P^+(k)$.

Remark 3.5.4. If we take a bundle with even particle hole symmetry, then $\widehat{H}(k) := 2P^+(k) - 1$ is unitary, self-adjoint and satisfies $\widehat{H}(\tau k)\Sigma = -\Sigma \widehat{H}(k)$, i.e. it has all the properties that the original family of Hamiltonians had.

Remark 3.5.5. Because $\Sigma^2 = 1$ is bijective, Σ defines a bijection between $(E_k)^+$ and $(E_{\tau k})^-$. Thus $\mathrm{rank}(E^+) = \dim E^+(k) = \dim E^-(\tau k) = \mathrm{rank}(E^-)$. So $\mathrm{rank}(E) = N =: 2n$ is even.

Remark 3.5.6. We can write $\Sigma = SC$ for a unitary S and a complex conjugation C. Moreover, we may assume that

$$S = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}.$$

Proof. The condition $\Sigma^2 = 1$ implies that S is symmetric. Thus by Autonne-Takagi factorisation, there is $Q_1 \in U(2n)$ such that $Q_1 S Q_1^T = \mathbb{I}_{2n}$. Let

$$Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_n & i\mathbb{I}_n \\ \mathbb{I}_n & -i\mathbb{I}_n \end{pmatrix}.$$

Changing basis by $Q_2 Q_1$ brings the matrix S into the desired form. □

From now on we will assume that $S = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$. First, let us study the situation in 0D and 1D. Let $E = E^+ \oplus E^-$ be a bundle with even PHS in 0D or 1D. There is a global frame v^+ of

E^+ and we define a frame of E^- by $v^-(k) := \Sigma v^+(\tau k)$. Then $v(k) = (v^+(k), v^-(k))$ is a frame of $E = E^+ \oplus E^-$ satisfying

$$v(\tau k) = \Sigma v(k) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}. \tag{5}$$

Remark 3.5.7. Let v be a unitary frame satisfying Eq. (5). At the fixed points $\tau k^* = k^*$ where v is defined, the number $\det v(k^*) \in \{\pm 1\}$ is independent of the frame. So for a 0D-bundle E in class D with $v \in F(E)$ satisfying Eq. (5), the 0D-index is given through $\boldsymbol{\mathcal{I}}_D^0(E) = \det v$. The 1D-index for a bundle E over S^1 with symmetry D can be defined through $\boldsymbol{\mathcal{I}}_D^1(E) = \det(v(0))\det(v(\pi))$, where v is any frame satisfying Eq. (5). Note that the indices $\boldsymbol{\mathcal{I}}_D^0$ and $\boldsymbol{\mathcal{I}}_D^1$ are related. The 1D-index is trivial if and only if the 0D-indices at the fixed points are equal.

Proof. Suppose that we have two unitary frames v, w as in Eq. (5), let k^* be a fixed point of τ. By assumption $\Sigma = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} C$, so taking the determinant of Eq. (5) gives $\det v(k^*) = \overline{\det v(k^*)}$. Since v is unitary, $\det v(k^*) \in \{\pm 1\}$. The frames have the form $w(k^*) = (w^+(k^*), \Sigma w^+(k^*))$ and $v(k^*) = (v^+(k^*), \Sigma v^+(k^*))$. There is a unitary matrix $U \in U(n, \mathbb{C})$ such that $w^+(k^*) = v^+(k^*)U$. Therefore,

$$w(k^*) = v(k^*) \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix}.$$

Note that $\det \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} = 1$. Thus, $\det w(k^*) = \det v(k^*)$. ☐

Now we move on to the 2D case. Our aim is to define an index for bundles with even PHS over $\mathbb{T} = S^1 \times S^1$. The idea is to look at the cut torus $\dot{\mathbb{T}}$ as in [2]. Let E be a bundle with even PHS. It induces a bundle $\dot{E} = \dot{E}^+ \oplus \dot{E}^-$ on $\dot{\mathbb{T}}$. By Proposition 3.1.9, there is a frame $v^+ : \dot{\mathbb{T}} \to F(\dot{E}^+)$ from which we obtain $v^-(k) := \Sigma v^+(\tau k)$, a frame of $\dot{E}^-$. By setting $v(k) = (v^+(k), v^-(k))$ we obtain a frame $v : \dot{\mathbb{T}} \to F(\dot{E})$ satisfying Eq. (5). Let $v^\pm(\pi, k_2) = v^\pm(-\pi, k_2)T^\pm(k_2)$ and $v(\pi, k_2) = v(-\pi, k_2)T(k_2)$. Note that

$$T(k_2) = \begin{pmatrix} T^+(k_2) & 0 \\ 0 & T^-(k_2) \end{pmatrix}.$$

Proposition 3.5.8.

$$T^-(k_2) = (\overline{T^+(-k_2)})^{-1}$$

Proof. We have $v^+(\pi, k_2) = v^+(-\pi, k_2)T^+(k_2)$. Applying Σ gives $v^-(-\pi, -k_2) = \Sigma v^+(\pi, k_2) = \Sigma(v^+(-\pi, k_2)T^+(k_2)) = (\Sigma v^+(-\pi, k_2))\overline{T^+(k_2)} = v^-(\pi, -k_2)\overline{T^+(k_2)}$. Thus,

$$v^-(\pi, -k_2) = v^-(-\pi, -k_2)(\overline{T^+(k_2)})^{-1} = v^-(-\pi, -k_2)T^-(-k_2).$$

Hence, $T^-(-k_2) = (\overline{T^+(k_2)})^{-1}$. ☐

Remark 3.5.9. We may always assume the frames to be unitary. For unitary frames, Proposition 3.5.8 simplifies to $T^-(k_2) = T^+(-k_2)^T$.

Definition 3.5.10. Define the index of E as $\boldsymbol{\mathcal{I}}_D^2(E) := \boldsymbol{\mathcal{W}}(\det(T^-))$.

Remark 3.5.11. Note that for the index $\boldsymbol{\mathcal{I}}_A^2$ for symmetry A we have

$$\boldsymbol{\mathcal{I}}_D^2(E) = \boldsymbol{\mathcal{I}}_A^2(E).$$

In particular, $\boldsymbol{\mathcal{I}}_D^2$ is well defined, i.e. independent of the frame v^+.

Lemma 3.5.12. *There is a unitary frame $v : \dot{\mathbb{T}} \to F(\dot{E})$ satisfying*

$$v(\tau k) = \Sigma v(k) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix},$$

and $v(-\pi, k_2^*) = v(\pi, k_2^*)$ for $k_2^* \in \{0, \pi\}$.

Proof. Let $v : \dot{\mathbb{T}} \to F(\dot{E})$ be a unitary frame with

$$v(\tau k) = \Sigma v(k) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} v(-\pi, 0) &= (v^+(-\pi, 0), v^-(-\pi, 0)), \\ v(-\pi, \pi) &= (v^+(-\pi, \pi), v^-(-\pi, \pi)). \end{aligned}$$

Note that $v^-(-\pi, 0)$ and $\Sigma v^+(-\pi, 0)$ both lie in $F(E^-_{(-\pi,0)})$. So there is a $G(0) \in GL(n)$ such that $v^-(-\pi, 0)G(0) = \Sigma v^+(-\pi, 0)$. Analogously, there is $G(-\pi) \in GL(n)$ such that $v^-(-\pi, -\pi)G(-\pi) = \Sigma v^+(-\pi, -\pi)$. Since $GL(n)$ is path connected, we can find a path $\gamma : [-\pi, 0] \to GL(n)$ connecting $G(-\pi)$ and $G(0)$. Let $\widetilde{\gamma} : S^1 \to GL(n)$ be an extension of γ defined through $\widetilde{\gamma}(k) = \gamma(-k)$ for $k \in [0, \pi]$. Then $\det \widetilde{\gamma}$ has winding number zero. Thus, there is a homotopy $\widetilde{G} : [-\pi, 0] \times S^1 \to GL(n)$ such that for all $k_2 \in S^1$ we have $\widetilde{G}(-\pi, k_2) = \widetilde{\gamma}(k_2)$ and $\widetilde{G}(0, k_2) = \mathbb{I}_n$. For $k = (k_1, k_2) \in [-\pi, 0] \times S^1$ now set

$$w(k) := v(k) \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{G}(k_1, k_2) \end{pmatrix}$$

Then

$$\begin{aligned} w(0, k_2) &= v(0, k_2), \\ w(-\pi, 0) &= (v^+(-\pi, 0), \Sigma v^+(-\pi, 0)) \text{ and} \\ w(-\pi, \pi) &= (v^+(-\pi, \pi), \Sigma v^+(-\pi, \pi)). \end{aligned}$$

Now extend w to $\dot{\mathbb{T}}$ through

$$w(k_1, k_2) := \Sigma w(\tau k) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$$

for $k_1 \in [0, \pi]$. This is well defined, because $w(0, k_2) = v(0, k_2) = \Sigma v(0, -k_2) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$. Moreover,

$$\begin{aligned} w(\pi,0) &= (w^+(-\pi,0), \Sigma w^+(-\pi,0)) = w(-\pi,0) \quad \text{and} \\ w(\pi,\pi) &= (w^+(-\pi,\pi), \Sigma w^+(-\pi,\pi)) = w(-\pi,\pi). \end{aligned}$$

So w is a frame with all the properties we wanted. □

Given a bundle E with even PHS over the 2-torus $\mathbb{T}$, one can obtain bundles with even PHS over S^1 by restricting the base space. Let $\gamma_1 := \{0\} \times S^1, \gamma_2 := S^1 \times \{\pi\}, \gamma_3 := \{\pi\} \times S^1, \gamma_4 := S^1 \times \{0\}$. We call γ_i and γ_{i+2} parallel for $i = 1, 2$. For all i, the bundle E_{γ_i} has even PHS over S^1 and we can thus look at $\mathcal{I}_D^1(E_{\gamma_i})$. Note that $\prod_{i=1}^4 \mathcal{I}_D^1(E_{\gamma_i}) = 1$, because every fixed point $\tau k^* = k^*$ is counted twice. Thus up to cyclic permutation, we have the following four possibilities for $(\mathcal{I}_D^1(E_{\gamma_i}))_{i=1}^4$: $(1,1,1,1)$, $(-1,-1,-1,-1)$, $(1,-1,1,-1)$ and $(1,1,-1,-1)$. In the first three cases, the 1D-index agrees on parallel γ_i. In the last case, the 1D-index is different on parallel γ_i.

Theorem 3.5.13. *Let E be a bundle over $\mathbb{T}$ with even PHS. If the 1D-index $\mathcal{I}_D^1(E)$ agrees on parallel γ_i then the 2D-index $\mathcal{I}_D^2(E)$ is even. If the 1D-index is different for parallel γ_i then the 2D-index is odd.*

Proof. Let v be a unitary frame as in Lemma 3.5.12. On $[-\pi, 0] \times S^1$ define

$$w(k_1,k_2) := (v^+(k_1,k_2), v_1^-(k_1,k_2), ..., v_{n-1}^-(k_1,k_2), v_n^-(k_1,k_2) \det v(k_1,k_2)^{-1} \det v(0,k_2)).$$

Then $\det w(k_1,k_2) = \det w(0,k_2)$ and $w(0,k_2) = v(0,k_2)$ for all $k_1 \in [-\pi,0]$ and $k_2 \in S^1$. Note that for $k^* \in \{0,\pi\}$ we have

$$w(-\pi,k^*) = v(-\pi,k^*) \begin{pmatrix} \mathbb{I}_{2n-1} & 0 \\ 0 & \mathcal{I}_D^1(E|_{S^1\times\{k^*\}}) \end{pmatrix}.$$

Extend w to $\dot{\mathbb{T}}$ through

$$w(k_1,k_2) := \Sigma w(\tau k) \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$$

for $(k_1,k_2) \in [0,\pi] \times S^1$. Note that w is well defined at $k_1 = 0$ and that $\det w(k_1,k_2) = \det w(0,k_2)$ holds also for $k_1 \in [0,\pi]$. By Proposition 3.5.8, the transition matrix $T(k_2)$ with $w(-\pi,k_2)T(k_2) = w(\pi,k_2)$ is of the form

$$T(k_2) = \begin{pmatrix} (\overline{T^-(-k_2)})^{-1} & 0 \\ 0 & T^-(k_2) \end{pmatrix}.$$

Since w is unitary, $\det T^-(k_2) = e^{i\alpha(k_2)}$ for some continuous $\alpha : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Moreover, for all $k_2 \in S^1$ we have $\det T(k_2) = 1$ because $\det w(-\pi,k_2) = \det w(0,k_2) = \det w(\pi,k_2)$. This implies that $1 = \det T^-(k_2) \det(\overline{T^-(-k_2)})^{-1} = e^{i\alpha(k_2)} e^{i\alpha(-k_2)}$, i.e. $\alpha(k_2) \equiv -\alpha(-k_2) \mod 2\pi\mathbb{Z}$. Hence $\det T^-$ winds by the same amount from $-\pi$ to 0 as from 0 to π, i.e. $\mathcal{W}_{[-\pi,0]}(\det T^-) =$

$\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$. For $k^* \in \{0,\pi\}$ we have

$$\begin{aligned}
w(\pi,k^*) &= \Sigma w(-\pi,k^*)\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix} = \Sigma v(-\pi,k^*)\begin{pmatrix}\mathbb{I}_{2n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\\
&= v(\pi,k^*)\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\begin{pmatrix}\mathbb{I}_{2n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\\
&= v(-\pi,k^*)\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\begin{pmatrix}\mathbb{I}_{2n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\\
&= w(-\pi,k^*)\begin{pmatrix}\mathbb{I}_{2n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix}\begin{pmatrix}\mathbb{I}_{2n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_n & 0\end{pmatrix},
\end{aligned}$$

where we used that $v(\pi,k^*) = v(-\pi,k^*)$ by assumption. Thus,

$$T^-(k^*) = \begin{pmatrix}\mathbb{I}_{n-1} & 0\\ 0 & \boldsymbol{\mathcal{I}}_D^1(E|_{S^1\times\{k^*\}})\end{pmatrix}.$$

In particular, if the 1D-index agrees on parallel γ_i, then $\det T^-(0) = \det T^-(\pi)$. Then, the winding number $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$ is an integer and thus $\boldsymbol{\mathcal{I}}_D^2(E) = 2\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$ is even. If the 1D-index differs on parallel γ_i, then $\det T^-(0) = -\det T^-(\pi)$ and $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-) \equiv \frac{1}{2} \mod \mathbb{Z}$. Hence, $\boldsymbol{\mathcal{I}}_D^2(E)$ is odd in this case. □

Remark 3.5.14. Theorem 3.5.13 describes the relation of the $\mathbb{Z}_2$-index in 1D and the $\mathbb{Z}$-index in 2D for symmetry D. In [12] this relation is viewed from a different perspective. For any symmetry, any $\mathbb{Z}_2$-index in dimension d can be defined from the $\mathbb{Z}$- or the $\mathbb{Z}_2$-index $\boldsymbol{\mathcal{I}}^{d+1}$ in dimension $d+1$ through so-called dimensional reduction. The idea is as follows: Given d-dimensional vector bundles E_1, E_2 in a certain symmetry class, it is possible to construct a $(d+1)$-dimensional bundle $\widetilde{E}(E_1,E_2)$ in the same symmetry class, which agrees with E_1 over $\{0\}\times\mathbb{T}^d$ and is equal to E_2 over $\{\pi\}\times\mathbb{T}^d$. The resulting $(d+1)$-dimensional bundle $\widetilde{E}$ is not unique, but the parity of $\boldsymbol{\mathcal{I}}^{d+1}(\widetilde{E})$ is. Let E_0 denote a trivial d-dimensional vector bundle in the symmetry class, e.g. a vector bundle corresponding to constant Fermi projections. One can define $\boldsymbol{\mathcal{I}}^d(E_1) := \boldsymbol{\mathcal{I}}^{d+1}(\widetilde{E}(E_1,E_0)) \mod 2$.

In this context, Theorem 3.5.13 shows that $\boldsymbol{\mathcal{I}}_D^1$ is precisely the index one can obtain from $\boldsymbol{\mathcal{I}}_D^2$ through dimensional reduction. Remarks 3.5.7 and 3.4.6 provide examples for this relationship between 0D- and 1D-indices.

3.6 DIII

Definition 3.6.1. Let $H = \{H(k) : k \in \mathbb{T}^d\}$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H belongs to symmetry class DIII if there are operators $\Theta, \Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ, Σ are antiunitary,

(ii) $\Theta^2 = -1$ and $\Sigma^2 = 1$,

(iii) for all $k \in \mathbb{T}$,

$$H(\tau k)\Sigma = -\Sigma H(k) \quad \text{and} \quad H(\tau k)\Theta = \Theta H(k),$$

(iv) $\Theta\Sigma = \Sigma\Theta = \Pi$.

Note that Π is unitary, $\Pi^2 = -1$ and $H(k)\Pi = -\Pi H(k)$.

Remark 3.6.2. Let H be in symmetry class DIII. Let P denote the Fermi projection. Then $\widehat{H}(k) := 2P(k) - 1$ is self-adjoint, unitary and squares to 1 and belongs to the class DIII. Moreover, any self-adjoint, unitary matrix K with $K^2 = 1$ can be written as $K = 2P - 1$ for some projection P.

Remark 3.6.3. Let H be in class DIII. Then H is equivalent to the corresponding $\widehat{H}$.

Proof. Note that the homotopy F constructed in the proof of Remark 3.1.3 satisfies $F(\tau k, t)\Pi = -\Pi F(k, t)$. □

Definition 3.6.4. A bundle in class DIII is a vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$ with antilinear maps $\Sigma, \Theta : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$,

(ii) $\Sigma^2 = 1$ and $\Theta^2 = -1$ and $\Pi := \Sigma\Theta = \Theta\Sigma$,

(iii) the orthogonal projections $P^\pm(k)$ onto $E^\pm(k)$ satisfy $P^-(\tau k)\Sigma = \Sigma P^+(k)$ and $P^+(\tau k)\Theta = \Theta P^+(k)$.

Note that Π is unitary, $\Pi^2 = -1$ and $P^-(k)\Pi = \Pi P^+(k)$.

Remark 3.6.5. Let H be in class DIII. Let $P^-(k)$ be the Fermi projection and $P^+(k)$ the projection associated to $\sigma(H) \cap (0, \infty)$. Let $E^\pm(k) = \{(k, P^\pm(k)(\mathbb{C}^N))\}$. Then we obtain a bundle in class DIII. Conversely, from a bundle in class DIII, we can define $H(k) := 2P^+(k) - 1$ which is a continuous family of self-adjoint operators in class DIII.

Remark 3.6.6. Let H be in symmetry class DIII. By Proposition 3.2.2 we can choose a basis such that $\Pi = i\mathbb{I}_n \oplus -i\mathbb{I}_n$, hence $N = 2n$. Since $\Theta\Pi = \Pi\Theta$, if $\Pi v = iv$, then $\Pi\Theta v = \Theta\Pi v = \Theta iv = -i\Theta v$. Thus from a basis $B = (b_1, ..., b_n)$ of the eigenspace $E_{\Pi,i}$, we obtain a basis $(-\Theta b_1, ..., -\Theta b_n)$ of $E_{\Pi,-i}$. In this basis, we have $\Theta = UC$ for the complex conjugation C and

$$U = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

Moreover, $\{\Pi, \widehat{H}(k)\} = 0$ implies

$$\widehat{H}(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}, \tag{6}$$

for some continuous $h : \mathbb{T}^d \to U(n)$. The condition $\Theta\widehat{H}(k) = \widehat{H}(\tau k)\Theta$ gives

$$h(\tau k) = -h(k)^T. \tag{7}$$

Note that for given symmetries Θ, Σ, Π we can choose a basis such that the family H belongs to class DIII if and only if $\widehat{H}$ satisfies Eq. (6) and (7).

We want to classify the continuous families H satisfying Eq. (6) and (7) in dimensions 0, 1, and 2. In 0D, the condition in Eq. (7) is equivalent to h being antisymmetric. Since h has to be invertible, it must have even rank $n = 2m$. First, we need the following result.

Proposition 3.6.7. *The set of antisymmetric matrices in $U(2m)$ is path-connected.*

Proof. Let ϵ be the matrix with m blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal. Let A_1, A_2 be two antisymmetric matrices in $U(2m)$. Then there exist $Q_1, Q_2 \in U(2m)$ such that $A_i = Q_i \epsilon Q_i^T$ for $i = 1, 2$ [15]. Since $U(2m)$ is path-connected we can also choose a path $Q(t)$ from Q_1 to Q_2 in $U(2m)$. Then $A(t) = Q(t)\epsilon Q(t)^T$ is a path form A_1 to A_2 in the set of antisymmetric matrices in $U(2m)$. $\square$

Thus, the 0D-index for DIII vanishes. In 1D we want to define an index for bundles in class DIII.

Proposition 3.6.8. *Let $E = E^+ \oplus E^-$ be a bundle in class DIII in 1D. Note that* $\operatorname{rank}(E) = 4m$. *Then, there exists a unitary frame $v^+ : S^1 \to F(E^+)$ satisfying*

$$\Theta v^+(k) \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} = v^+(\tau k). \tag{8}$$

Proof. First, we show that we can pick such a frame at the fixed points $k^* \in \{0, \pi\}$. Let $v \in F(E^+_{k^*})$. Then, also $\Theta v \in F(E^+_{k^*})$ and the vectors v and Θv are orthogonal because $\Theta^2 = -1$. Therefore, we can inductively choose unitary frames of $E^+_{k^*}$ of the form

$$(v_1(k^*), ..., v_m(k^*), \Theta v_1(k^*), ..., \Theta v_m(k^*)).$$

Now we can interpolate with a section $v^+ : [0, \pi] \to F(E^+)$ between $v^+(0)$ and $v^+(\pi)$. Then define v^+ on $[-\pi, 0]$ by Eq. (8). For details see the proof of Proposition 3.3.8. $\square$

Let $E = E^+ \oplus E^-$ be a bundle in class DIII. We may pick a basis such that $\Pi = i\mathbb{I}_{2m} \oplus -i\mathbb{I}_{2m}$. Then $H(k) := 2P^+(k) - 1$ has the form $H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}$ for $h(k) : \mathbb{T}^d \to U(2m)$. Thus, any (local) frame v^+ of E^+ is of the form

$$v^+(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} w(k) \\ h(k)w(k) \end{pmatrix},$$

for some continuous $w(k) \in U(2m)$.

In the case $d = 1$, pick a global frame as in Proposition 3.6.8. The corresponding $w : S^1 \to U(2m)$ then satisfies

$$h(\tau k)w(\tau k) = -\overline{w(k)} \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix}. \tag{9}$$

We define the 1D-index as follows.

Definition 3.6.9. Let $E = E^+ \oplus E^-$ be a 1D-bundle in class DIII. Pick a basis B such that $\Pi = i\mathbb{I}_{2m} \oplus -i\mathbb{I}_{2m}$ and choose a global frame v^+ of E^+ as in Proposition 3.6.8. We then have $v^+(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} w(k) \\ \widehat{w}(k)\end{pmatrix}$, where $w(k) : S^1 \to U(2m)$. Define the index as

$$\boldsymbol{\mathcal{I}}^1_{DIII}(E) := \boldsymbol{\mathcal{W}}(\det w) \mod 2.$$

Proposition 3.6.10. *The index $\boldsymbol{\mathcal{I}}^1_{DIII}$ is well defined, i.e. independent of the choice of B and v^+.*

Proof. Let $E = E^+ \oplus E^-$ be a 1D bundle in class DIII. A different choice of the basis B amounts to a change of basis by $M = \begin{pmatrix} A & 0 \\ 0 & \widetilde{A}\end{pmatrix} \in U(2m) \oplus U(2m)$, because Π has to stay the same. In the new basis, then $w(k) = A^{-1}w_{old}(k)A$, thus leaving $\boldsymbol{\mathcal{W}}(\det w)$ unchanged. The index is also independent of the choice of v^+: Suppose $v^+(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} w(k) \\ \widehat{w}(k)\end{pmatrix}$ and $\widetilde{v}^+(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} z(k) \\ \widehat{z}(k)\end{pmatrix}$ are two unitary frames of E^+ as in Proposition 3.6.8. Then there is $G : S^1 \to U(2m)$ such that $w(k)G(k) = z(k)$. Then by Eq. (9), we have

$$\begin{aligned} -\overline{w(k)}\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0\end{pmatrix} G(\tau k) &= h(\tau k)w(\tau k)G(\tau k) = h(\tau k)z(\tau k) \\ &= -\overline{z(k)}\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0\end{pmatrix} = -\overline{w(k)G(k)}\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0\end{pmatrix}. \end{aligned}$$

Thus

$$\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0\end{pmatrix} G(\tau k) = \overline{G(k)}\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0\end{pmatrix}.$$

At the fixed points G is symplectic, so $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det G) \in \mathbb{Z}$. Moreover, $\det G(\tau k) = \det \overline{G(k)}$. Thus, $\boldsymbol{\mathcal{W}}(\det G) = 2\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det G) \in 2\mathbb{Z}$ and $\boldsymbol{\mathcal{W}}(\det z) = \boldsymbol{\mathcal{W}}(\det w) + \boldsymbol{\mathcal{W}}(\det G) \equiv \boldsymbol{\mathcal{W}}(\det w) \mod 2$. Therefore, the index is well defined. □

In [12] a 1D-index for DIII is defined directly for the continuous family H. It can be calculated as follows. Let $h : S^1 \to U(2m)$ be a block corresponding to H. The symmetry condition in Eq. (7) implies that h is antisymmetric at the fixed points, so the Pfaffian $\mathrm{Pf}(h(k^*))$ is well-defined. Choose a continuous path $\alpha : [0, \pi] \to \mathbb{C} \setminus \{0\}$ such that $\alpha(k)^2 = \det(h(k))$. Then, the index is given by $(-1)^{\widehat{\boldsymbol{\mathcal{I}}}(H)} = \frac{\mathrm{Pf}(h(\pi))}{\alpha(\pi)}\frac{\alpha(0)}{\mathrm{Pf}(h(0))}$.

Remark 3.6.11. Let H be a family in DIII and let $h : S^1 \to U(2m)$ be a corresponding block. Then H is equivalent to a $\widetilde{H}$ which admits a block $\widetilde{h} : S^1 \to U(2m)$ with $\det \widetilde{h}(k) = 1$.

Proof. Let $\alpha : [0, \pi] \to \mathbb{C} \setminus \{0\}$ be continuous and such that $\alpha(k)^{2m} = \det(h(k))^{-1}$. Let $f(k, t)$ be a homotopy between the constant map 1 and $\alpha(k)$. Then

$$F(k, t) = \begin{cases} h(k)f(k, t) & \text{for } k \in [0, \pi] \\ h(k)f(\tau k, t) & \text{for } k \in [\pi, 2\pi] \end{cases}$$

defines a homotopy respecting the symmetry condition in Eq. (7) and $\widetilde{h} := F(\cdot, 1)$ has the desired properties. $\square$

Proposition 3.6.12. *The indices $\mathcal{I}^1_{DIII}$ and $\widehat{\mathcal{I}}$ agree.*

Proof. Let H be a family in DIII and let $h : S^1 \to U(2m)$ be a corresponding block. By Remark 3.6.11 it suffices to show that the two indices agree if $\det h(k) = 1$. In this case we have $(-1)^{\widehat{\mathcal{I}}(H)} = \frac{\mathrm{Pf}(h(\pi))}{\mathrm{Pf}(h(0))}$, which indeed takes values in $\{\pm 1\}$ since $\mathrm{Pf}(h(k))^2 = \det h(k) = 1$. On the other hand, Eq. (9) implies $\det w(\tau k) = \overline{\det w(k)}$. So $\mathcal{W}(\det w) = 2\mathcal{W}_{[0,\pi]}(\det w)$, i.e. $\mathcal{W}(\det w)$ is even or odd iff $\frac{\det w(\pi)}{\det w(0)} = 1$ or -1 respectively. This means that $(-1)^{\mathcal{W}(\det w)} = \frac{\det w(\pi)}{\det w(0)}$. Moreover, Eq. (9) also implies

$$h(\tau k) = -\overline{w(k)} \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} \overline{w(\tau k)}^T.$$

Using that $\mathrm{Pf}(BAB^T) = \det(B)\mathrm{Pf}(A)$ we obtain $\mathrm{Pf}(h(\tau k)) = \overline{\det(w(k))}\mathrm{Pf}\begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix}$. Thus, $\frac{\det w(\pi)}{\det w(0)} = \frac{\mathrm{Pf}(h(\pi))}{\mathrm{Pf}(h(0))}$, and we have $(-1)^{\mathcal{I}^1_{DIII}(H)} = (-1)^{\mathcal{W}(\det w)} = \frac{\mathrm{Pf}(h(\pi))}{\mathrm{Pf}(h(0))} = (-1)^{\widehat{\mathcal{I}}(H)}$. Therefore, the two $\mathbb{Z}_2$-indices agree. $\square$

Now we consider the situation in 2D for DIII.

Definition 3.6.13. Let $E = E^+ \oplus E^-$ be a 2D-bundle in class DIII. In particular, E has time reversal symmetry. So we can define

$$\mathcal{I}^2_{DIII}(E) := \mathcal{I}^2_{AII}(E),$$

where $\mathcal{I}^2_{AII}$ is defined in Section 3.7.

Proposition 3.6.14. *Let $E = E^+ \oplus E^-$ be a 2D-bundle in class DIII. There is a unitary frame $v : \dot{\mathbb{T}} \to F(\dot{E}^+)$ satisfying*

$$\theta v(k) \begin{pmatrix} 0 & -\mathbb{I}_m \\ \mathbb{I}_m & 0 \end{pmatrix} = v(\tau k)$$

and $v(-\pi, k^*) = v(\pi, k^*)$ for $k^* \in \{0, \pi\}$.

Proof. Let $\mathrm{rank}(E) = 4m$. Pick sections $v : \{0\} \times S^1 \to F(\dot{E}^+)$ and $w : \{\pi\} \times S^1 \to F(\dot{E}^+)$ as in Proposition 3.6.8. We may extend the v to a section $v : [0,\pi] \times S^1 \to F(\dot{E}^+)$. Then for $k_2^* \in \{0, \pi\}$ there are $G(k_2^*) \in U(2m)$ such that $v(\pi, k_2^*)G(k_2^*) = w(\pi, k_2^*)$. Let $G : [0, \pi] \to U(2m)$ be a path connecting $G(0)$ and $G(\pi)$ and for $k \in [-\pi, 0]$ let $G(k) = G(-k)$. Then $\mathcal{W}(\det G) = 0$ and thus there is a homotopy $\widetilde{G} : [0, \pi] \times S^1 \to U(2m)$ between $\mathbb{I}_{2m}$ and G. Now for $k \in [0, \pi] \times S^1$ set $\widetilde{v}(k) = v(k)G(k)$. Then $\widetilde{v}(\pi, k_2^*) = w(\pi, k_2^*)$ and $\widetilde{v}(0, \cdot) = v(0, \cdot)$. Now extend $\widetilde{v}$ to $\dot{\mathbb{T}}$ through $\widetilde{v}(\tau k) = \theta \widetilde{v}(k) \begin{pmatrix} 0 & -\mathbb{I}_m \\ \mathbb{I}_m & 0 \end{pmatrix}$. Then $\widetilde{v}(-\pi, k_2^*) = \widetilde{v}(\pi, k_2^*)$, so $\widetilde{v}$ is a frame of the form we wanted. $\square$

Theorem 3.6.15. *Let $E = E^+ \oplus E^-$ be a 2D-bundle in class DIII. Then*

$$\mathcal{I}^2_{DIII}(E) = \mathcal{I}^1_{DIII}(E|_{\gamma_2}) + \mathcal{I}^1_{DIII}(E|_{\gamma_4}) \mod 2.$$

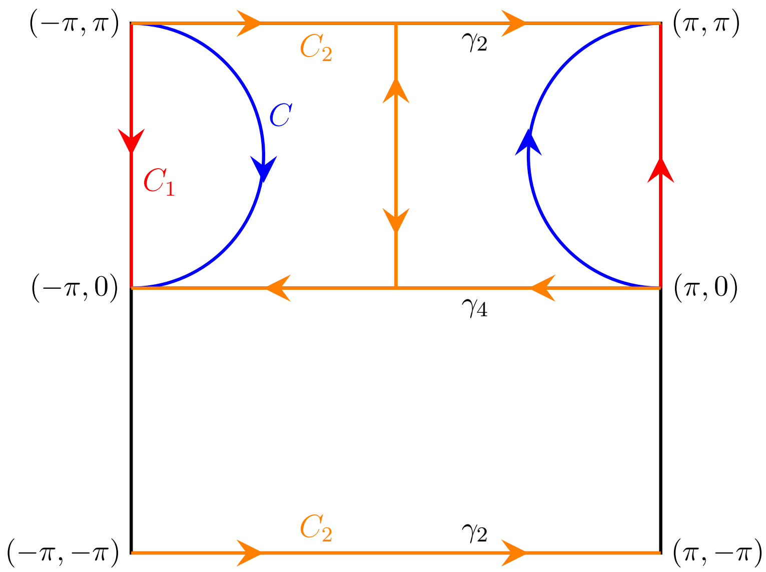


Figure 1: The curves C, C_1 and C_2 on the cut torus $\dot{\mathbb{T}} = [-\pi,\pi] \times S^1$ with $(-\pi,k^*)$ and (π,k^*) identified for $k^* \in \{0,\pi\}$.

So the 2D-index is trivial iff the 1D-indices on parallel γ_i agree.

Proof. Let $\mathrm{rank}(E) = 4m$ and let v be a frame of E^+ as in Proposition 3.6.14. Let $T : S^1 \to U(2m)$ be given by $v(-\pi,k_2)T(k_2) = v(\pi,k_2)$. Then $T(0) = T(\pi) = \mathbb{I}_{2m}$ and by Remark 3.7.6, the 2D-index is $\mathcal{I}^2_{DIII}(E) = \mathcal{I}^2_{AII}(E) = \mathcal{W}_{[0,\pi]}(\det T) \mod 2$. We can write $v(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} w(k) \\ \widehat{w}(k)\end{pmatrix}$, for $w(k) \in U(2m)$. Note that $w(-\pi,k_2)T(k_2) = w(\pi,k_2)$ and thus $\det T(k_2) = \det w(\pi,k_2) \det w(-\pi,k_2)^{-1}$. Hence,

$$\mathcal{I}^2_{DIII}(E) = \mathcal{W}_{[0,\pi]}(\det(w(\pi,\cdot))) - \mathcal{W}_{[0,\pi]}(\det(w(-\pi,\cdot))) \mod 2.$$

We can view w as being defined on the cut torus $\dot{\mathbb{T}}$ with the points $(-\pi,k^*)$ and (π,k^*) identified for $k^* \in \{0,\pi\}$.

Consider the curve C as in Figure 1. Let W_C be the winding number of $\det w$ along C. By shrinking the curve C to the curve C_1 depicted in Figure 1, we see that

$$W_C = \mathcal{W}_{[0,\pi]}(\det(w(\pi,\cdot))) - \mathcal{W}_{[0,\pi]}(\det(w(-\pi,\cdot))).$$

In particular, $I^2_{DIII}(E) \equiv W_C \mod 2$. By enlarging the curve C to C_2 as in Figure 1, we note that

$$W_C = \mathcal{W}(\det w|_{\gamma_2}) - \mathcal{W}(\det w|_{\gamma_4}).$$

Hence, $W_C \equiv \mathcal{I}^1_{DIII}(E|_{\gamma_2}) - \mathcal{I}^1_{DIII}(E|_{\gamma_4}) \mod 2$. So we have

$$\mathcal{I}^2_{DIII}(E) \equiv W_C \mod 2 \equiv \mathcal{I}^1_{DIII}(E|_{\gamma_2}) + \mathcal{I}^1_{DIII}(E|_{\gamma_4}) \mod 2,$$

which proves the Theorem. □

3.7 AII

This symmetry class is treated in [2]. Here we only mention the definitions and properties relevant to symmetry DIII.

Definition 3.7.1 (Cf. Definition 2.4. in [2])**.** Let $H(k)$ for $k \in \mathbb{T}^d$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $\mu \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H has odd time-reversal symmetry if there is an operator $\Theta : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ is antiunitary,

(ii) $\Theta^2 = -1$,

(iii) for all $k \in \mathbb{T}$,
$$\Theta H(k) = H(\tau k)\Theta.$$

Definition 3.7.2 (Cf. Section 4.3. in [2])**.** A bundle with odd time-reversal symmetry is a vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$ with an antilinear map $\Theta : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$

(ii) $\Theta^2 = -1$

(iii) the orthogonal projections $P^\pm(k)$ onto $E^\pm(k)$ satisfy $P^\pm(\tau k)\Theta = \Theta P^\pm(k)$

Lemma 3.7.3 (Lemma 4.5 in [2])**.** *Let E be a bundle in class AII in 2D. Let ϵ be the matrix with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal. There are unitary frames $v^\pm : \dot{\mathbb{T}} \to F(E^\pm)$ satisfying*
$$v^\pm(\tau k) = \Theta v^\pm(k)\epsilon.$$

Let $n = \operatorname{rank}(E^+)$ and let $T : S^1 \to U(n)$ be the transition matrix determined by $v^+(\pi, k_2) = v^+(-\pi, k_2)T(k_2)$. The $\mathbb{Z}_2$-index $\boldsymbol{\mathcal{I}}$ defined in [2] for bundles in class AII only depends on the winding of the eigenvalues of T, see Eq. (16) and (25) in [2].

Remark 3.7.4. By reordering the basis vectors we can obtain unitary frames $v^\pm : \dot{\mathbb{T}} \to F(\dot{E}^\pm)$ satisfying
$$v^\pm(\tau k) = \Theta v^\pm(k) \begin{pmatrix} 0 & \mathbb{I}_{m^\pm} \\ -\mathbb{I}_{m^\pm} & 0 \end{pmatrix},$$
where $2m^\pm = \operatorname{rank}(E^\pm)$. This leads to a slightly different transition function $T_D = PTP^{-1}$, where P is the permutation matrix with columns $(e_1, e_{m^++1}, e_2, e_{m^++2}, ..., e_{m^+}, e_{2m^+})$, where e_i denotes the i-th standard basis vector of $\mathbb{C}^{2m^+}$. Since T_D and T are conjugate, they have the same eigenvalues and thus $\boldsymbol{\mathcal{I}}(T_D) = \boldsymbol{\mathcal{I}}(T) = \boldsymbol{\mathcal{I}}(E)$.

Definition 3.7.5. Let $(-1)^{\boldsymbol{\mathcal{I}}^2_{AII}(E)} := \boldsymbol{\mathcal{I}}(E)$.

Remark 3.7.6. From the definitions in [2] it follows that if $T(0) = T(\pi)$, then
$$\boldsymbol{\mathcal{I}}^2_{AII}(E) = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T) \mod 2 = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T_D) \mod 2.$$

3.8 CII

Definition 3.8.1. Let $H = \{H(k) : k \in \mathbb{T}^d\}$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H belongs to symmetry class CII if there are operators $\Theta, \Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ, Σ are antiunitary,

(ii) $\Theta^2 = \Sigma^2 = -1$,

(iii) for all $k \in \mathbb{T}$,
$$H(\tau k)\Sigma = -\Sigma H(k) \quad \text{and} \quad H(\tau k)\Theta = \Theta H(k),$$

(iv) $\Theta\Sigma = \Sigma\Theta = \Pi$.

Note that Π is unitary, squares to the identity and $H(k)\Pi = -\Pi H(k)$.

Remark 3.8.2. Let H be in symmetry class CII. Let P denote the Fermi projection. Then $\widehat{H}(k) := 2P(k) - 1$ is self-adjoint, unitary and squares to 1 and belongs to the class CII. Moreover, any self-adjoint, unitary matrix K with $K^2 = 1$ can be written as $K = 2P - 1$ for some projection P.

Remark 3.8.3. Let H be in class CII. Then H is equivalent to the corresponding $\widehat{H}$.

Proof. Note that the homotopy F constructed in the proof of Remark 3.1.3 satisfies $F(\tau k, t)\Sigma = -\Sigma F(k, t)$ and $F(\tau k, t)\Theta = \Theta F(k, t)$. □

Remark 3.8.4. Let H be in symmetry class CII, let $\widehat{H}$ be as in Remark 3.8.2. By Proposition 3.2.2 we can choose a basis such that $\Pi = \mathbb{I}_n \oplus -\mathbb{I}_n$, hence $N = 2n$. Since $\Sigma\Pi = \Pi\Sigma$, if $\Pi v = v$, then also $\Pi\Sigma v = \Sigma\Pi v = \Sigma v$. Because $\Sigma^2 = -1$, the vectors v and Σv are orthogonal. Thus one can choose a orthonormal basis of the eigenspace $E_{\Pi,1}$ of the form $(v_1, ..., v_m, \Sigma v_1, ..., \Sigma v_m)$ and similarly for $E_{\Pi,-1}$ we can choose a basis $(w_1, ..., w_m, -\Sigma w_1, ..., -\Sigma w_m)$. Thus $N = 4m$ and in this basis, we have $\Sigma = SC$ for the complex conjugation C and

$$S = \begin{pmatrix} 0 & -\mathbb{I}_m & 0 & 0 \\ \mathbb{I}_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_m \\ 0 & 0 & -\mathbb{I}_m & 0 \end{pmatrix}.$$

Moreover, $\{\Pi, \widehat{H}(k)\} = 0$ and $\widehat{H}(k)^2 = 1$ implies

$$\widehat{H}(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}, \tag{10}$$

for some continuous $h : \mathbb{T}^d \to U(2m)$. The condition $\Sigma\widehat{H}(k) = -\widehat{H}(\tau k)\Sigma$ gives

$$h(\tau k) \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} \overline{h(k)}. \tag{11}$$

Note that for given symmetries Θ, Σ, Π we can choose a basis such that the family H belongs to class CII if and only if $\widehat{H}$ satisfies Eq. (10) and (11) in some basis.

We want to classify the continuous families H satisfying Eq. (10) and (11) in dimensions 0, 1, and 2. In 0D, the condition in Eq. (11) is equivalent to h^T being symplectic, i.e.

$$h \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} h^T = \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix}.$$

Since the symplectic group is connected [5], the 0D-index for CII vanishes. In 1D, an index can be defined through

Definition 3.8.5.

$$\boldsymbol{\mathcal{I}}^1_{CII}(H) := \boldsymbol{\mathcal{W}}(\det h).$$

This is very similar to the situation for class BDI. However, the index can take only even values.

Proposition 3.8.6. *For 1-dimensional H in class CII, the index $\boldsymbol{\mathcal{I}}^1_{CII}(H)$ is even.*

Proof. At a fixed point $k^* \in \{0, \pi\}$, h is symplectic and thus has $\det h(k^*) = 1$. Hence, $\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h)$ is an integer. Moreover, $\det(h(\tau k)) = \det(\overline{h(k)})$ implies that $\boldsymbol{\mathcal{W}}|_{[-\pi,0]}(\det h) = \boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h)$. Thus, $\boldsymbol{\mathcal{I}}^1_{CII}(H) := \boldsymbol{\mathcal{W}}(\det h) = 2\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h) \in 2\mathbb{Z}$. $\square$

Example 3.8.7. Note that

$$h(k) = \begin{pmatrix} e^{ikl} & 0 & 0 & 0 \\ 0 & \mathbb{I}_{m-1} & 0 & 0 \\ 0 & 0 & e^{ikl} & 0 \\ 0 & 0 & 0 & \mathbb{I}_{m-1} \end{pmatrix}$$

satisfies Eq. (11) for any $l \in \mathbb{Z}$. The family H given by Eq. (10) then has index $\boldsymbol{\mathcal{I}}^1_{CII}(H) = 2l$. Thus, the index $\boldsymbol{\mathcal{I}}^1_{CII}$ can attain all values in $2\mathbb{Z}$.

Proposition 3.8.8. *The 2D-index for CII vanishes, explaining the entry 0 in Table 1.*

Proof. Suppose H_1 and H_2 lie in the class CII in 2D and that the lower dimensional indices agree, i.e. the 1D-index agrees on γ_i for $i \in \{1, 2, 3, 4\}$. There is a map $G : \mathbb{T} \to U(2m)$ such that $G(k)h_2(k) = h_1(k)$.

Eq. (11) implies that $\boldsymbol{\mathcal{W}}|_{[-\pi,0]}(\det h_1|_{\gamma_i}) = \boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_1|_{\gamma_i})$. Thus,

$$2\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_1|_{\gamma_i}) = \boldsymbol{\mathcal{I}}^1_{CII}(h_1|_{\gamma_i}) = \boldsymbol{\mathcal{I}}^1_{CII}(h_2|_{\gamma_i}) = 2\boldsymbol{\mathcal{W}}|_{[0,\pi]}(\det h_2|_{\gamma_i}),$$

and we conclude that $\boldsymbol{\mathcal{W}}_{[-\pi,0]}(\det G|_{\gamma_i}) = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det G|_{\gamma_i}) = 0$.

Thus, G satisfies the assumptions of Proposition 2.3.3 with $X = U(2m)$, $I = \mathbb{I}_{2m}$ and

$$f(A) = \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} \overline{A} \begin{pmatrix} 0 & -\mathbb{I}_m \\ \mathbb{I}_m & 0 \end{pmatrix}.$$

Therefore, we obtain a homotopy F_1, which by the symplectic condition at fixed points k^* has $\det F_1(k^*, t) = 1$. Hence, for the winding number we have

$$\mathcal{W}_{[-\pi,0]}(\det F_1(\cdot, t)|_{\gamma_i}) = \mathcal{W}_{[0,\pi]}(\det F_1(\cdot, t)|_{\gamma_i}) = \mathcal{W}_{[0,\pi]}(\det G|_{\gamma_i}) = 0$$

for every t. Thus, $\widetilde{G} = F_1(\cdot, 1)$ satisfies the assumption of part (ii) of Proposition 2.3.3 and since $\pi_2(U(2m, \mathbb{C})) = 0$, we obtain a homotopy F_2 between G and $\mathbb{I}_n$ satisfying $F_2(\tau k, t) = f(F_2(k, t))$. Then, we can define $h(k, t) := F(k, t)h_2(k)$, which defines a homotopy between h_1 and h_2 satisfying $h(\tau k, t) = f(h(k, t))$. Thus, H_1 can be deformed into H_2 while keeping the symmetry and the spectral gap intact. Hence, the 2D-index for CII vanishes. □

3.9 C

Definition 3.9.1. Let $H(k)$ for $k \in \mathbb{T}^d$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H has odd particle-hole symmetry if there is an operator $\Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Σ is antiunitary,

(ii) $\Sigma^2 = -1$,

(iii) for all $k \in \mathbb{T}$,

$$\Sigma H(k) = -H(\tau k)\Sigma.$$

Remark 3.9.2. The treatment of symmetry C is very similar to symmetry D. The definitions and proofs work analogously. For proofs that are identical in case C and D we will refer to symmetry D.

Let $P^-(k)$ be the Fermi projection and $P^+(k)$ the projection associated to $\sigma(H) \cap (0, \infty)$. Let $E^\pm(k) = \{(k, P^\pm(k)(\mathbb{C}^N))\}$.

Proposition 3.9.3. *The projections satisfy* $P^-(\tau k)\Sigma = \Sigma P^+(k)$.

Proof. Identical to the proof of Proposition 3.5.2. □

Thus a family of Hamiltonians with odd PHS induces a bundle satisfying the following definition.

Definition 3.9.4. A bundle with odd particle hole symmetry is a vector bundle of the form $E = E^+ \oplus E^- = \mathbb{T}^d \times \mathbb{C}^N$ with an antilinear map $\Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) the fibres $E^+(k)$ and $E^-(k)$ are orthogonal subspaces of $\mathbb{C}^N$ for every $k \in \mathbb{T}^d$,

(ii) $\Sigma^2 = -1$,

(iii) the orthogonal projections $P^\pm(k)$ onto $E^\pm(k)$ satisfy $P^-(\tau k)\Sigma = \Sigma P^+(k)$.

Remark 3.9.5. Because $\Sigma^2 = -1$ is bijective, Σ defines a bijection between $E^+(k)$ and $E^-(\tau k)$. Thus $\operatorname{rank}(E^+) = \dim E^+(k) = \dim E^-(\tau k) = \operatorname{rank}(E^-)$. So $\operatorname{rank}(E) = N =: 2n$ is even.

Remark 3.9.6. We can write $\Sigma = SC$ for a unitary S and a complex conjugation C. Moreover, we may assume that

$$S = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}.$$

Proof. The condition $\Sigma^2 = -1$ implies that S is antisymmetric. Let ϵ be the matrix with n blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal. Then there exists $Q \in U(2n)$ such that $QSQ^T = \epsilon$ [15]. Let P be the matrix $P = i(e_1, e_{n+1}, e_2, e_{n+2}, ..., e_n, e_{2n})$, where e_j denotes the j-th standard basis vector. Then,

$$P\epsilon P^T = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}.$$

Thus, changing basis by PQ brings the matrix S into the desired form. □

From now on we will assume that $S = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$.

Remark 3.9.7. If we take a bundle with odd PHS, then $\widehat{H}(k) := 2P^+(k) - 1$ is unitary, self-adjoint and satisfies $\widehat{H}(\tau k)\Sigma = -\Sigma\widehat{H}(k)$, i.e. it has all the properties that the original Hamiltonian had.

Our aim is to define an index for bundles with odd PHS over $\mathbb{T} = S^1 \times S^1$. The idea is to look at the cut torus $\dot{\mathbb{T}}$ as in [2]. Let E be a bundle with odd PHS. It induces a bundle $\dot{E} = \dot{E}^+ \oplus \dot{E}^-$ on $\dot{\mathbb{T}}$, which by Proposition 3.1.9 admits a frame $v^+ : \dot{\mathbb{T}} \to F(\dot{E}^+)$. Then we define a frame v^- of $\dot{E}^-$ through $v^-(k) := \Sigma v^+(\tau k)$. By setting $v(k) = (v^+(k), v^-(k))$ we obtain frame v of $\dot{E}$ satisfying

$$v(\tau k) = \Sigma v(k) \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

Let $v^\pm(\pi, k_2) = v^\pm(-\pi, k_2)T^\pm(k_2)$ and $v(\pi, k_2) = v(-\pi, k_2)T(k_2)$. Note that

$$T(k_2) = \begin{pmatrix} T^+(k_2) & 0 \\ 0 & T^-(k_2) \end{pmatrix}.$$

Proposition 3.9.8.

$$T^-(k_2) = (\overline{T^+(-k_2)})^{-1}$$

Proof. Identical to the proof of Proposition 3.5.8. □

Definition 3.9.9. Define the index of E as $\mathcal{I}_C^2(E) := \mathcal{W}(\det(T^-))$.

Remark 3.9.10. Note that for the index $\mathcal{I}_A^2$ for symmetry A we have

$$\mathcal{I}_C^2(E) = \mathcal{I}_A^2(E).$$

In particular, $\mathcal{I}_C^2$ is well defined, i.e. independent of the frame v^+.

Remark 3.9.11. We may always assume our frames to be unitary.

Lemma 3.9.12. *For a unitary frame v satisfying*

$$v(\tau k) = \Sigma v(k) \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}, \tag{12}$$

we have $\det v(k^*) = 1$ *for all fixed points* $\tau k^* = k^*$ *where v is defined.*

Proof. At all fixed points k^* we have

$$v(k^*) = \Sigma v(k^*) \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \overline{v(k^*)} \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

We can rearrange this as

$$v(k^*)^T \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix} v(k^*) = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

Thus $v(k^*)$ is a symplectic matrix and thus has $\det v(k^*) = 1$. □

Remark 3.9.13. Let $d \in \{0, 1\}$. Let $\Lambda = \mathrm{diag}(\lambda_i)$, where $\lambda_i : \mathbb{T}^d \to \mathbb{R}_{>0}$ for $1 \leq i \leq n$. If $v : \mathbb{T}^d \to U(2n)$ satisfies Eq. (12), then $H = v(k) \begin{pmatrix} \Lambda(k) & 0 \\ 0 & -\Lambda(\tau k) \end{pmatrix} v(k)^*$ is in class C. Conversely, any H in class C in dimension d can be written in this form.

In 0D, Eq. (12) means that the frame v is a symplectic matrix. Since the symplectic group is connected [5], we may deform any $H = v(k) \begin{pmatrix} \Lambda(k) & 0 \\ 0 & -\Lambda(\tau k) \end{pmatrix} v(k)^*$ in class C to $\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$, while keeping the symmetry and the spectral gap intact. This justifies the entry 0 in Table 1.

Let H be in class C in 1D. We may write $H(k) = v(k) \begin{pmatrix} \Lambda(k) & 0 \\ 0 & -\Lambda(\tau k) \end{pmatrix} v(k)^*$ for some $v : S^1 \to U(2n)$ satisfying Eq. (12). By Proposition 2.3.2 we can deform v in such a way that $v(0) = v(\pi) = \mathbb{I}_{2n}$ while keeping the symmetry property intact. Let $l = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det v)$. For $k \in [0, \pi]$ let $\widetilde{v}(k) = v(k) \begin{pmatrix} e^{-ikl} & 0 \\ 0 & \mathbb{I}_{2n-1} \end{pmatrix}$ and extend $\widetilde{v}$ to S^1 such that it satisfies Eq. (12). We may replace v by $\widetilde{v}$ without changing H. By Proposition 2.3.2 (ii) we may deform $\widetilde{v}$ to $\mathbb{I}_{2n}$ while keeping the symmetry property intact. So H can be again deformed to $\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$ while keeping the symmetry and the spectral gap intact, justifying the entry 0 in Table 1.

Now we want to show that the index I_C^2 can only take even values.

Lemma 3.9.14. *There is a unitary frame $v : \dot{\mathbb{T}} \to F(\dot{E})$ satisfying Eq. (12) and $v(-\pi, k_2^*) = v(\pi, k_2^*)$ for $k_2^* \in \{0, \pi\}$.*

Proof. Identical to the proof of Lemma 3.5.12 up to replacing all $\begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$ with $\begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$. □

Theorem 3.9.15. *In 2D, for any bundle E with odd PHS the index $\boldsymbol{\mathcal{I}}_C^2(E)$ is even.*

Proof. Let v be a frame as in Lemma 3.9.14. On $[-\pi, 0] \times S^1$ define

$$w(k_1, k_2) := (v^+(k_1, k_2), v_1^-(k_1, k_2), ..., v_{n-1}^-(k_1, k_2), v_n^-(k_1, k_2) \det v(k_1, k_2)^{-1} \det v(0, k_2)).$$

Then $\det w(k_1, k_2) = \det w(0, k_2)$ and $w(0, k_2) = v(0, k_2)$ for all $k_1 \in [-\pi, 0]$ and $k_2 \in S^1$. Note that $w(-\pi, 0) = v(-\pi, 0)$ and $w(-\pi, \pi) = v(-\pi, \pi)$ by Lemma 3.9.12. Extend w to $\dot{\mathbb{T}}$ through

$$w(k_1, k_2) := \Sigma w(\tau k) \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

for $(k_1, k_2) \in [0, \pi] \times S^1$. Note that w is well defined at $k_1 = 0$ and that w is again a unitary frame of the form as in Lemma 3.9.14. By Proposition 3.9.8, the transition matrix $T(k_2)$ with $w(-\pi, k_2)T(k_2) = w(\pi, k_2)$ is of the form

$$T(k_2) = \begin{pmatrix} (\overline{T^-(-k_2)})^{-1} & 0 \\ 0 & T^-(k_2) \end{pmatrix}.$$

Since w is unitary, $\det T^-(k_2) = e^{i\alpha(k_2)}$ for some continuous $\alpha : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Moreover, for all $k_2 \in S^1$ we have $\det T(k_2) = 1$ because $\det w(-\pi, k_2) = \det w(0, k_2) = \det w(\pi, k_2)$. This implies that $1 = \det T^-(k_2) \det(\overline{T^-(-k_2)})^{-1} = e^{i\alpha(k_2)} e^{i\alpha(-k_2)}$, i.e. $\alpha(k_2) \equiv -\alpha(-k_2) \mod 2\pi\mathbb{Z}$. Hence $\det T^-$ winds by the same amount from $-\pi$ to 0 as from 0 to π, i.e. $\boldsymbol{\mathcal{W}}_{[-\pi,0]}(\det T^-) = \boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$. We have $w(-\pi, 0) = w(\pi, 0)$ and $w(-\pi, \pi) = w(\pi, \pi)$. Thus, $T^-(0) = T^-(\pi) = \mathbb{I}_n$ and $\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$ is an integer. So $\boldsymbol{\mathcal{I}}_C^2(E) = 2\boldsymbol{\mathcal{W}}_{[0,\pi]}(\det T^-)$ is even. □

3.10 CI

Definition 3.10.1. Let $H = \{H(k) : k \in \mathbb{T}^d\}$ be a continuous family of self-adjoint operators on $\mathbb{C}^N$ with spectral gap $0 \notin \sigma(H(k))$ for all $k \in \mathbb{T}^d$. We say that H belongs to symmetry class CI if there are operators $\Theta, \Sigma : \mathbb{C}^N \to \mathbb{C}^N$ such that

(i) Θ, Σ are antiunitary,

(ii) $\Theta^2 = 1$ and $\Sigma^2 = -1$,

(iii) for all $k \in \mathbb{T}$,
$$H(\tau k)\Sigma = -\Sigma H(k) \quad \text{and} \quad H(\tau k)\Theta = \Theta H(k),$$

(iv) $\Theta\Sigma = \Sigma\Theta = \Pi$.

Note that Π is unitary, $\Pi^2 = -1$ and $H(k)\Pi = -\Pi H(k)$.

Remark 3.10.2. Let H be in symmetry class CI. By Proposition 3.2.2 we can choose a basis such that $\Pi = i\mathbb{I}_n \oplus -i\mathbb{I}_n$, hence $N = 2n$. Since $\Sigma\Pi = \Pi\Sigma$, if $\Pi v = iv$, then $\Pi\Sigma v = \Sigma\Pi v = \Sigma iv = -i\Sigma v$. Thus from a basis $B = (b_1, ..., b_n)$ of the eigenspace $E_{\Pi,i}$, be obtain a basis $(-\Sigma b_1, ..., -\Sigma b_n)$ of $E_{\Pi,-i}$. In this basis, we have $\Sigma = SC$ for the complex conjugation C and

$$S = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

Moreover, $\{\Pi, H(k)\} = 0$ implies

$$H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix}, \tag{13}$$

for some continuous $h : \mathbb{T}^d \to GL(n)$. The condition $\Sigma H(k) = -H(\tau k)\Sigma$ gives

$$h(\tau k) = h(k)^T. \tag{14}$$

Note that for given symmetries Θ, Σ, Π we can choose a basis such that the family H belongs to class CI if and only if H satisfies Eq. (13) and (14).

We want to classify the continuous families H satisfying Eq. (13) and (14) in dimensions 0, 1, and 2. First we need the following result.

Proposition 3.10.3. *The set of symmetric matrices in $GL(n, \mathbb{C})$ is path-connected.*

Proof. Let A_1, A_2 be two symmetric matrices in $GL(n, \mathbb{C})$. By Autonne-Takagi factorisation, we can write $A_1 = Q_1\Lambda_1 Q_1^T$ and $A_2 = Q_2\Lambda_2 Q_2^T$ for diagonal matrices Λ_1, Λ_2 and $Q_1, Q_2 \in U(n)$. For every $1 \leq i \leq n$ choose a path α_i from $(\Lambda_1)_{ii}$ to $(\Lambda_2)_{ii}$ in $\mathbb{C} \setminus \{0\}$. Since $U(n)$ is path-connected we can also choose a path $Q(t)$ from Q_1 to Q_2 in $U(n)$. Then $A(t) = Q(t)\mathrm{diag}(\alpha_i(t))Q(t)^T$ is a path form A_1 to A_2 in the set of symmetric matrices in $GL(n, \mathbb{C})$. □

In 0D, the condition in Eq. (14) is equivalent to h being symmetric. By Proposition 3.10.3 thus the 0D-index for CI vanishes. Also in 1D we want to show that the index for CI vanishes.

Proposition 3.10.4. *Let H be a one dimensional family in class CI and let $h : S^1 \to GL(n)$ be the corresponding block. Then there is a homotopy $F : S^1 \times [0, 1] \to GL(n)$ between h and $\mathbb{I}_n$ satisfying $F(\tau k, t) = F(k, t)^T$.*

Proof. By Proposition 2.3.2 we may assume that $h(k^*) = \mathbb{I}_n$ for $k^* \in \{0, \pi\}$. Let $l = \mathbb{W}_{[0,\pi]}(\det h)$, then there is a homotopy $F : [0, \pi] \times [0, 1/2] \to GL(n)$ between $h|_{[0,\pi]}$ and $\begin{pmatrix} e^{ikl} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$ relative to the endpoints. Extending F to S^1 via $F(\tau k, t) = F(k, t)^T$, we see that $F(k, 1/2) = \begin{pmatrix} e^{i|k|l} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$ for $k \in [-\pi, \pi]$. For $t \in [1/2, 1]$ let

$$F(k, t) = \begin{pmatrix} e^{2(1-t)i|k|l} & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}.$$

Then F is a homotopy of the required form. □

Proposition 3.10.5. *The 2D-index for class CI vanishes, justifying the entry 0 in Table 1.*

Proof. Suppose H lies in the class CI. Let $F : S^1 \times [0, \pi] \times \{0\} \cup (\gamma_2 \cup \gamma_4) \times [0, 1] \to GL(n)$ be given through $F(k_1, k_2, 0) = h(k_1, k_2)$, $F|_{\gamma_2 \times [0,1]}(k, t) = h|_{\gamma_2}(k)$ and let $F(k_1, 0, t)$ be a homotopy between $h|_{\gamma_4}$ and $\mathbb{I}_n$ as in Proposition 3.10.4. Since the domain of F is a retract of $S^1 \times [0, \pi] \times [0, 1]$, we can extend F to $S^1 \times [0, \pi] \times [0, 1]$ and via $F(\tau k, t) = F(k, t)^T$ also to $\mathbb{T} \times [0, 1]$. Thus we may assume, that $h|_{\gamma_4} = \mathbb{I}_n$. Now let $f_1 : S^1 \times [0,1] \to GL(n)$ be a homotopy between $h|_{\gamma_1}$ and $\mathbb{I}_n$ as in Proposition 3.10.4. Note that we can choose $f_1(0,t) = \mathbb{I}_n$ by inspecting the proof of Proposition 3.10.4. Similarly, let $f_3 : S^1 \times [0,1] \to GL(n)$ be a homotopy between $h|_{\gamma_3}$ and $\mathbb{I}_n$ as in Proposition 3.10.4 with $f_3(0,t) = \mathbb{I}_n$. We can find a homotopy $F_3 : [0,\pi] \times S^1 \times [0,1] \to GL(n)$ such that $F_3(k_1,k_2,0) = h(k_1,k_2)$, $F_3(0,k_2,t) = f_1(k_2,t)$, $F_3(\pi,k_2,t) = f_3(k_2,t)$ and $F_3(k_1,0,t) = \mathbb{I}_n$. Note that $F_3(\cdot,1)$ is constantly equal to $\mathbb{I}_n$ on $\gamma_1 \cup \gamma_3 \cup \gamma_4|_{[0,\pi]}$. Since $\pi_2(GL(n)) = 0$, there is a homotopy F_4 between $F_3(\cdot,1)$ and $\mathbb{I}_n$. Extending both F_3 and F_4 to the torus via Eq. (14), we see that we can deform h to $\mathbb{I}_n$ while keeping the symmetry property intact. Thus the 2D-index for CI vanishes. For details see the proof of Proposition 2.3.3 (ii). □

4 Mathematical examples

The aim of this Section is to show that the 2D-indices defined in Section 3 for symmetry classes A, D, DIII and C indeed are non-trivial. We construct examples with non-vanishing indices.

4.1 Examples for class A

We want to find a continuous family $H : \mathbb{T} \to \mathbb{C}^{N\times N}$ of self-adjoint matrices with spectral gap at $\mu \in \mathbb{R}$, for which $\boldsymbol{\mathcal{I}}_A^2(H) \neq 0$. If $H : \mathbb{T} \to \mathbb{C}^{N\times N}$ belongs to class A and either $\operatorname{rank}(E^+) = 0$ or $\operatorname{rank}(E^-) = 0$, then $\boldsymbol{\mathcal{I}}_A^2(H) = 0$. Thus for $N = 1$ the index $\boldsymbol{\mathcal{I}}_A^2$ is always trivial. Let us consider $N = 2$. We can express a family $H_2 : \mathbb{T} \to \mathbb{C}^{2\times 2}$ of self-adjoint matrices through

$$H_2(k) = h_0(k)\mathbb{I}_2 + \vec{h}(k) \cdot \vec{\sigma},$$

for $h_0 : \mathbb{T} \to \mathbb{R}$, $\vec{h} : \mathbb{T} \to \mathbb{R}^3$ and $\vec{h}(k) \cdot \vec{\sigma} = \sum_{i=1}^3 h_i(k)\sigma_i$ for the Pauli matrices σ_i. A calculation shows that the eigenvalues of $H_2(k)$ are

$$\lambda_\pm(k) = h_0(k) \pm \left\|\vec{h}(k)\right\|.$$

Let us choose $h_0(k) = \mu$, then the gap condition is satisfied if and only if $\left\|\vec{h}(k)\right\| \neq 0$. In that case one can define $e(k) := -\frac{\vec{h}(k)}{\|\vec{h}(k)\|}$, which gives a continuous map $e : \mathbb{T} \to S^2$. In Section 8.4. of [13] it is shown that

$$\boldsymbol{\mathcal{I}}_A^2(H_2) = \deg(e),$$

where $\deg(e)$ is the degree of the map e. The quotient $\mathbb{T}/(\gamma_2 \cup \gamma_3)$ is homeomorphic to S^2, inducing a map $q : \mathbb{T} \to S^2$ with $\deg(q) = 1$. For every $l \in \mathbb{Z}$ pick a map $f_l : S^2 \to S^2$ with $\deg(f_l) = l$. Then choosing $\vec{h} = -f_l \circ q$, the corresponding H_2 has index $\boldsymbol{\mathcal{I}}_A^2(H_2) = l$. Hence, every value in $\mathbb{Z}$ can be attained by this index. We can extend this example to $N > 2$, by considering the direct sum $H = H_2 \oplus \mathbb{I}_{N-2}$, which has index $\boldsymbol{\mathcal{I}}_A^2(H) = \boldsymbol{\mathcal{I}}_A^2(H_2)$.

A physical model belonging to class A in 2D is the Haldane model [3]. This model has been realised experimentally [6].

4.2 Examples for class D

Let N be a positive even integer. We want to show that for every $l \in \mathbb{Z}$ there is a $H : \mathbb{T} \to GL(N)$ in class D with $\boldsymbol{\mathcal{I}}^2_D(H) = l$. Then also the 1D-index $\boldsymbol{\mathcal{I}}^1_D$ can take non-trivial values by Theorem 3.5.13.

For $n \in \mathbb{Z}_{>0}$ let $H_n(k) = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$ be a trivial family in class D. If $H_0 : \mathbb{T} \to GL(2)$ belongs to class D, then also $H = H_0 \oplus H_n$ belongs to D and $\boldsymbol{\mathcal{I}}^2_D(H) = \boldsymbol{\mathcal{I}}^2_D(H_0)$. Thus, it is enough to consider the case $N = 2$.

For $N = 2$ we can suppose that $\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C$, where C denotes the complex conjugation. Let us choose the ansatz $H(k) = \vec{h}(k) \cdot \vec{\sigma}$ for $\vec{h} : \mathbb{T}^d \to S^2$. Then H belongs to symmetry class D if and only if

$$(h_1, h_2, h_3)(\tau k) = (-h_1, -h_2, h_3)(k). \tag{15}$$

Therefore, h_1 and h_2 vanish at the fixed points of τ. For $d = 1$ the index is trivial if $h_3(0) = h_3(\pi)$ and non-trivial if $h_3(0) = -h_3(\pi)$.

For $d = 2$ the parity of the index depends on the configuration of 1D-indices on γ_i by Theorem 3.5.13. As discussed in Section 3.5, there are essentially four different configurations. Since the index is given by the Chern number, it is equal to the degree of $\vec{h}$ up to a change of sign. In Figure 2 we sketch maps $\vec{h}$ leading to a non-trivial index for every configuration in 1D. It is enough to define $\vec{h}$ on half of the torus, e.g. on $[0, \pi] \times S^1$ and to make sure that the symmetry condition is satisfied on the boundary. Then $\vec{h}$ can be extended uniquely to the torus $\mathbb{T}$ via the symmetry condition in Eq. (15).

4.3 Examples for class DIII

We want to find examples in 1D and 2D for which the indices $\mathbb{I}^1_{DIII}$ and $\mathbb{I}^2_{DIII}$ do not vanish. Since the rank has to be divisible by 4, we first consider $N = 4$. Following the discussion in Section 3.6 we want to find a family

$$H(k) = \begin{pmatrix} 0 & h(k)^* \\ h(k) & 0 \end{pmatrix},$$

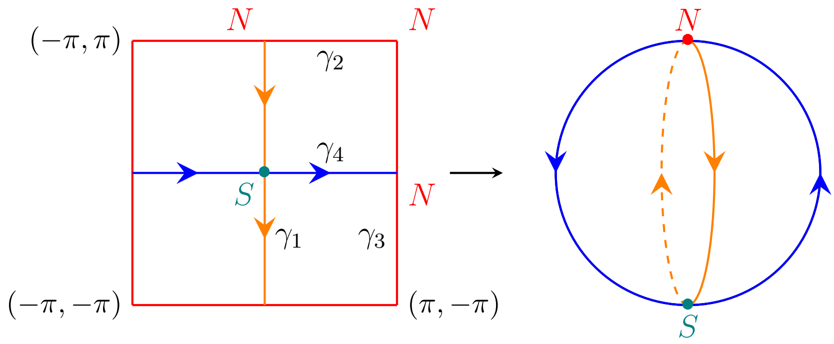
where $h : \mathbb{T}^d \to U(2)$ satisfies $h(\tau k) = -h(k)^T$. The symmetry operators are given by $\Pi = i \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$ and $\Theta = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} C$.

By Remark 3.6.11 we may assume that $\det h(k) = 1$, which means that there is a map $f : \mathbb{T} \to S^3 \subset \mathbb{R}^4$ with components $f = (a, b, c, d)$ such that

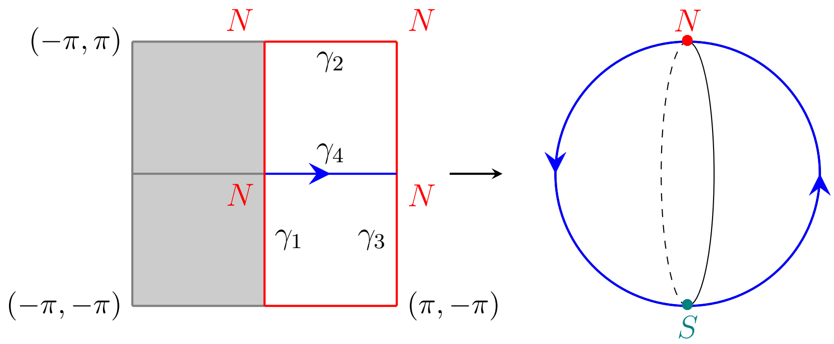
$$h(k) = \begin{pmatrix} a(k) + id(k) & b(k) + ic(k) \\ -b(k) + ic(k) & a(k) - id(k) \end{pmatrix}.$$

The condition $h(\tau k) = -h(k)^T$ implies that $b(\tau k) = b(k)$ and $(a, c, d)(\tau k) = -(a, c, d)(k)$.

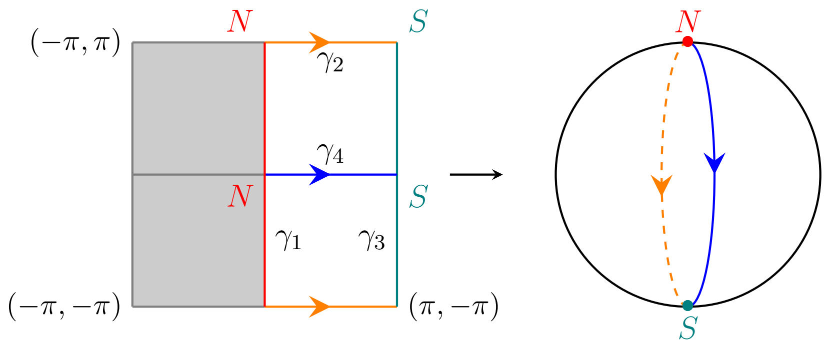
In 1D, the index will be trivial if $b(0) = b(\pi)$ and nontrivial if $b(0) = -b(\pi)$. Concretely, we



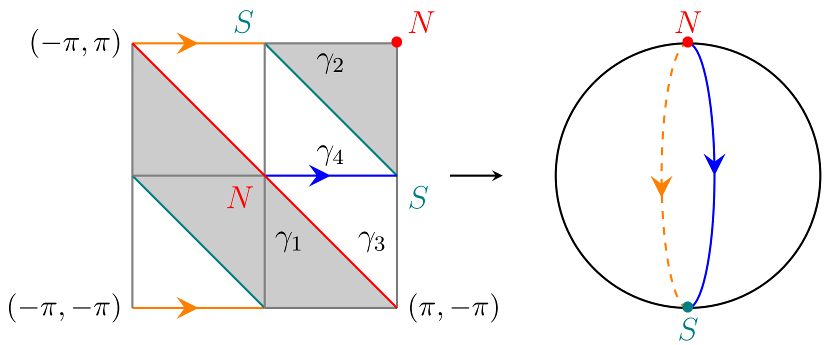
(a) $(\mathcal{I}_D^1(E_{\gamma_i}))_{i=1}^4 = (-1,1,1,-1)$. The map $\vec{h}$ sketched here has degree 1.



(b) $(\mathcal{I}_D^1(E_{\gamma_i}))_{i=1}^4 = (1,1,1,1)$. The map $\vec{h}$ sketched here has degree 2.



(c) $(\mathcal{I}_D^1(E_{\gamma_i}))_{i=1}^4 = (1,-1,1,-1)$. The map $\vec{h}$ sketched here has degree 2.



(d) $(\mathcal{I}_D^1(E_{\gamma_i}))_{i=1}^4 = (-1,-1,-1,-1)$. The map $\vec{h}$ sketched here has degree 2.

Figure 2: For each of the four different configurations of 1D-indices we sketch a map $\vec{h} : \mathbb{T} \to S^2$ with non-zero degree satisfying Eq. (15). We interpret the square on the left as $\mathbb{T}$ by identifying opposite edges. The sketch suggests how to define $\vec{h} : \mathbb{T} \to S^2$ on the unshaded part of the tours. Then $\vec{h}$ can be extended uniquely to all of $\mathbb{T}$ by the symmetry condition in Eq. (15).

can choose $b(k) = \cos(k)$, $c(k) = \sin(k)$ and $a(k) = d(k) = 0$. Then, $h(k) = \begin{pmatrix} 0 & e^{ik} \\ -e^{-ik} & 0 \end{pmatrix}$ and a frame as in Proposition 3.6.8 is given by

$$v^+(k) = \frac{1}{\sqrt{2}} \left(v_1^+(k), \Theta v_1^+(\tau k) \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -e^{ik} \\ 0 & -1 \\ -e^{-ik} & 0 \end{pmatrix}.$$

We read off that $w(k) = \begin{pmatrix} 1 & 0 \\ 0 & -e^{ik} \end{pmatrix}$ and thus $\boldsymbol{\mathcal{W}}(\det w) = -1 \equiv 1 \mod 2$. Hence, the index $\boldsymbol{\mathcal{I}}^1_{DIII}(H) \equiv 1 \in \mathbb{Z}_2$ is non-trivial. On the other hand, if we choose $b(k) = 1$ and $a(k) = c(k) = d(k) = 0$, then $\boldsymbol{\mathcal{I}}^1_{DIII}(H) = 0 \in \mathbb{Z}_2$.

In order to construct an example in 2D with non-trivial index, by Theorem 3.6.15 it suffices to find $h : \mathbb{T} \to U(2)$ such that $h(\cdot, \pi)$ is constant and $h(\cdot, 0)$ is non-trivial in 1D. Let $h(\cdot, 0)$ be non-trivial in 1D. Then $h(\cdot, 0) = (a, b, c, d) : S^1 \to S^3$ defines a loop in S^3. Since $\pi_1(S^3) = 0$, there is a homotopy $h : S^1 \times [0, \pi] \to U(2)$ between $h(\cdot, 0)$ and the constant map $h(0, 0) = h(\cdot, \pi)$. Extend h to the torus via $h(\tau k) = -h(k)^T$ for $k \in [-\pi, 0]$. Then by construction h induces a non-trivial index $\boldsymbol{\mathcal{I}}^2_{DIII}(H) \equiv 1 \in \mathbb{Z}_2$.

For dimensions $d = 1, 2$ and $N = 4m > 4$ we may choose $h(k) = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$, with

$$A = \begin{pmatrix} h_1(k) & 0 \\ 0 & 0_{m-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} h_2(k) & 0 \\ 0 & \mathbb{I}_{m-1} \end{pmatrix},$$

where $h_1, h_2 : \mathbb{T}^d \to \mathbb{C}$ are such that $\widehat{h}(k) = \begin{pmatrix} h_1(k) & h_2(k) \\ -\overline{h_2(k)} & \overline{h_1(k)} \end{pmatrix}$ induces a Hamiltonian H_4 of rank 4 with non-trivial index in dimension d. Then $\boldsymbol{\mathcal{I}}^d_{DIII}(H) = \boldsymbol{\mathcal{I}}^d_{DIII}(H_4)$ is non-trivial.

4.4 Examples for class C

By Theorem 3.9.15, we know that the index $\boldsymbol{\mathcal{I}}^2_C$ can only take even values. Let N be a positive even integer. We want to show that for every even integer $2l$ there is a $H : \mathbb{T} \to GL(N)$ in class C with $\boldsymbol{\mathcal{I}}^2_C(H) = 2l$. As for class D, it is enough to consider $N = 2$.

For $N = 2$ we can suppose that $\Sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C$, where C denotes the complex conjugation. Let us choose the ansatz $H(k) = \vec{h}(k) \cdot \vec{\sigma}$ for $\vec{h} : \mathbb{T} \to S^2$. Then H belongs to symmetry class C if and only if $\vec{h}(\tau k) = \vec{h}(k)$. The 2D-index is again given by the Chern number and thus equal to $\deg(-\vec{h})$. Let $q : \mathbb{T} \to \mathbb{T}/(k \sim \tau k)$ be the quotient map. Note that $\mathbb{T}/(k \sim \tau k) \cong S^2$ as sketched in Figure 3 and that $\deg(q) = 2$. For every $l \in \mathbb{Z}$ there is a map $f_l : \mathbb{T}/(k \sim \tau k) \to S^2$ with $\deg(f_l) = l$. Then $\vec{h} := f_l \circ q$ satisfies $\vec{h}(\tau k) = \vec{h}(k)$ and $\deg(\vec{h}) = 2l$. Thus the corresponding H has index $\boldsymbol{\mathcal{I}}^2_C(H) = -2l$.

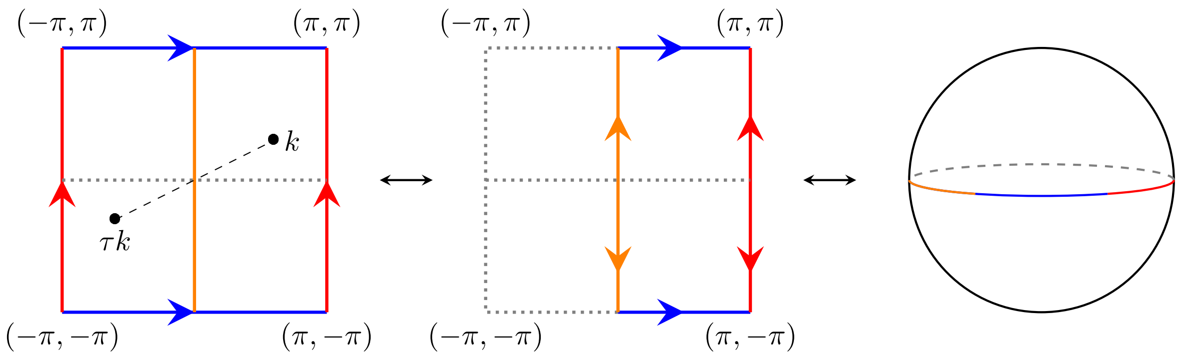


Figure 3: Edges of the same colour are identified. Left: A sketch of $\mathbb{T}$ where we want to identify $\tau k \sim k$. Middle: $\mathbb{T}/(\tau k \sim k)$ corresponds to the right half of the square with the edges identified as sketched. After gluing the edges accordingly, we obtain a sphere (right).

5 References

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