ETH Zürich

Semester paper

Principal bundles, Hopf bundles and Eigenbundles

by

Barbara Roos

supervised by

Prof. Dr. Giovanni Felder

and

Dr. Iuliya Beloshapka

June 18, 2018

Abstract

Principal bundles connect differential geometry, algebraic topology and physics. We first explain the most important concepts about principal bundles and study them on the example of the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$. We show that the Hopf bundle is non-trivial by calculating its first Chern class. Then we describe how magnetic monopoles and electromagnetism can be phrased in terms of principal bundles. Motivated by quantum mechanics we study normalised eigenbundles of Hamiltonians. Finally, we show that different Hopf bundles arise from normalised eigenbundles of 2×2 matrices.

Contents

1 Introduction

The study of principal bundles brings together differential geometry, algebraic topology and physics. One can think of a principal bundle as a manifold with a certain group attached at every point, i.e. it is a fibre bundle where the fibre is a group. Moreover, there is a well defined group action which can move a point in some fibre to any other point in the same fibre. However, this group action does not move points form one fibre to another. One very important example of a principal bundle is the Hopf bundle. For the Hopf bundle we attach the group of unit complex numbers $U(1)$ to the sphere S^2. But we attach it in such a way that we do not get $S^2 \times U(1)$ but actually the sphere S^3. Since $U(1)$ is just the circle S^1, we write the Hopf bundle as $S^1 \hookrightarrow S^3 \to S^2$. Analogous constructions are possible if instead of complex numbers we take real numbers, quaternions or octonions. With methods form algebraic topology it is possible to show that the Hopf bundle is indeed different from the trivial bundle $S^2 \times U(1)$.

In Section 2 the most important definitions connected to principal bundles are given. We study the different concepts on the example of the Hopf bundle. In Section 3 we define associated bundles and Chern classes and show that the Hopf bundle is non-trivial. Then in Section 4 we look at different settings where principal bundles show up in physics such as magnetic monopoles, Maxwell's theory of electromagnetism and eigenstates of Hamiltonians in quantum mechanics. Finally, we present a way of constructing Hopf bundles via eigenvectors of 2×2 matrices in Section 5. Here, we also encounter the Hopf bundles $S^0 \hookrightarrow S^1 \to S^1$ and $S^3 \hookrightarrow S^7 \to S^4$ arising form real numbers and quaternions, respectively.

I thank Prof. Dr. Giovanni Felder and Dr. Iuliya Beloshapka for taking their time to supervise my work and for our enlightening discussions.

2 Principal bundles

2.1 Preliminaries

Definition 2.1. Let M and F be topological spaces. A *fibre bundle over M with fibre F* is a tuple (E, π, M, F) where

i) E is a topological space and π is a continuous surjective map $\pi : E \to M$

ii) There is an open cover $\{U_i\}_{i \in I}$ of M such that for every $i \in I$ there is a homeomorphism $\Phi_i : U_i \times F \to \pi^{-1}(U_i) \subset E$ mapping $\{p\} \times F$ to $\pi^{-1}(p)$ for every $p \in U_i$.

E is called the *total space*, M is called the *base space*, for every $p \in M$ the preimage $E_p := \pi^{-1}(p)$ is called *fibre at p* and Φ_i is a *local trivialisation*.

For a fibre bundle (E, π, M, F) we may also write $\pi : E \to M$ or $F \hookrightarrow E \to M$ and take the other information as implicitly given.

Definition 2.2. Let G be a topological group. A *principal G-bundle* is a fibre bundle $\pi : E \to M$ together with a continuous right action $E \times G \to E$ such that

i) the action preserves the fibres, i.e. for all $g \in G$ and $p \in M$ we have $E_p g = E_p$

ii) on every fibre the action is free and transitive

iii) there is an open cover $\{U_i\}_{i \in I}$ of M such that for every $i \in I$ there is a homeomorphism $\Phi_i : U_i \times G \to \pi^{-1}(U_i) \subset E$ mapping $\{p\} \times G$ to $\pi^{-1}(p)$ for every $p \in U_i$ and satisfying $\Phi_i((p, g)h) = \Phi_i((p, g))h$ for all $g, h \in G$ and $p \in U_i$, where $(p, g)h := (p, gh)$.

If additionally G is a Lie group, E and M are smooth manifolds and π, Φ_i and the right action are smooth, then $\pi : E \to M$ is a *smooth principal G-bundle.*

In this chapter most statements can be formulated for both continuous and smooth principal bundles. The changes one needs to make for smooth bundles are indicated in brackets.

Remark 2.3. For a principal G-bundle $\pi : E \to M$

i) The map π induces a homeomorphism $E/G \to M$.

ii) Each fibre of a (smooth) principal G-bundle is homeomorphic (diffeomorphic) to G. To see this, pick an element $u \in \pi^{-1}(p) = E_p$ and look at the map $G \to E_p, g \mapsto ug$. This is a homeomorphism (diffeomorphism) because G acts freely on E_p.

See Dupont (1978) Chapter 3 for more details.

Definition 2.4. Let $\pi_F : F \to N$ and $\pi_E : E \to M$ be (smooth) principal G-bundles. A *bundle map* is a pair $(\bar{f}, f)$ of continuous (smooth) maps such that $\bar{f}$ is G-equivariant, i.e. $\bar{f}(ug) = \bar{f}(u)g$ and the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\bar{f}"] \arrow[d, "\pi_F"'] & E \arrow[d, "\pi_E"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

A bundle map is a *bundle isomorphism* if there is another bundle map forming a two sided inverse.

Example 2.5. Let M and F be topological spaces. Then $E = M \times F$ together with $\pi : M \times F \to M$, $(p, f) \mapsto p$ is a fibre bundle over M with fibre F. This fibre bundle is called the *trivial bundle.* Any bundle isomorphic to the trivial bundle is called trivial.

Definition 2.6. Let $\pi : E \to M$ be a (smooth) principal G-bundle. For an open subset U of M a *local section* defined on U is a continuous (smooth) map $\sigma : U \to E$ such that $\pi \circ \sigma = id|_U$. A *global section* is a continuous (smooth) map $\sigma : M \to E$ such that $\pi \circ \sigma = id|_M$.

Proposition 2.7. *A principal bundle admits a global section iff it is trivial.*

Proof. This proof is based on Nakahara (2003) Section 9.4.3. Let (E, π, M, G) be a principal bundle. Suppose $s : M \to E$ is a global section. For an element $u \in E$ we observe the following: For $p = \pi(u)$, the two elements $s(p)$ and u both lie in the same fibre $\pi^{-1}(p)$. Since G acts freely on the fibre, there is a unique element $g \in G$ such that $u = s(p)g$. Hence, there is a well defined bundle isomorphism $\phi : E \to M \times G$ mapping $u = s(p)g$ to (p, g). Thus E is trivial.

Conversely, if $\phi : E \to M \times G$ is a bundle isomorphism, we can define a global section $s : M \to E$ through $s(p) = \phi^{-1}(p, e)$. □

Remark 2.8. Let M and N be homeomorphic (diffeomorphic) manifolds. Any principal G-bundle over M is isomorphic to some principal G-bundle over N and vice versa.

Proof. Let $\pi_F : F \to N$ be a principal G-bundle for some topological (or a Lie) group G and let $f : N \to M$ be a homeomorphism (diffeomorphism). Then $\pi_E := f \circ \pi_F : F \to M$ is a (smooth) principal G-bundle over M. The map (id_F, f) is a bundle isomorphism. Analogously, we can construct a principal bundle over N from a principal bundle over M. □

Example 2.9. The complex projective space $\mathbb{C}P^1$ is defined as the quotient space $\mathbb{C}P^1 := (\mathbb{C}^2 \setminus \{0\})/\sim$, where $\alpha \sim \beta$ iff $\alpha = \lambda\beta$ for some $\lambda \in \mathbb{C}$. The equivalence class of (z_1, z_2) in $\mathbb{C}P^1$ is denoted by $(z_1 : z_2)$. The set $\mathbb{C}P^1$ is a one dimensional complex manifold and the differentiable structure is given through $\mathbb{C}P^1 = U_1 \cup U_2$ where

$$U_1 = \{(z_1 : z_2)|z_1 \neq 0\} = \{(1 : z)|z \in \mathbb{C}\} \cong \mathbb{C}$$
$$U_2 = \{(z_1 : z_2)|z_2 \neq 0\} = \{(z : 1)|z \in \mathbb{C}\} \cong \mathbb{C}.$$

The space $\mathbb{C}P^1$ is diffeomorphic to $\mathbb{C} \cup \{\infty\}$ through the map $(z_1 : z_2) \mapsto z_1/z_2$. Moreover, S^2 is diffeomorphic to $\mathbb{C} \cup \{\infty\}$ through stereographic projection. Thus, $\mathbb{C}P^1$ is diffeomorphic to S^2 and let g denote the homeomorphism induced by the maps above. Consider the projection $f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1, (z_1, z_2) \mapsto (z_1 : z_2)$. Viewing S^3 as a subset of $\mathbb{C}^2$ his gives us a map $h := g \circ f|_{S^3} : S^3 \to S^2$, called the Hopf map. Explicitly, the Hopf map is given through

$$h(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

We claim that $h : S^3 \to S^2$ is a smooth principal $U(1)$-bundle.

Proof. First, we have a group action $S^3 \times U(1) \to S^3, (z_1, z_2, \lambda) \mapsto (z_1\lambda, z_2\lambda)$, which is well defined since $|\lambda| = 1$. Second, we have to check the locality condition, i.e. that $\pi^{-1}(U_i) \cong U_i \times U(1)$ for some open cover $\{U_i\}$ of S^2. By Remark 2.8 we can verify this for the map $f : S^3 \to \mathbb{C}P^1$ instead.

We construct diffeomorphisms ϕ_1 and ϕ_2

$$\begin{aligned}\phi_1 : U_1 \times U(1) \to \{(z_1, z_2) \in S^3 | z_1 \neq 0\} = f^{-1}(U_1)\\ \phi_2 : U_2 \times U(1) \to \{(z_1, z_2) \in S^3 | z_2 \neq 0\} = f^{-1}(U_2)\end{aligned}$$

which are compatible with the action of $U(1)$. Let us take

$$\begin{aligned}\phi_1 :&((1 : z), \lambda) \mapsto \frac{1}{\sqrt{1 + |z|^2}}(\lambda, z\lambda)\\ \phi_2 :&((z : 1), \lambda) \mapsto \frac{1}{\sqrt{1 + |z|^2}}(z\lambda, \lambda)\end{aligned}$$

These maps are invertible since their inverse is given by

$$\begin{aligned}\phi_1^{-1} :&(z_1, z_2) \mapsto ((1 : z_2/z_1), z_1/|z_1|)\\ \phi_2^{-1} :&(z_1, z_2) \mapsto ((z_1/z_2 : 1), z_2/|z_2|)\end{aligned}$$

Moreover, the maps are compatible with the right action, since

$$\phi_1^{-1}((z_1, z_2)\lambda) = ((1 : z_2\lambda/z_1\lambda), z_1\lambda/|z_1\lambda|) = ((1 : z_2/z_1), z_1\lambda/|z_1|) = \phi_1^{-1}((z_1, z_2))\lambda$$

where we used that $|\lambda| = 1$. Analogously for ϕ_2. □

Definition 2.10. Suppose Φ_i and Φ_j are local trivialisations of a principal G-bundle $\pi : E \to M$.

$$\begin{aligned}\Phi_i :&U_i \times G \to \pi^{-1}(U_i)\\ \Phi_j :&U_j \times G \to \pi^{-1}(U_j)\end{aligned}$$

For $p \in U_i \cap U_j$ and the identity element $e \in G$ both $\Phi_i(p, e)$ and $\Phi_j(p, e)$ lie on the same fibre $\pi^{-1}(p)$. Hence there is an element $t_{ji}(p) \in G$ such that $\Phi_i(p, e) = \Phi_j(p, e)t_{ji}(p) = \Phi_j(p, t_{ji}(p))$. We call $t_{ji} : U_i \cap U_j \to G$ a *transition function.* The transition function is continuous since $t_{ji}(p) = \pi_2 \circ \Phi_j^{-1} \circ \Phi_i \circ \iota(p)$ where $\pi_2 : U_j \times G \to G$ denotes the projection onto the second component and ι is the inclusion $\iota : U_i \to U_i \times G, p \mapsto (p, e)$. Note that for arbitrary $g \in G$ we have $\Phi_i(p, g) = \Phi_j(p, t_{ji}(p)g)$. For smooth principal bundles the transition functions are smooth.

Example 2.11. We now want to calculate the transition functions for the local trivialisations we chose for the Hopf bundle in Example 2.9. By definition, for $x \in U_1 \cap U_2$: $\phi_1(x,1) = \phi_2(x,1)t_{21}(x)$, where t_{21} is one of the transition functions we are looking for. Elements of $U_1 \cap U_2$ are of the form $(z_1 : z_2)$ where $z_1, z_2 \neq 0$. We have

$$\begin{aligned}\phi_1((z_1 : z_2), 1) &= \frac{1}{\sqrt{1+|z_2|^2/|z_1|^2}}(1, z_2/z_1)\\ \phi_2((z_1 : z_2), 1) &= \frac{1}{\sqrt{1+|z_1|^2/|z_2|^2}}(z_1/z_2, 1)\end{aligned}$$

Thus we get for $z = z_2/z_1$

$$t_{21}((1:z)) = t_{21}((z_1 : z_2)) = \frac{\sqrt{1+|z_1|^2/|z_2|^2}}{\sqrt{1+|z_2|^2/|z_1|^2}}\frac{z_2}{z_1} = \frac{|z_1|}{z_1}\frac{z_2}{|z_2|} = \frac{z}{|z|}$$

and hence $t_{12}((1:z)) = |z|/z$.

Remark 2.12. (see Dupont (1978) Chapter 3) Let G be a Lie group and M a smooth manifold. Suppose there is an open cover $\{U_\alpha\}$ of M together with transition functions satisfying $t_{\gamma\beta}t_{\beta\alpha} = t_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and $t_{\alpha\alpha} = 1$ on U_α. Then there is a principal bundle with total space

$$E := \left(\coprod_\alpha U_\alpha \times G\right) / \sim$$

where $(p, g) \in U_\alpha \times G \sim (p, t_{\beta\alpha}(p)g) \in U_\beta \times G$ for all $p \in U_\alpha \cap U_\beta$ and $g \in G$.

Definition 2.13. Let $\pi : E \to M$ be a (smooth) principal G-bundle, N a topological space (smooth manifold) and $f : N \to M$ a continuous (smooth) map. The *pullback* is the (smooth) principal G-bundle $f^*\pi : f^*E \to N$ with $f^*E = \{(q, e) \mid f(q) = \pi(e)\} \subset N \times E$ and $f^*\pi(q, e) = q$, where the right G-action is given by $(q, e)g = (q, eg)$.

Proposition 2.14. *Let $\pi : E \to M$ be a principal G-bundle and $f_1, f_2 : N \to M$ homotopic maps. Then the pullback bundles f_1^*E and f_2^*E are isomorphic.*

For a proof see Cohen (2002) Chapter 2.1.

One can show that for homotopy equivalent spaces X and Y the principal G-bundles over X are in 1 to 1 correspondence with the principal G-bundles over Y. One can formulate this by introducing universal G-bundles. Let $Prin_G(X)$ denote the set of isomorphism classes of principal G-bundles over X. Let $\pi : E \to B$ be a principal G-bundle, where B is a connected space. For every continuous map $f : X \to B$ the pullback construction gives an element of $Prin_G(X)$. By Proposition 2.14 the pullback construction induces a map $[X, B] \to Prin_G(X)$.

Definition 2.15. A principal G-bundle $p : E \to B$ is called *universal* if for every space X the pullback construction

$$[X, B] \to Prin_G(X)$$

is a bijection.

The hard bit is then to prove the existence of a universal bundle.

Theorem 2.16. *For any topological group G there exists a universal G-bundle.*

This is Theorem 2.21 in Cohen (2002). From this we obtain the following result.

Corollary 2.17. *If X and Y are homotopy equivalent spaces and G is a topological group then the sets $Prin_G(X)$ and $Prin_G(Y)$ are isomorphic.*

Proof. By Theorem 2.16 there is a universal G-bundle $p : E \to B$. By the universal property we have $Prin_G(X) \cong [X, B]$ and $Prin_G(Y) \cong [Y, B]$. Since X and Y are homotopy equivalent, we have $[X, B] \cong [Y, B]$ and thus $Prin_G(X) \cong Prin_G(Y)$. □

From now on we implicitly assume all principal bundles to be smooth.

Definition 2.18. Let H and G be two Lie groups and suppose $\alpha : H \to G$ is a Lie group homomorphism. Let $\pi : F \to M$ be a principal H-bundle and let $\zeta : E \to M$ be a principal G-bundle. We say that F is a *reduction* of E or E is an *extension* of F relative to α if there is a differentiable map $\varphi : F \to E$ such that

i) $\varphi(F_p) \subset E_p$ for all $p \in M$ and

ii) $\varphi(uh) = \varphi(u)\alpha(h)$ for all $h \in H$ and $u \in F$.

Proposition 2.19. *A principal $U(n)$-bundle can be extended to a principal $GL(n, \mathbb{C})$-bundle.*

Proof. Let $\pi : F \to M$ be a principal $U(n)$-bundle with local trivialisations $\phi_i : U_i \times U(n) \to \pi^{-1}(U_i) \subset F$ and transition functions $t_{ji} : U_i \cap U_j \to U(n)$. The inclusion $\iota : U(n) \to GL(n, \mathbb{C})$ is a Lie group homomorphism. Let $\iota_i : U_i \times U(n) \to U_i \times GL(n, \mathbb{C})$ denote the inclusions. Fist we want to construct a principal $GL(n, \mathbb{C})$-bundle. Define $E := (\coprod_i U_i \times GL(n, \mathbb{C})) / \sim$ where $(p, g) \in U_i \times GL(n, \mathbb{C}) \sim (p, t_{ji}(p)g) \in U_j \times GL(n, \mathbb{C})$ for all $p \in U_i \cap U_j$ and $g \in GL(n, \mathbb{C})$. The right action of $GL(n, \mathbb{C})$ on E is defined through $[(p, g)]h = [(p, gh)]$ and a projection ζ from E to M is given through $[(p, g)] \mapsto p$. This makes E a principal $GL(n, \mathbb{C})$-bundle with local trivialisations $\psi_i : U_i \times GL(n, \mathbb{C}) \to \zeta^{-1}(U_i), (p, g) \mapsto [(p, g)]$. The transition functions for the ψ_i are the same as for the local trivialisations ϕ_i of F.

Now define $\varphi_i : \pi^{-1}(U_i) \to \zeta_i^{-1}(U_i)$ as $\varphi_i := \psi_i \circ \iota_i \circ \phi_i^{-1}$. For $u \in \pi^{-1}(U_i \cap U_j)$ write $\phi_i^{-1}(u) = (p, g)$. By definition of the transition functions we have $\phi_j^{-1}(u) = (p, t_{ji}(p)g)$. Hence, $\varphi_j(u) = \psi_j(p, t_{ji}(p)g) = \psi_i(p, g) = \varphi_i(u)$. Thus we get a well defined map $\varphi : F \to E$ with $\varphi|_{U_i} = \varphi_i$. This map φ satisfies the properties from Definition 2.18, so E is an extension of F. □

2.2 Connections

Definition 2.20. Let E be a principal G-bundle. Let $\mathfrak{g}$ be the Lie algebra of G. For any $a \in \mathfrak{g}$ and $u \in E$, we have a curve through u defined by $u \exp(ta)$. Hence, $\xi_u(a) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \exp(ta) \in T_uE$. The vector field $\xi(a)$ is called the *fundamental vector field* generated by a.

Remark 2.21. The curve $u \exp(ta)$ lies in one fibre E_p, where $\pi(u) = p$. Thus $\xi_u(a)$ belongs to $T_u(E_p)$. Moreover, note that ξ_u is linear. Since G acts freely on E_p, the map ξ_u is injective. Now $T_u(E_p)$ and $\mathfrak{g}$ have the same dimension, thus $\xi_u : \mathfrak{g} \to T_u(E_p)$ is an isomorphism. We denote its inverse by ι_u.

Proposition 2.22. *The map $\xi : \mathfrak{g} \to \Gamma(TE)$ is a Lie algebra homomorphism, i.e.* $\xi([a, b]) = [\xi(a), \xi(b)]$.

For a proof see Dupont (1978) Chapter 3.

Example 2.23. We want to calculate the fundamental vector field of the Hopf bundle. Let $z \in S^3$. If we write $z = (z_1, z_2)$ for complex numbers z_1 and z_2, the action of $U(1)$ is given by $(z_1, z_2)\lambda = (z_1\lambda, z_2\lambda)$. We can express z through real coordinates, $z = (x_1, x_2, x_3, x_4) = (\mathrm{Re}(z_1), \mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2))$. Then the action is given by $(x_1, x_2, x_3, x_4)(a + ib) = (x_1a - x_2b, x_1b + x_2a, x_3a - x_4b, x_3b + x_4a)$. Moreover, note that $Lie(U(1)) = i\mathbb{R}$.

Now we calculate the fundamental vector field. Let $a \in \mathbb{R}$ and $(x_1, x_2, x_3, x_4) \in S^3$. Using $\exp(ita) = \cos(ta) + i \sin(ta)$ we get

$$\begin{aligned}
&\xi(ia)((x_1, x_2, x_3, x_4)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (x_1, x_2, x_3, x_4) \exp(ita) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (x_1 \cos(ta) - x_2 \sin(ta),\\
&x_1 \sin(ta) + x_2 \cos(ta), x_3 \cos(ta) - x_4 \sin(ta), x_3 \sin(ta) + x_4 \cos(ta))\\
&= a(-x_2, x_1, -x_4, x_3) = a\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}\right)
\end{aligned}$$

Definition 2.24. Let E be a principal G-bundle. Let R_g be the right action $R_g : E \to E, u \mapsto ug$. Let ad_g be the action on $\mathfrak{g}$ induced by the adjoint action on G.

Definition 2.25. Let E be a principal G-bundle. Let $u \in E$ and $p = \pi(u)$. The *vertical subspace* V_uE is the subspace of T_uE which is tangential to the fibre E_p.

With this notation Remark 2.21 states that $\xi_u : \mathfrak{g} \to V_uE$ is an isomorphism.

Definition 2.26. Let E be a principal G-bundle. A *connection on E* is a unique separation of the tangent space T_uE into the vertical subspace V_uE and the horizontal subspace H_uE such that

(i) $T_uE = H_uE \bigoplus V_uE$ for all $u \in E$

(ii) A smooth vector field X on E is separated into smooth vector fields $X_u^H \in H_u E$ and $X_u^V \in V_u E$ as $X = X^H + X^V$

(iii) $H_{ug}E = (H_u E)g = R_{g*}(H_u E)$ for all $u \in E$ and $g \in G$.

Definition 2.27. Let E be a principal G-bundle and let $\mathfrak{g}$ be the Lie algebra of G. A connection form ω is a $\mathfrak{g}$ valued one form, i.e. $\omega \in \mathfrak{g} \otimes T^*E$ such that

(i) $\omega(\xi(a)) = a$ for all $a \in \mathfrak{g}$

(ii) $R_g^* \omega = Ad_{g^{-1}}\omega$

Note that $V_u E = \{\xi(\omega(v)) \mid v \in T_u E\}$. Interpreting ω as projection onto the vertical subspace gives a link between connection forms and connections:

Proposition 2.28. *Connection forms and connections are equivalent in the following sense: A connection form ω defines a connection through $H_u E := \{v \in T_u E \mid \omega(v) = 0\}$. Conversely, given a connection on a principal G-bundle E, we can define a connection one form through $\omega_u := \iota_u \circ V_u$ where $V_u : T_u E \to V_u E$ is the projection onto the vertical subspace.*

For a proof see Westenholz (1978) Chapter 11 or Nakahara (2003) Section 10.1.

Remark 2.29. For principal bundles over paracompact differentiable manifolds connections always exists. This can be shown making use of the local triviality condition and a partition of unity (see Corollary 3.11 in Dupont (1978)).

We now want to define a connection on the Hopf bundle. A connection corresponds to a projection onto the vertical subspace. In this case, the vertical subspace is one dimensional and spanned by the vector $-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$, as calculated in Example 2.23. So up to some factor we want our connection to be the dual of this vector, i.e. $-x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2 - x_4\mathrm{d}x_3 + x_3\mathrm{d}x_4$.

Example 2.30. Let $\tilde{\omega} = i(x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1 + x_3\mathrm{d}x_4 - x_4\mathrm{d}x_3)$ be a $Lie(U(1)) = i\mathbb{R}$ valued 1-form on $\mathbb{R}^4$. We restrict $\tilde{\omega}$ to S^3 and call it ω. We claim that ω is a connection on the Hopf bundle.

Proof. We have to show that

i) $\omega(\xi(a)) = a$ for all $a \in i\mathbb{R}$ and that

ii) $R_g^*\omega = Ad_{g^{-1}}\omega$.

For i) recall from Example 2.23 that $\xi(a) = -ia\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}\right)$. Thus, $\omega(\xi(a)) = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) = a$.

For ii) we choose $x = (x_1, x_2, x_3, x_4) \in S^3$, $v = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} + v_4 \frac{\partial}{\partial x_4} \in T_x S^3$ and $\lambda = a + ib \in U(1)$. We observe that $(Ad_{\lambda^{-1}} \omega)_x(v) = \omega_x(v) = i(x_1 v_2 - x_2 v_1 + x_3 v_4 - x_4 v_3)$. On the other hand we have $x \cdot \lambda = (x_1 a - x_2 b, x_1 b + x_2 a, x_3 a - x_4 b, x_3 b + x_4 a)$. Thus $R_{\lambda *}(v) = (v_1 a - v_2 b, v_1 b + v_2 a, v_3 a - v_4 b, v_3 b + v_4 a)$. Hence $\omega_{x \cdot \lambda}(R_{\lambda *}(v)) = i((x_1 a - x_2 b)(v_1 b + v_2 a) - (x_1 b + x_2 a)(v_1 a - v_2 b) + (x_3 a - x_4 b)(v_3 b + v_4 a) - (x_3 b + x_4 a)(v_3 a - v_4 b)) = i(x_1 v_2 - x_2 v_1 + x_3 v_4 - x_4 v_3)$, where we used $a^2 + b^2 = 1$. Thus ω is a connection. □

Let $\pi : E \to M$ be a principal G-bundle and let $\{U_i\}_{i \in I}$ be a open cover of M such that there are local trivialisations $\phi_i : U_i \times G \to \pi^{-1}(U_i)$.

Definition 2.31. Suppose ω is a connection 1-form on E. Let $\sigma_i : U_i \to E$ be a local section. We define the *local connection 1-form* A_i on U_i as the pullback $A_i := \sigma_i^* \omega$.

Proposition 2.32. *Suppose ω is a connection 1-form on E. Let $\sigma_i : U_i \to E$ and $\sigma_j : U_j \to E$ be local sections. For all $p \in U_i \cap U_j$ the local forms $A_i = \sigma_i^* \omega$ and $A_j = \sigma_j^* \omega$ satisfy the compatibility condition*

$$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} \mathrm{d} t_{ij}, \tag{1}$$

where $t_{ij} : U_i \cap U_j \to G$ is the transition function.

For a proof see Nakahara (2003) Section 10.1.

Theorem 2.33. *Suppose that for all $i \in I$ there are local $\mathfrak{g}$-valued 1-forms A_i on U_i. Define the functions $g_i : \pi^{-1}(U_i) \to G$ through $\phi_i^{-1}(u) = (p, g_i(u))$. On $\pi^{-1}(U_i)$ define the 1-form $\omega_i = g_i^{-1} \pi^* A_i g_i + g_i^{-1} \mathrm{d} g_i$.*

i) If A_i and A_j satisfy Equation (1) for $p \in U_i \cap U_j$ then $\omega_i = \omega_j$ on $\pi^{-1}(U_i \cap U_j)$.

ii) Suppose that for every pair $i, j \in I$ for $p \in U_i \cap U_j$ Equation (1) is satisfied. Then there is a connection 1-form ω such that for all $i \in I$ we have $\sigma_i^* \omega = A_i$ and $\omega|_{\pi^{-1}(U_i)} = \omega_i$.

For a proof see Nakahara (2003) Section 10.1.

Example 2.34. For the Hopf bundle $h : S^3 \to S^2$ local connection forms are given by

$$A_1 = \frac{i}{2}(1 - \cos\theta)\mathrm{d}\varphi \quad \text{and} \quad A_2 = -\frac{i}{2}(1 + \cos\theta)\mathrm{d}\varphi$$

in spherical coordinates $(\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ on $U_1 := S^2 \setminus \{(0, 0, -1)\}$ and $U_2 := S^2 \setminus \{(0, 0, 1)\}$, respectively. For a proof see Shnir (2005) Section 3.3.

2.3 Curvature

Let $\Omega^r(M)$ denote the space of all differential r-forms on a manifold M. For a vector space V the space of V-valued r-forms on M is denoted by $V \otimes \Omega^r(M)$. The exterior derivative on differential forms can be extended to vector valued forms as follows: For pure tensors we set $\mathrm{d}(v \otimes \phi) = v \otimes \mathrm{d}\phi$ and extend this by linearity.

Definition 2.35. Let E be a principal G-bundle over M with a given connection. Let $\phi \in \mathfrak{g} \otimes \Omega^r(E)$ be a Lie algebra valued r-form on E. Let $u \in E$ and $X_1, ..., X_{r+1} \in T_uE$. Let $h_u : T_uE \to H_uE$ denote the projection onto the horizontal subspace. The *covariant derivative* of ϕ is given by

$$\mathrm{D}\phi(X_1, ..., X_{r+1}) = \mathrm{d}\phi(h(X_1), ..., h(X_{r+1}))$$

Definition 2.36. Let E be a principal G-bundle over M with a given connection. The *curvature 2-form* $\Omega \in \mathfrak{g} \otimes \Omega^2(E)$ is the covariant derivative of the connection one form ω.

Theorem 2.37. *(Cartan's structure equation) For $X, Y \in T_uE$ we have*

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)]. \tag{2}$$

For the proof see for example Nakahara (2003) Section 10.3.2.

Example 2.38. In Example 2.30 we saw that $\omega = i(x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 + x_3 \mathrm{d}x_4 - x_4 \mathrm{d}x_3)$ defines a connection 1-form on the Hopf bundle. We want to calculate the curvature 2-form using Cartan's structure equation. Since in this example the Lie algebra $U(1)$ is commutative the equation reduces to $\Omega = \mathrm{d}\omega$. Hence,

$$\begin{aligned}\Omega &= 2i(\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4) \\ &= \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_1 + \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_2,\end{aligned}$$

where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$.

Definition 2.39. Let U be a chart of M and $\sigma : U \to E$ a local section. The local form F of the curvature Ω is defined as $F = \sigma^*\Omega$.

Remark 2.40. Let $A = \sigma^*\omega$ be the local connection form. Since $\sigma^*\mathrm{d}\omega = \mathrm{d}(\sigma^*\omega)$ it follows from Equation (2) that

$$F(X, Y) = \mathrm{d}A(X, Y) + [A(X), A(Y)].$$

For more details see Nakahara (2003) Section 10.3.4.

Remark 2.41. Suppose the curvature form Ω can be written as $\Omega = \pi^*F$ for some $F \in \mathfrak{g} \otimes \Omega^2(M)$. Then for any local section $\sigma : U \to E$ we get $\sigma^*\Omega = (\pi \circ \sigma)^*F = F$ on U.

Theorem 2.42. *Let U_i and U_j be two charts of M and let F_i and F_j be local curvature forms on U_i and U_j coming from the same connection. Then on $U_i \cap U_j$ the following* *compatibility condition is satisfied:*

$$F_j = t_{ij}^{-1} F_i t_{ij},$$

where t_{ij} is the transition function.

This is shown in Nakahara (2003) Section 10.3.4.

Example 2.43. We now want to calculate a local curvature form on the Hopf bundle, namely for the chart $U_2 \cong \mathbb{C}$. Recall the projection $g : S^3 \to \mathbb{C}P^1, (z_1, z_2) \mapsto (z_1 : z_2)$. For $z_2 \neq 0$ we may write $(z_1 : z_2) = (z_1/z_2 : 1) = (z : 1)$.

We make the ansatz $F = f(z)\mathrm{d}\bar{z} \wedge \mathrm{d}z$ and want $\Omega = g^*F$. We calculate $g^*\mathrm{d}z = \mathrm{d}(z_1/z_2) = (z_2\mathrm{d}z_1 - z_1\mathrm{d}z_2)/z_2^2$. Thus $g^*(\mathrm{d}\bar{z} \wedge \mathrm{d}z) = (|z_2|^2\mathrm{d}\bar{z_1} \wedge \mathrm{d}z_1 + |z_1|^2\mathrm{d}\bar{z_2} \wedge \mathrm{d}z_2 - \bar{z_2}z_1\mathrm{d}\bar{z_1} \wedge \mathrm{d}z_2 - \bar{z_1}z_2\mathrm{d}\bar{z_2} \wedge \mathrm{d}z_1)/|z_2|^4$. Now note that $1 = |z_1|^2 + |z_2|^2 = z_1\bar{z_1} + z_2\bar{z_2}$ implies $0 = z_1\mathrm{d}\bar{z_1} + \bar{z_1}\mathrm{d}z_1 + z_2\mathrm{d}\bar{z_2} + \bar{z_2}\mathrm{d}z_2$. Thus we get $\bar{z_2}z_1\mathrm{d}\bar{z_1} \wedge \mathrm{d}z_2 = -\bar{z_2}\bar{z_1}\mathrm{d}z_1 \wedge \mathrm{d}z_2 - |z_2|^2\mathrm{d}\bar{z_2} \wedge \mathrm{d}z_2$ and $\bar{z_1}z_2\mathrm{d}\bar{z_2} \wedge \mathrm{d}z_1 = -|z_1|^2\mathrm{d}\bar{z_1} \wedge \mathrm{d}z_1 - \bar{z}_1\bar{z}_2\mathrm{d}z_2 \wedge \mathrm{d}z_1$. So we actually have $g^*(\mathrm{d}\bar{z} \wedge \mathrm{d}z) = (|z_1|^2 + |z_2|^2)(\mathrm{d}\bar{z_1} \wedge \mathrm{d}z_1 + \mathrm{d}\bar{z_2} \wedge \mathrm{d}z_2)/|z_2|^4 = (|z_1|^2 + |z_2|^2)\Omega/|z_2|^4$. Since $1 = |z_1|^2 + |z_2|^2$ we can multiply the equation by this to get $g^*(\mathrm{d}\bar{z} \wedge \mathrm{d}z) = (|z_1/z_2|^2 + 1)^2\Omega = (|z|^2 + 1)^2\Omega$. Therefore, choosing

$$F = \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{(1 + |z|^2)^2}$$

gives us the desired property $\Omega = g^*F$.

3 Associated bundles

Proposition 3.1. *Suppose G is a Lie group acting on the left on some manifold X. Let E be a principal bundle. We define an equivalence relation on $E \times X$ as follows: for any $g \in G$: $(u, x) \sim (ug^{-1}, gx)$. Then the quotient space $\chi := E \times_G X := (E \times X)/\sim$ is a fibre bundle with fibre X. This fibre bundle is called the associated fibre bundle.*

For a proof see (Westenholz; 1978) Chapter 6.

Definition 3.2. A *vector bundle* is a fibre bundle where the fibre is a vector space and every local trivialisation Φ_i is a linear isomorphism of vector spaces.

Proposition 3.3. *Suppose E is a principal G-bundle and V is a vector space. Let $\rho : G \to GL(V)$ be a representation of G. Then $E \times_\rho V := (E \times V)/\sim$ where $\forall g \in G$: $(u, v) \sim (ug^{-1}, \rho(g)v)$ is a vector bundle with fibre V. It is called the associated vector bundle.*

This is shown in Nakahara (2003) Section 9.4.2.

We can also construct a principal bundle from a vector bundle:

Definition 3.4. Suppose $\pi : P \to M$ is a vector bundle with fibres $P_p \cong \mathbb{R}^k$. For every fibre P_p of P we denote the set of all (ordered) bases by F_p. We define the *frame bundle* as $E := \bigcup_{p \in M} \{p\} \times F_p$ together with the projection $\pi_E : E \to M$ onto the first component. The differentiable structure on E is given as follows. First note that the bases of $\mathbb{R}^k$ are exactly the elements of $GL(k, \mathbb{R})$ when we identify the basis vectors with the columns of a matrix. Thus if $\phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is a local trivialisation of P we get a map $\phi_E : U \times GL(k, \mathbb{R}) \to \pi_E^{-1}(U)$ given by $(p, (v_1, .., v_k)) \mapsto (p, (b_1, ..., b_k))$, where $\phi(p, v_i) = b_i$. Demanding the ϕ_E to be local trivialistations of E (i.e. diffeomorphisms) gives us the differentiable structure on E.

Proposition 3.5. *Let P be a vector bundle over M with fibre V. Then the frame bundle is a principal $GL(V)$-bundle over M, where $GL(V)$ acts on the bases by matrix multiplication from the right.*

This is shown in Westenholz (1978) Chapter 6.

Remark 3.6. If the vector bundle carries a metric one can also define orthonormal and unitary frame bundles by restricting to orthonormal bases and $O(k)$ or unitary bases and $U(k)$, respectively. These bundles then are reductions of the $GL(k, \mathbb{R})$ frame bundle. (see Proposition 2.19).

Example 3.7. Let E be the Hopf bundle and $V = \mathbb{C}$. We want to examine the associated bundle $\chi = S^3 \times_{U(1)} \mathbb{C}$. According to Proposition 3.3, χ is a one dimensional complex vector bundle over S^2. Recall f and U_1 from Example 2.9. We define the map $\bar{f} : S^3 \times_{U(1)} \mathbb{C} \to \mathbb{C}P^1, [(p, z)] \mapsto f(p)$. This map is well defined, since $f(\lambda p) = f(p)$ for $\lambda \in \mathbb{C}$. We have $\bar{f}^{-1}(U_1) = \{[(z_1, z_2, z)] \mid (z_1, z_2) \in S^3, z \in \mathbb{C}, z_1 \neq 0\}$. A local trivialisation is given by

$$\begin{aligned} \Phi_1 : \quad & U_1 \times \mathbb{C} && \to \bar{f}^{-1}(U_1) \\ & ((z_1 : z_2), z) && \mapsto \left[\left(\frac{1}{\sqrt{1 + |z_2|^2/|z_1|^2}}, \frac{z_2}{z_1 \sqrt{1 + |z_2|^2/|z_1|^2}}, z \right) \right] \end{aligned}$$

The two sided inverse is given by

$$\Phi_1^{-1} : [(z_1, z_2, z)] \mapsto \left((z_1 : z_2), \frac{z_1 z}{|z_1|} \right)$$

The inverse is well defined since

$$\Phi_1^{-1}([(z_1 \lambda, z_2 \lambda, z)]) = \left((z_1 \lambda : z_2 \lambda), \frac{z_1 \lambda z}{|z_1 \lambda|} \right) = \left((z_1 : z_2), \frac{z_1 \lambda z}{|z_1|} \right) = \Phi_1^{-1}([(z_1, z_2, \lambda z)])$$

Analogously one can find a local trivialisation $\Phi_2 : U_2 \times \mathbb{C} \to \bar{f}^{-1}(U_2)$.

3.1 Chern classes

Definition 3.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $I^k(G)$ be the vector space of all symmetric multilinear maps $f : \mathfrak{g}^k \to \mathbb{R}$ which satisfy $f(ad_g a_1, ..., ad_g a_k) = f(a_1, ..., a_k)$ for all $g \in G$ and $a_i \in \mathfrak{g}$.

Now set $I(G) := \bigoplus_{k=0}^{\infty} I^k(G)$. For $f \in I^k(G)$ and $g \in I^l(g)$ we define their product as

$$fg(a_1, ..., a_{k+l}) := \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} f(a_{\sigma(1)}, ...a_{\sigma(k)}) g(a_{\sigma(k+1)}, ..., a_{\sigma(k+l)}),$$

where S_{k+l} is the group of permutations of $k+l$ elements. This turns $I(G)$ into a commutative algebra.

For $f \in I^k(G)$ and a 2-form Ω on M define the $2k$-form $f(\Omega)$ on M as

$$f(\Omega)(X_1, ..., X_{2k}) := \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} \operatorname{sgn}(\sigma) f(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), ..., \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})).$$

Theorem 3.9. *(Chern-Weil) Let $\pi : E \to M$ be a principal G-bundle with connection 1-form ω and corresponding curvature 2-form Ω.*

i) For every $f_k \in I^k(G)$ there is a unique closed $2k$-form $\bar{f}(\Omega)$ on M such that $f_k(\Omega) = \pi^* \bar{f}(\Omega)$.

ii) The cohomology class in the de Rham cohomology $[\bar{f}(\Omega)] \in H^{2k}(M)$ is independent of the choice of the connection on E.

iii) The map $w_E : I(G) \to H^*(M), f \mapsto [\bar{f}(\Omega)]$ is a natural algebra homomorphism, i.e. if $g : F \to E$ is a bundle homomorphism of principal G-bundles, then $g^* w_E = w_F$.

For the proof see (Kobayashi and Nomizu; 1969) Chapter 12 or (Greub et al.; 1973) Chapter 6.

Definition 3.10. Let $a_1, ..., a_r$ be a basis of $\mathfrak{g}$. A map $p : \mathfrak{g} \to \mathbb{R}$ is called invariant polynomial function if $p(\sum_{i=1}^{r} t_i a_i)$ is a polynomial in $t_1, ..., t_r$ and if $p(ad_g v) = p(v)$ for all $g \in G$ and $v \in \mathfrak{g}$.

The set $P(\mathfrak{g})$ of invariant polynomial functions on $\mathfrak{g}$ forms an algebra.

Proposition 3.11. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The algebra of invariant symmetric multilinear mappings $I(G)$ is isomorphic to the algebra $P(\mathfrak{g})$ of invariant polynomial functions on $\mathfrak{g}$.*

A proof is given in Kobayashi and Nomizu (1969) Chapter 12. The idea is to consider the algebra homomorphism $\phi : I(G) \to P(\mathfrak{g})$ given by $(\phi f)(v) = f(v, ..., v)$ for $f \in I^k(G)$ and $v \in \mathfrak{g}$. Through a polarization argument an inverse to ϕ can be defined.

Example 3.12. Suppose $G < GL(r;\mathbb{C})$ is a matrix Lie group with Lie algebra $\mathfrak{g}$. Then $p(A) := \det(I + \frac{i}{2\pi}A)$ is an invariant polynomial. Invariance follows form $p(ad_g A) = \det(I + \frac{i}{2\pi}ad_g A) = \det(g(I + \frac{i}{2\pi}A)g^{-1}) = \det(I + \frac{i}{2\pi}A) = p(A)$ and p is a polynomial since the determinant is a polynomial in the entries of the matrix it acts on.

Definition 3.13. Let $\pi : E \to M$ be a principal $G = GL(r, \mathbb{C})$-bundle. We can define invariant symmetric multilinear maps $f_k \in I^k(G)$ for $0 \leq k \leq r$ through

$$\det\left(I + \frac{i}{2\pi}A\right) = \sum_{k=0}^{r}(\phi f_k)(A)$$

for $A \in \mathfrak{g}$. For $k > r$ we let $f_k = 0$. We call $c_k(E) := w_E(f_k)$ the *k'th Chern class.* For a complex vector bundle we define the Chern class as the Chern class of the associated frame bundle. For a principal H-bundle $\zeta : F \to M$ which can be extended to a principal G-bundle relative to $\alpha : H \to G$, we define the k'th Chern class as $c_k(F) := w_F(\alpha^* f_k)$, where $\alpha^* f_k(v_1, ..., v_k) = f_k(\alpha_* v_1, ..., \alpha_* v_k)$ for all v_i in the Lie algebra of H.

Corollary 3.14. *The Chern classes c_k of a trivial bundle are zero for $k > 0$.*

Proof. This proof is based on Nakahara (2003) Section 11.1. Let $\pi : P \to M$ be a trivial principal G-bundle. There is a principal bundle isomorphism f from P to the trivial bundle $E = M \times G$. Let $F = \{p\} \times G$ be the trivial principal G-bundle over $\{p\}$. Let $g : E \to F$ be the map given through $g(q, g) = (p, g)$. This is a principal bundle homomorphism. The space $\Omega^k(\{p\})$ of differential forms over $\{p\}$ is zero for $k \geq 1$. Thus also the Chern classes $c_k(F) = 0$. By Theorem 3.9 we have $c_k(P) = (g \circ f)^* c_k(F) = 0$, because $(g \circ f)^*$ is a ring homomorphism. $\square$

Example 3.15. By Proposition 2.19 the Hopf bundle E can be extended to a principal $GL(1, \mathbb{C}) = \mathbb{C}^*$-bundle and hence we can calculate its Chern classes. Let Ω denote the curvature 2-form on E from Example 2.38. We have $\det(1 + \frac{i}{2\pi}v) = 1 + \frac{i}{2\pi}v$ for $v \in \mathbb{C} = Lie(\mathbb{C}^*)$. Thus $f_0 = 1$ and $f_1(v) = \frac{i}{2\pi}v$ and $f_k(v) = 0$ for $k > 1$. By definition we have $c_1(P) = w_P(\iota^* f_1)$, where $\iota^* f_1(v) = f_1(\iota_* v) = f_1(v)$ for $v \in i\mathbb{R}$. Moreover, $f_1(\Omega)(v_1, v_2) = \frac{1}{2}(f_1(\Omega(v_1, v_2)) - f_1(\Omega(v_2, v_1)))$$=f_1(\Omega(v_1, v_2))$, where we used that $\Omega(v_1, v_2) = -\Omega(v_2, v_1)$. Let $\theta \in \Omega^2(\mathbb{C}P^1)$ be the unique 2-form with $\pi^*\theta(v_1, v_2) = f_1(\Omega)(v_1, v_2) = \frac{i}{2\pi}\Omega(v_1, v_2)$. We see that $\theta = \frac{i}{2\pi}F$ where F is the 2-form calculated in Example 2.43 satisfying $\pi^* F = \Omega$.

For the Chern classes we thus get that $c_0(P) = 1$, $c_1(P)$ is the cohomology class of $\frac{i}{2\pi}F$ and for $k > 1$ we have $c_k(P) = 0$.

We now want to show that $c_1(P)$ is not trivial. If two elements α, β of $\Omega^2(\mathbb{C}P^1)$ lie in the same cohomology class, they differ by a boundary, i.e. $\alpha - \beta = \mathrm{d}\eta$. Thus by Stokes' Theorem we have

$$\int_{\mathbb{C}P^1} \alpha - \int_{\mathbb{C}P^1} \beta = \int_{\mathbb{C}P^1} \mathrm{d}\eta = \int_{\partial\mathbb{C}P^1} \eta = 0,$$

since $\mathbb{C}P^1$ has empty boundary as it is diffeomorphic to S^2. Therefore, to prove that the first Chern class of the Hopf bundle is not zero it is enough to show

$$\int\limits_{\mathbb{C}P^1} \theta \neq 0.$$

From Example 2.43 we know that on $\mathbb{C}P^1 \setminus \{(z:0)\} = U_2 \cong \mathbb{C}$ we have $F = \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{(1+|z|^2)^2}$. With $z = re^{i\varphi}$ we have $\mathrm{d}z = e^{i\varphi}\mathrm{d}r + ire^{i\varphi}\mathrm{d}\varphi$ and $\mathrm{d}\bar{z} = e^{-i\varphi}\mathrm{d}r - ire^{-i\varphi}\mathrm{d}\varphi$ which gives $\mathrm{d}\bar{z} \wedge \mathrm{d}z = 2ir\mathrm{d}r \wedge \mathrm{d}\varphi$. Since $\{(z:0)\}$ is just a point in $\mathbb{C}P^1$ we get

$$\begin{aligned}\int\limits_{\mathbb{C}P^1} \theta = \frac{i}{2\pi} \int\limits_{U_1} F = \frac{i}{2\pi} \int\limits_0^\infty \int\limits_0^{2\pi} \frac{2ir}{(1+|z|^2)^2} \mathrm{d}\phi \mathrm{d}r = -2 \int\limits_0^\infty \frac{r}{(1+r^2)^2} \mathrm{d}r \\ \overset{s=r^2}{=} - \int\limits_0^\infty \frac{1}{(1+s)^2} \mathrm{d}s = \left[\frac{1}{1+s} \right]_{s=0}^{s=\infty} = -1.\end{aligned}$$

Since $c_1(E)$ is not trivial, the Hopf bundle is not trivial by Corollary 3.14.

3.2 Covariant derivative

Definition 3.16. A *covariant derivative* on a vector bundle $\pi : E \to M$ is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), (X,Y) \mapsto \nabla_X Y := \nabla(X,Y)$ with the following properties:

i) $C^\infty(M)$-linearity in the first component

ii) Linearity in the second component

iii) Leibnitz rule: For $f \in C^\infty(M)$ $\nabla_X fY = f\nabla_X Y + X(f)Y$

Now let $\pi : E \to M$ be a principal G-bundle with connection 1-form ω.

Definition 3.17. Let $\gamma : [0,1] \to M$ be a curve in M. A curve $\tilde{\gamma}$ is called *horizontal lift* of γ if $\pi \circ \tilde{\gamma} = \gamma$ and for all $t \in [0,1]$ the tangent vector $\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}(t)$ lies in $H_{\tilde{\gamma}(t)}E$.

The following two Theorems are proved in Nakahara (2003) Section 10.4.

Theorem 3.18. *Let $\gamma : [0,1] \to M$ be a curve. For every $u \in \pi^{-1}(\gamma(0))$ there exists a unique horizontal lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = u$.*

Theorem 3.19. *If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two horizontal lifts of γ, they satisfy $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$ for some $g \in G$. Then for all $t \in [0,1]$ also $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$.*

Let $E = P \times_\rho V$ be a vector bundle associated to P.

Definition 3.20. Let $\gamma : [0,1] \to M$ be a curve in M. Parallel transport along γ is a map $\tau_\gamma(t) : \pi_E^{-1}(\gamma(0)) \to \pi_E^{-1}(\gamma(t))$ given through

$$\tau_\gamma(t)([\tilde{\gamma}(0), v]) = [\tilde{\gamma}(t), v],$$

where $\tilde{\gamma}$ is a horizontal lift of γ.

Remark 3.21. The parallel transport is well defined, i.e. it is independent of the choice of the horizontal lift: If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two horizontal lifts of γ then $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$ by Theorem 3.19. Thus if $u = [\tilde{\gamma}'(0), v] = [\tilde{\gamma}(0)g, v] = [\tilde{\gamma}(0), \rho(g)v]$ then $\tau_\gamma(t)(u) = \tau_\gamma(t)([\tilde{\gamma}'(0), v]) = [\tilde{\gamma}'(t), v] = [\tilde{\gamma}(t), \rho(g)v] = \tau_\gamma(t)([\tilde{\gamma}(0), \rho(g)v]) = \tau_\gamma(t)(u)$. Moreover, parallel transport is a linear isomorphism for every t and hence invertible.

Definition 3.22. Let $p \in M$ let $X \in \Gamma(TM)$ and $Y \in \Gamma(E)$ be vector fields and let $\gamma : [0,1] \to M$ be an integral curve of X_p such that $\gamma(0) = p$. The covariant derivative of Y with respect to X at the point p is defined as

$$(\nabla_X Y)_p = \lim_{t \to 0} \frac{1}{t} \left(\tau_{\gamma(t)}^{-1} Y(\gamma(t)) - Y(\gamma(0)) \right)$$

Theorem 3.23. *The covariant derivative from Definition 3.22 satisfies the properties of a covariant derivative on a vector bundle given in Definition 3.16.*

This is proved in Morrison (2000).

4 Principal bundles in physics

4.1 Magnetic monopoles

This section follows some ideas presented in Nakahara (2003) Sections 1.9 and 10.5. For a magnetic monopole the magnetic field would satisfy $\nabla \cdot \vec{B}(\vec{x}) = 4\pi g \delta^3(\vec{x})$. The solution is $\vec{B} = g\frac{\vec{r}}{r^3}$. Now we want to calculate the magnetic vector potential such that $\vec{B} = \nabla \times \vec{A}$. There cannot be any global expression for $\vec{A}$ with no singularities since we would get a contradiction:

$$4\pi g = \int\limits_{S^2} \nabla \cdot \vec{B} \mathrm{d}^3 x = \int\limits_{S^2} \nabla \cdot (\nabla \times \vec{A}) \mathrm{d}^3 x = 0.$$

It is possible, however, to write down local solutions $\vec{A}_1$ and $\vec{A}_2$ on $U_1 := \mathbb{R}^3 \setminus \{(0,0,z) | z \leq 0\}$ and $U_2 := \mathbb{R}^3 \setminus \{(0,0,z) | z \geq 0\}$, respectively. They are given by

$$\vec{A}_1 = \frac{g}{r(r+z)} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{g(1 - \cos\theta)}{r \sin\theta} e_\varphi$$

and

$$\vec{A}_2 = \frac{g}{r(r-z)}\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} = -\frac{g(1+\cos\theta)}{r\sin\theta}e_\varphi,$$

where $e_\varphi = -\sin\varphi e_x + \cos\varphi e_y$ for the unit vectors e_x and e_y in x and y direction, respectively. On $U_1 \cap U_2$ we have $A_1 - A_2 = \nabla(2g\varphi)$. Now consider a particle of mass m and charge e in this field. The time independent Schrödinger equation is $\frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2\psi(r) = E\psi(r)$. Under the change $\vec{A} \to \vec{A} + \nabla\Lambda$ the wave function changes as $\psi \to \exp(ie\Lambda/c\hbar)\psi$. So in our case with $\Lambda = 2g\varphi$,

$$\psi_1\Big|_{\varphi=0} = \psi_2\Big|_{\varphi=0} = \psi_2\Big|_{\varphi=2\pi} = \exp(ie2g2\pi/\hbar c)\psi_1\Big|_{\varphi=2\pi} = \exp(4\pi ge/\hbar c)\psi_1\Big|_{\varphi=0}.$$

Thus we obtain the condition that $4\pi ge/\hbar c = 2\pi k$ for some $k \in \mathbb{Z}$.

Now we want to formulate this situation using principal bundles. As base space we take $\mathbb{R}^3 \setminus \{0\}$ which is homotopy equivalent to S^2. We want the local connection forms to be

$$A_1 = i\widetilde{g}(1-\cos\theta)\mathrm{d}\varphi$$

$$A_2 = -i\widetilde{g}(1+\cos\theta)\mathrm{d}\varphi$$

where $\widetilde{g} = ge/\hbar c$. Up to the factor $e/\hbar c$ these 1-forms correspond to the vector potentials $\vec{A}_1$ and $\vec{A}_2$ since

$$r\sin\theta\mathrm{d}\varphi = -\sin\varphi\mathrm{d}x + \cos\varphi\mathrm{d}y.$$

Suppose A_1 and A_2 are local connection forms of a principal $U(1)$-bundle over S^2 with transition function $t_{12}(r,\theta,\varphi) = \exp(if(r,\theta,\varphi))$. The compatibility condition then implies $A_2 - A_1 = t_{12}^{-1}\mathrm{d}t_{12} = i\frac{\partial f}{\partial\varphi}\mathrm{d}\varphi + i\frac{\partial f}{\partial r}\mathrm{d}r + i\frac{\partial f}{\partial\theta}\mathrm{d}\theta$. But on the other hand we have $A_2 - A_1 = 2i\widetilde{g}\mathrm{d}\varphi$. Hence, $t_{12}(r,\theta,\varphi) = \exp(i2\varphi\widetilde{g})$ and thus $4\phi\widetilde{g} = 2\pi k$ for some $k \in \mathbb{Z}$, which is exactly the condition we derived from the physical perspective. The principal bundle is given by $E = U_1 \times U(1) \coprod U_2 \times U(1)/\sim$, where $U_1 \times U(1) \ni (x,\lambda) \sim (x, t_{21}(x)\lambda) \in U_2 \times U(1)$. The magnetic potential corresponds to the local curvature of this bundle.

Remark 4.1. The principal $U(1)$-bundles over S^2 can be classified by the homotopy class of the transition functions, see for example Theorem 2.7 in Cohen (2002). The construction of the monopole bundles hence produces all possible principal $U(1)$-bundles over S^2 up to isomorphism. For $\widetilde{g} = 1/2$ the principal bundle E is exactly the Hopf bundle, see Example 2.34. For $\widetilde{g} = 0$ this gives the trivial bundle.

4.2 Electromagnetism

Electromagnetism allows for a formulation in terms of principal bundles. We take the Minkowski space-time $\mathbb{R}^4$ as basespace and consider a $U(1)$-bundle E over it. Since $\mathbb{R}^4$

is contractible, the bundle is trivial $E = \mathbb{R}^4 \times U(1)$. A local connection form A is just a $i\mathbb{R}$-valued 1-form on $\mathbb{R}^4$:

$$A = \sum_{\mu} A_\mu \mathrm{d}x^\mu$$

The local curvature is then given by

$$F = \mathrm{d}A = \frac{1}{2}\sum_{\mu,\nu}\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right)\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu,$$

where $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. We know that $\mathrm{d}F = \mathrm{d}^2 A = 0$. Let us name the components of $F_{\mu\nu}$ in the following way:

$$F_{\mu\nu} = i\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

Then the equation $\mathrm{d}F = 0$ implies the two vacuum Maxwell equations

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{and} \quad \nabla \cdot \vec{B} = 0.$$

So thinking of the magnetic and the electric field as components of the curvature gives us two of the four vacuum Maxwell equations for free. One can derive the other two from minimizing the Maxwell action

$$S(A) = \frac{1}{4}\int_{\mathbb{R}^4} F_{\mu\nu}F^{\mu\nu}\mathrm{d}^4x,$$

see for example Nakahara (2003) Section 10.5.

4.3 Berry connection

In quantum mechanics one is interested in eigenstates of Hamiltonians. Multiplying a state with an element of $U(1)$ does not change the physical meaning. This gives us the motivation to study the following construction.

Proposition 4.2. *Let M be a smooth manifold and H a complex Hermitian (or real symmetric) $n \times n$ matrix depending smoothly on the parameter $p \in M$. Since the eigenvalues of $H(p)$ are real they can be ordered. Assume that for some open subset U of M for every $p \in U$ the eigenspace to the kth eigenvalue of $H(p)$ is one dimensional. Then one can define a principal $U(1)$-bundle (or an S^0-bundle) consisting of normalised eigenvectors to the kth eigenvalue of $H(p)$ for all $p \in U$.*

We call this bundle a *normalised eigenbundle* or *NE-bundle*.

Proof. Let us denote the eigenvalues of $H(p)$ by $\lambda_j(p)$ for $j \in \{1, ..., n\}$. The coefficients of the characteristic polynomial χ_p of $H(p)$ depend smoothly on $p \in M$. Since $\lambda_k(p)$ is a simple root of this polynomial, it is smooth by the implicit function theorem. The rational function $f_p(x) = \chi_p(x)/(x - \lambda_k(p))$ depends smoothly on p. Since $\chi_p(\lambda_k(p)) = 0$, the rational function $f_p(x)$ has no pole and is holomorphic. Moreover, we can write $f_p(x) = \prod_{j \neq k} (x - \lambda_j(p))$. Note that $\chi_p'(\lambda_k) = \prod_{j \neq k} (\lambda_k(p) - \lambda_j(p))$. Following an idea in Schwinger (2001) Chapter 1.7, for $p \in U$ we define the matrix

$$P(p) = \frac{f_p(H(p))}{\chi_p'(\lambda_k(p))} = \prod_{j \neq k} \frac{H(p) - I\lambda_j(p)}{\lambda_k(p) - \lambda_j(p)},$$

where I is the unit matrix of size n. Note that the different factors commute, so this product is well defined. This matrix $P(p)$ depends smoothly on $p \in U$ by our considerations above. Let E_p be the eigenspace to the eigenvalue λ_k at $p \in U$. Applying $P(p)$ to eigenvectors v_j of $H(p)$, we see that $P(p)v_k = v_k$ and $P(p)v_j = 0$ for $j \neq k$. Since the eigenvectors form a basis of $V = \mathbb{C}^n$ (or $\mathbb{R}^n$ in the real case) we see that the matrix $P(p)$ acts as a projection onto E_p.

Now let $s \in U$ and pick an eigenvector $v \in E_s$. Then $\|P(s)v\| = \|v\| > 0$ and since $\|P(p)v\|$ is a continuous function of p, there is an open neighbourhood $O \subset U$ of s, such that $P(p)v \neq 0$ for all $p \in O$. Now pick a $p \in O$. Since E_s and E_p are one dimensional, and $P(p)\big|_{E_s} : E_s \to E_p$ is a non-zero linear map, it is actually an isomorphism. We therefore obtain an isomorphism

$$O \times E_s \to \coprod_{p \in O} E_p, \quad (p, v) \mapsto P(p)v.$$

This is a local trivialisation of $E := \coprod_{p \in U} E_p$. Since $s \in U$ was arbitrary, we can find local trivialisations on all of U and thus E is a vector bundle. Now as described in Section 3 we can consider the unitary (or orthonormal) frame bundle associated to E. This gives us exactly the NE-bundle. □

Let $\pi : E \to M$ be a NE-bundle for some complex Hermitian matrix. For any local section $\sigma : U \to E$ we can define the 1-form $A(p) = \sigma(p)^\dagger \mathrm{d}\sigma(p)$ (see Nakahara (2003) Section 10.6).

Proposition 4.3. *Let $\{U_i\}$ be an open cover of M together with local sections $\sigma_i : U_i \to E$ and let $A_i(p) = \sigma_i(p)^\dagger \mathrm{d}\sigma_i(p)$.*

i) There exists a connection 1-form ω such that $\sigma_i^* \omega = A_i$, i.e. the 1-forms A_i are local connection 1-forms.

ii) The connection ω is independent of the choice of the cover $\{U_i\}$ and the connections σ_i.

The connection ω is called *Berry connection.*

Proof. Let $\sigma : U \to E$ be any local section of the NE-bundle (not necessarily equal to any σ_i). We have $0 = \mathrm{d}(\sigma(p)^\dagger\sigma(p)) = (\mathrm{d}\sigma(p))^\dagger\sigma(p) + \sigma(p)^\dagger\mathrm{d}\sigma(p) = \bar{A}(p) + A(p)$. Thus A is an $i\mathbb{R}$-valued 1-form. Let $\tau : V \to E$ be another local section (not necessarily equal to any σ_i) and $B(p) = \tau(p)^\dagger\mathrm{d}\tau(p)$. For $p \in U \cap V$ we can write $\tau(p) = \sigma(p)t(p)$ for the transition function t. Then $B(p) = \tau(p)^\dagger\mathrm{d}\tau(p) = \bar{t}(p)\sigma(p)^\dagger(\mathrm{d}\sigma(p))t(p)+\bar{t}(p)\sigma(p)^\dagger\sigma(p)\mathrm{d}t(p) = A(p)+t^{-1}(p)\mathrm{d}t(p)$.

Applying this to $\sigma = \sigma_i$ and $\tau = \sigma_j$ shows that the assumptions of Theorem 2.33ii) are satisfied. Hence, there exists a connection 1-form ω such that $\sigma_i^*\omega = A_i$. Moreover, on U_i we have $\omega = \omega_i$ where ω_i is defined in Theorem 2.33.

To prove ii) consider another cover $\{V_\alpha\}$ of the NE-bundle with sections τ_α from which we obtain the connection 1-form $\widetilde{\omega}$ using Theorem 2.33ii). Applying our calculation above to $\sigma = \sigma_i$ and $\tau = \tau_\alpha$ for any i and α shows that the assumptions of Theorem 2.33i) are satisfied. Thus $\omega_i = \widetilde{\omega}_\alpha$ on $\pi^{-1}(U_i \cap V_\alpha)$ for all α and i. Hence, the 1-forms ω and $\widetilde{\omega}$ actually agree on E. $\square$

4.4 Eigenbundles for Hamiltonians

Lemma 4.4. *Let $H(p)$ be a complex Hermitian $n \times n$ matrix depending smoothly on $p \in \mathbb{R}^3$. Assume that at $p = 0$ two eigenvalues $\lambda_1(p)$ and $\lambda_2(p)$ are equal, and that there is an open neighbourhood V of 0 such that all the eigenvalues are pairwise distinct on $V \setminus \{0\}$. There exists a unitary matrix $u(p)$ depending continuously on p such that $u(p)^\dagger H(p)u(p)$ has block form $H_2(p) \oplus H_{n-2}(p)$ where $H_{n-2}(p)$ is diagonal and $H_2(p)$ is a 2×2 matrix with eigenvalues λ_1 and λ_2 and the NE-bundles of H for λ_1 and λ_2 are isomorphic to the NE-bundles of H_2.*

Proof. Note that the eigenvalues of $H(p)$ are real. Choose an open ball U such that $\bar{U} \subset V$ with centre 0. We can assume without loss of generality that $\lambda_1(p) > \lambda_2(p)$ for all $p \in U \setminus \{0\}$. Let $\lambda_j(p)$ denote the eigenvalues of $H(p)$. Set $B_1 = \mathbb{R}^3 \setminus \{(0,0,z)|z \leq 0\}$ and $B_2 = \mathbb{R}^3 \setminus \{(0,0,z)|z \geq 0\}$. Consider the NE-bundles E^j for the eigenvalues λ_j. For $j < 3$ this is a principal $U(1)$-bundle over $U \setminus \{0\}$, for $j \geq 3$ the bundle E^j is actually well defined on all of U. Since U is contractible E^j is trivial for $j \geq 3$ and hence admits a global section $s^j : U \to E^j$. For $j = 1$ we cover $U \setminus \{0\}$ with $U_1 = U \cap B_1$ and $U_2 = U \cap B_2$. We choose local sections $s^1_\alpha : U_\alpha \to E^1$. The transition function $t^1_{\alpha\beta}$ is given through $s^1_\beta(p) = s^1_\alpha(p)t^1_{\alpha\beta}(p)$ for all $p \in U_\alpha \cap U_\beta$. For $j = 2$, pick any section $\widetilde{s}_\alpha$ of E^2 on U_α. Now define a new section through $s^2_\alpha = \widetilde{s}_\alpha \det(s^1_\alpha, \widetilde{s}_\alpha, s^3, ..., s^n)^{-1}$. Then the matrix

$A_\alpha = (s^1_\alpha, s^2_\alpha, s^3, ..., s^n)$ is a unitary matrix with determinant one. For $p \in U_\alpha \cap U_\beta$ the transition function is given through $s^2_\beta(p) = s^2_\alpha(p) t^2_{\alpha\beta}(p)$. For $p \in U_\alpha \cap U_\beta$ we have

$$\begin{aligned} 1 = \det A_\alpha = \det(s^1_\alpha, s^2_\alpha, s^3, ..., s^n) &= \det(s^1_\beta t^1_{\beta\alpha}(p), s^2_\beta t^2_{\beta\alpha}(p), s^3, ..., s^n) \\ &= t^1_{\beta\alpha}(p) t^2_{\beta\alpha}(p) \det A_\beta = t^1_{\beta\alpha}(p) t^2_{\beta\alpha}(p) \end{aligned} \tag{3}$$

Now let θ denote the azimutal angle of p. Consider the following vector fields

$$\begin{aligned} \widetilde{c}^1(p) &= \frac{1+\cos\theta}{2} s^1_1(p) + \frac{1-\cos\theta}{2} s^2_2(p) \\ \widetilde{c}^2(p) &= -\frac{1-\cos\theta}{2} s^1_2(p) + \frac{1+\cos\theta}{2} s^2_1(p) \end{aligned}$$

Both are well defined on $U \backslash \{0\}$ because wherever a section is not well defined its prefactor vanishes. These vector fields are orthogonal, since by Equation 3

$$\langle \widetilde{c}^1(p), \widetilde{c}^2(p) \rangle = \frac{1-\cos^2\theta}{4} (-t^1_{12}(p) + t^2_{21}(p)) = 0.$$

We normalise them

$$c^i(p) = \frac{\widetilde{c}^i(p)}{\sqrt{\frac{2-\sin^2\theta}{2}}}$$

to get an orthonormal frame $u(p) = (c^1(p), c^2(p), s^3(p), ..., s^n(p))$ defined on $U \setminus \{0\}$. In this basis the Hamiltonian becomes diagonal apart form a 2×2 block H_2 which is given by

$$H_2(p) = \frac{1}{4 - 2\sin^2\theta} \begin{pmatrix} \lambda_1(p)(1+\cos\theta)^2 + \lambda_2(p)(1-\cos\theta)^2 & (1-\cos^2\theta) t^1_{12}(p)(\lambda_2(p) - \lambda_1(p)) \\ (1-\cos^2\theta) \bar{t}^1_{12}(p)(\lambda_2(p) - \lambda_1(p)) & \lambda_1(p)(1-\cos\theta)^2 + \lambda_2(p)(1+\cos\theta)^2 \end{pmatrix}$$

The NE-bundles for H_2 are isomorphic to the bundles E^1 and E^2. This can be seen for example by comparing H_2 to the matrix in Section 5.2. Let b, c, d and v_1 be as in Section 5.2. One gets

$$b + ic = \frac{1-\cos^2\theta}{2(1+\cos^2\theta)} \bar{t}^1_{12}(p)(\lambda_2(p) - \lambda_1(p))$$

and

$$d = \frac{\cos\theta}{1+\cos^2\theta} (\lambda_1(p) - \lambda_2(p)).$$

The condition $b = c = 0$ is equivalent to $\theta \in \{0, \pi\}$. Moreover, we have $d > 0$ for $\theta = 0$ and $d < 0$ for $\theta = \pi$. So for the larger eigenvalue λ_1 the local section obtained from v_1 by plugging in the above expressions for b, c and d is defined on U_1 (this is the U_1 from this Section), and the other local section is indeed defined on U_2. The transition function

$\frac{b-ic}{b^2+c^2}$ gives exactly $t^1_{12}(p)$. So both the NE-bundle of H_2 for λ_1 and E^1 have the same open cover and the same transition functions, hence they are isomorphic by Remark 2.12. The NE-bundle for λ_2 and E^2 are isomorphic as well, since they have again the same open cover with identical transition functions (which are just the inverse of the transition functions for λ_1 by Equation (3)). $\square$

5 Hopf bundles via 2×2 matrices

In this Section we consider symmetric or Hermitian 2×2 matrices with real, complex or quaternionic entries. We study how the normalised eigenvectors depend on the entries and view them as a principal bundle. It turns out that this gives a construction for different Hopf bundles.

5.1 Real matrices

Analogously to the complex Hopf bundle, the map $h : S^1 \to S^1, (x_1, x_2) \mapsto (2x_1x_2, x_1^2 - x_2^2)$ induces a principal S^0-bundle. We call this bundle the Hopf bundle $S^0 \hookrightarrow S^1 \to S^1$. Since the total space S^1 is connected and S^0 and $S^1 \times S^0$ are not, this Hopf bundle is non-trivial. One can show (see Theorem 2.7 in Cohen (2002)) that up to isomorphism there are only two principal S^0-bundles over S^1. This means that there is only the trivial bundle and this Hopf bundle.

Every symmetric real 2×2 matrix can be written as

$$A(a,b,c) = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$. Its eigenvalues are then given by $\lambda_\pm = a \pm \sqrt{c^2+b^2}$. Let $U_1 := \mathbb{R}^2 \setminus \{(0,c)|c \leq 0\}$ and $U_2 := \mathbb{R}^2 \setminus \{(0,c)|c \geq 0\}$. For $(a,b,c) \in \mathbb{R} \times U_1$ the vector

$$v_1(a,b,c) = \begin{pmatrix} c + \sqrt{b^2+c^2} \\ b \end{pmatrix} \frac{1}{\sqrt{2(b^2+c^2) + 2c\sqrt{c^2+b^2}}}$$

is a normalised eigenvector of $A(a,b,c)$ to the eigenvalue $\lambda_+(a,b,c)$. Note that the expression is not well defined for $b = 0$ and $c < 0$. However, there is another expression which is well defined in this case. For $(a,b,c) \in \mathbb{R} \times U_2$ the vector

$$v_2(a,b,c) = \begin{pmatrix} b \\ -c + \sqrt{b^2+c^2} \end{pmatrix} \frac{1}{\sqrt{2(b^2+c^2) - 2c\sqrt{c^2+b^2}}}$$

is a normalised eigenvector of $A(a,b,c)$ to the eigenvalue $\lambda_+(a,b,c)$. Notice that v_1 and

v_2 are independent of the parameter a and invariant under scaling $(b,c) \mapsto (rb, rc)$ for $r \in \mathbb{R}_{>0}$. The normalised eigenbundle thus is of the form $\pi : \mathbb{R}\times\mathbb{R}_{>0}\times E \to \mathbb{R}\times\mathbb{R}_{>0}\times S^1$, where E is a principal S^0-bundle over S^1.

We focus on E and view v_1 and v_2 as maps depending only on b and c. The maps v_1 and v_2 define local sections of E over S^1. The transition functions t_{ij} such that $v_i t_{ij} = v_j$ take values in $\{\pm 1\}$ and are defined on $\mathbb{R}^2 \setminus \{(0,c) | c \in \mathbb{R}\}$ which has two connected components. We see that $t_{12}(b,c) = 1$ for $b > 0$ and $t_{12}(b,c) = -1$ for $b < 0$. Thus this bundle has a connected total space and hence is non-trivial. Hence it is isomorphic to the Hopf bundle $S^0 \hookrightarrow S^1 \to S^1$.

5.2 Complex matrices

All complex Hermitian 2×2 matrices can be expressed as

$$A(a,b,c,d) = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix}$$

for some $a,b,c,d \in \mathbb{R}$. The eigenvalues are $\lambda_\pm = a \pm \sqrt{d^2+b^2+c^2}$. Again we will concentrate on λ_+ and define $U_1 := \mathbb{R}^3 \setminus \{(0,0,d) | d \le 0\}$ and $U_2 := \mathbb{R}^3 \setminus \{(0,0,d) | d \ge 0\}$. As before, the eigenvectors are independent of the parameter a. We have

$$v_1(b,c,d) = \begin{pmatrix} d + \sqrt{b^2+c^2+d^2} \\ b+ic \end{pmatrix} \frac{1}{\sqrt{2(b^2+c^2+d^2) + 2d\sqrt{d^2+b^2+c^2}}}$$

and

$$v_2(b,c,d) = \begin{pmatrix} b-ic \\ -d + \sqrt{b^2+c^2+d^2} \end{pmatrix} \frac{1}{\sqrt{2(b^2+c^2+d^2) - 2d\sqrt{b^2+c^2+d^2}}}$$

on U_1 and U_2 respectively. The maps v_1 and v_2 are local sections of a $U(1)$-bundle over S^2. The transition function is $t_{12} = \frac{b-ic}{b^2+c^2}$.

Now we can calculate the Berry connection. Switching to spherical coordinates

$$\begin{pmatrix} b \\ c \\ d \end{pmatrix} = r \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

we obtain

$$v_1(\varphi,\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\cos\theta} \\ e^{i\varphi}\sqrt{1-\cos\theta} \end{pmatrix}$$

and

$$v_2(\varphi,\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi}\sqrt{1+\cos\theta} \\ \sqrt{1-\cos\theta} \end{pmatrix}.$$

Hence for the Berry connection we have $A_1 = v_1^\dagger \mathrm{d}v_1 = \frac{i}{2}(1 - \cos\theta)\mathrm{d}\varphi$ and $A_2 = -\frac{i}{2}(1 + \cos\theta)\mathrm{d}\varphi$. By the calculation in Section 4.1, the local connection forms A_1 and A_2 fix the homotopy class of the transition function. By Remark 4.1, the principal bundle we obtained is exactly the same as the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$.

5.3 Quaternionic matrices

The skew field of quaternions is given by $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ where $i^2 = j^2 = k^2 = -1$ and $ij = k,\ jk = i$ and $ki = j$. For $q = a + bi + cj + dk$ the complex conjugate is given by $\bar{q} = a - bi - cj - dk$. Note that $|q|^2 = q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2$.

Definition 5.1. Suppose A is an $n \times n$ matrix with quaternionic entries. We call $q \in \mathbb{H}$ a *right eigenvalue* of A if there is a nonzero vector $v \in \mathbb{H}^n$ such that $Av = vq$.

Definition 5.2. We call a quaternionic matrix A Hermitian if $A^\dagger = A$, where $\dagger$ denotes the transpose complex conjugate as in the complex case.

Proposition 5.3. *If A is Hermitian and λ is a right eigenvalue of A, then λ is real.*

Proof. Let $Av = v\lambda$. We have $(Av)^\dagger v = v^\dagger A v = v^\dagger v \lambda$ and $(Av)^\dagger v = \bar{\lambda} v^\dagger v$. Since $v^\dagger v$ is real and non-zero, we obtain $\lambda = \bar{\lambda}$. $\square$

Any quaternionic Hermitian 2×2 matrix can be written as

$$A(a, b, c, d, e, f) = \begin{pmatrix} e + f & a + ib + jc + kd \\ a - ib - jc - kd & e - f \end{pmatrix}$$

for some $a, b, c, d, e, f \in \mathbb{R}$. The eigenvalues are $\lambda_\pm = e \pm \sqrt{a^2 + b^2 + c^2 + d^2 + f^2}$. Let us call $l = \sqrt{a^2 + b^2 + c^2 + d^2 + f^2}$. Again we will consider λ_+. Let $U_1 := \mathbb{R}^5 \setminus \{(0, 0, 0, 0, f) | f \leq 0\}$ and $U_2 := \mathbb{R}^5 \setminus \{(0, 0, 0, 0, f) | f \geq 0\}$. We find the following expressions for eigenvectors on U_1 and U_2, respectively:

$$v_1(a, b, c, d, f) = \begin{pmatrix} f + l \\ a - ib - jc - kd \end{pmatrix} \frac{1}{\sqrt{2l^2 + 2fl}}$$

and

$$v_2(a, b, c, d, f) = \begin{pmatrix} a + ib + jc + kd \\ -f + l \end{pmatrix} \frac{1}{\sqrt{2l^2 - 2fl}}$$

We now compare this to the Hopf bundle $S^3 \hookrightarrow S^7 \to S^4$ which can be constructed the following way. Identify $\mathbb{R}^8$ with $\mathbb{H}^2$ and view S^7 as a subset of $\mathbb{H}^2$. Consider the manifold $\mathbb{H}P^1$ which is the quotient of $\mathbb{H}^2 \setminus \{(0, 0)\}$ by the equivalence relation $(q_1\lambda, q_2\lambda) \sim (q_1, q_2)$ for all $\lambda \in \mathbb{H} \setminus \{0\}$. Let $g : S^7 \to \mathbb{H}P^1$ be the quotient map. We identify $\mathbb{R}^5$ with $\mathbb{H} \times \mathbb{R}$ through $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1 + x_2 i + x_3 j + x_4 k, x_5)$. Thus we can view S^4 as a subset of

$\mathbb{H} \times \mathbb{R}$. Let $\psi : \mathbb{H}P^1 \to S^4$ be the diffeomorphism $[(q_1, q_2)] \mapsto (2q_1\bar{q_2}, |q_1|^2 - |q_2|^2)/(|q_2|^2 + |q_1|^2)$. The Hopf map is given through $h = \psi \circ g : S^7 \to S^4$.

Let $U(\mathbb{H})$ denote the unit quaternions. Note that $U(\mathbb{H}) \cong S^3$.

Proposition 5.4. *There is a principal $U(\mathbb{H})$-bundle with the Hopf map $h : S^7 \to S^4$ as projection. We call it the quaternionic Hopf bundle.*

Proof. A smooth right $U(\mathbb{H})$-action on S^7 is given through $(q_1, q_2) \cdot \lambda = (q_1\lambda, q_2\lambda)$ for all $\lambda \in U(\mathbb{H})$ and $(q_1, q_2) \in S^7$. For every $p \in S^4$ the fibre $h^{-1}(p)$ is of the form $\{(q_1, q_2)\lambda \mid \lambda \in U(\mathbb{H})\}$ for any $(q_1, q_2) \in S^7$ with $h(q_1, q_2) = p$. The right $U(\mathbb{H})$-action thus preserves the fibres and is free and transitive on each fibre. One calculates that $h \circ v_i = id\big|_{U_i \cap S^4}$ for $i = 1, 2$. Hence we can define local trivialisations $\phi_i : (U_i \cap S^4) \times U(\mathbb{H}) \to h^{-1}(U_i \cap S^4), (p, \lambda) \mapsto v_i(p)\lambda$. These local trivialisations are compatible with the right action. $\square$

As we did in Section 2 for the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$ we now want to calculate a connection and the corresponding curvature for the quaternionic Hopf bundle $S^3 \hookrightarrow S^7 \to S^4$ and show that the bundle is non-trivial. The Lie algebra of $U(\mathbb{H})$ is given through the algebra of imaginary quaternions where the Lie bracket is the commutator. For the generators we thus have $[i, j] = 2k$, $[j, k] = 2i$ and $[k, i] = 2j$. For $v = ai + bj + ck$ and $u = (q_1, q_2) = (x_1 + x_2 i + x_3 j + x_4 k, y_1 + y_2 i + y_3 j + y_4 k) \in S^7$ the fundamental vector field can be calculated to be

$$\xi_u(v) = \begin{pmatrix} -x_2a - x_3b - x_4c + i(x_1a - x_4b + x_3c) + j(x_1b - x_2c + x_4a) + k(x_1c + x_2b - x_3a) \\ -y_2a - y_3b - y_4c + i(y_1a - y_4b + y_3c) + j(y_1b - y_2c + y_4a) + k(y_1c + y_2b - y_3a) \end{pmatrix}$$

Now we want to find a connection on the quaternionic Hopf bundle. Recall that a connection is a projection onto the vertical subspace. The vertical subspace at $u \in S^7$ is generated by the three orthonormal vectors $\xi_u(i)$, $\xi_u(j)$ and $\xi_u(k)$. The vector fields $\xi(i)$, $\xi(j)$ and $\xi(k)$ define a frame and the dual of this frame consists of 1-forms ω_i, ω_j and ω_k on S^7 that can be combined to give the projection we want. So the connection form will look like $\omega = \mu_i\omega_i + \mu_j\omega_j + \mu_k\omega_k$. Since $\omega(\xi(i)) = i$, the factor μ_i has to be equal to i. Analogously, $\mu_j = j$ and $\mu_k = k$. This leads to the following expression:

Proposition 5.5. *A connection on the quaternionic Hopf bundle is given by*

$$\begin{aligned} \omega =& (-x_2i - x_3j - x_4k)\mathrm{d}x_1 + (x_1i - x_4j + x_3k)\mathrm{d}x_2 \\ & + (x_4i + x_1j - x_2k)\mathrm{d}x_3 + (-x_3i + x_2j + x_1k)\mathrm{d}x_4 \\ & + \textit{the same with } y \textit{ instead of } x \\ =& \bar{q_1}\mathrm{d}q_1 + \bar{q_2}\mathrm{d}q_2 \end{aligned}$$

Proof. Expanding gives

$$\bar{q_1}\mathrm{d}q_1 + \bar{q_2}\mathrm{d}q_2 = \omega + \sum_{i=1}^{4} x_i \mathrm{d}x_i + y_i \mathrm{d}y_i.$$

Since $\sum_{i=1}^{4} x_i^2 + y_i^2 = 1$ we have $0 = 2\sum_{i=1}^{4} x_i \mathrm{d}x_i + y_i \mathrm{d}y_i$ which implies $\bar{q_1}\mathrm{d}q_1 + \bar{q_2}\mathrm{d}q_2 = \omega$. To show that ω is a connection, we calculate $\omega_u(\xi_u(ai + bj + ck)) = (ia + jb + kc)(\sum_{i=1}^{4} x_i^2 + y_i^2) = ai + bj + ck$. The second property of a connection follows from $Ad_{g^{-1}}\omega_q(v) = g^{-1}\omega_q g(v) = \sum_{l=1}^{2} g^{-1}(\bar{q_l}\mathrm{d}q_l(v))g = \sum_{l=1}^{2} \overline{(q_l g)}\mathrm{d}(q_l g)(R_{g*}(v)) = R_g^*\omega_{qg}$. $\square$

Now we want to calculate a local connection form on $V_2 := \mathbb{H} \cong \{[(q,1)] | q \in \mathbb{H}\} \subset \mathbb{H}P^1$. We pick the section $s : V_2 \to S^7$,

$$s(q) = \frac{(q,1)}{\sqrt{1+q\bar{q}}}.$$

Then we have

$$A_1 = \frac{\bar{q}}{\sqrt{1+q\bar{q}}}\mathrm{d}\left(\frac{q}{\sqrt{1+q\bar{q}}}\right) + \frac{1}{\sqrt{1+q\bar{q}}}\mathrm{d}\left(\frac{1}{\sqrt{1+q\bar{q}}}\right)$$

One calculates

$$\mathrm{d}\left(\frac{q}{\sqrt{1+q\bar{q}}}\right) = -\frac{q}{2}\frac{(\mathrm{d}q)\bar{q} + q\mathrm{d}\bar{q}}{(1+q\bar{q})^{3/2}} + \frac{\mathrm{d}q}{\sqrt{1+q\bar{q}}}$$

and

$$\mathrm{d}\left(\frac{1}{\sqrt{1+q\bar{q}}}\right) = -\frac{1}{2}\frac{(\mathrm{d}q)\bar{q} + q\mathrm{d}\bar{q}}{(1+q\bar{q})^{3/2}}.$$

Plugging this in we get

$$\begin{aligned} A_1 &= \frac{\bar{q}\mathrm{d}q}{1+q\bar{q}} - \left(\frac{1+\bar{q}q}{2}\right)\frac{(\mathrm{d}q)\bar{q} + q\mathrm{d}\bar{q}}{(1+q\bar{q})^2} \\ &= \frac{2\bar{q}\mathrm{d}q - (\mathrm{d}q)\bar{q} - q\mathrm{d}\bar{q}}{2(1+q\bar{q})}. \end{aligned}$$

Observe that $q\bar{q} = \bar{q}q$ implies $(\mathrm{d}q)\bar{q} + q\mathrm{d}\bar{q} = (\mathrm{d}\bar{q})q + \bar{q}\mathrm{d}q$. Hence we have

$$A_1 = \frac{\bar{q}\mathrm{d}q - (\mathrm{d}\bar{q})q}{2(1+q\bar{q})} = \mathrm{Im}\left(\frac{\bar{q}\mathrm{d}q}{(1+q\bar{q})}\right).$$

The curvature is given by

$$F = \mathrm{A}_1 + \frac{1}{2}A_1 \wedge A_1 = \frac{1}{(1+|q|^2)^2}\mathrm{d}\bar{q} \wedge \mathrm{d}q,$$

as shown in Naber (1997) on p. 288.

We can look at the quaternionic Hopf bundle as an $su(2)$-bundle by identifying i, j, k

with the Pauli matrices $I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. With $q = x_1 + x_2 I + x_3 J + x_4 K$ and $|q|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ the curvature then takes the form

$$\begin{aligned} F =& \frac{2}{(1+|q|^2)^2} \left((\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge \mathrm{d}x_4) I + (\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4) J \right. \\ & \left. + (\mathrm{d}x_1 \wedge \mathrm{d}x_4 - \mathrm{d}x_2 \wedge \mathrm{d}x_3) K \right) \end{aligned}$$

As shown for example in Nakahara (2003) Section 11.2, the second Chern class is given by

$$c_2(F) = \frac{1}{2} \left(\frac{i}{2\pi} \right)^2 (\mathrm{tr} F \wedge \mathrm{tr} F - \mathrm{tr}(F \wedge F)),$$

where the trace and the wedge product are defined as follows. For pure tensors $a \otimes \eta$ and $b \otimes \omega \in \mathfrak{g} \otimes \Omega(M)$ let $\mathrm{tr}(a \otimes \eta) = \mathrm{tr}(a) \otimes \eta$ and $(a \otimes \eta) \wedge (b \otimes \omega) = ab \otimes (\eta \wedge \omega)$. Then extend these definitions by linearity to all of $\mathfrak{g} \otimes \Omega(M)$.

Since the Pauli matrices have trace zero we get $\mathrm{tr} F = 0$ and hence also $\mathrm{tr} F \wedge \mathrm{tr} F = 0$. We calculate

$$\begin{aligned} F \wedge F &= \frac{4}{(1+|q|^2)^4} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 (-I^2 - I^2 - J^2 - J^2 - K^2 - K^2) \\ &= \frac{24}{(1+|q|^2)^4} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, $\mathrm{tr}(F \wedge F) = \frac{48}{(1+|q|^2)^4} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$. Now following the calculations in Naber (1997) on p. 320 we have

$$\int_{\mathbb{H}P^1} c_2(F) = \frac{6}{\pi^2} \int_{\mathbb{R}^4} \frac{1}{(1+|q|^2)^4} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$$

We introduce spherical coordinates

$$x_1 = r \sin\chi \sin\varphi \cos\theta, \quad x_2 = r \sin\chi \sin\varphi \sin\theta, \quad x_3 = r \sin\chi \cos\varphi, \quad x_4 = r \cos\chi,$$

where $r \in [0, \infty), \theta \in [0, 2\pi]$ and $\chi, \varphi \in [0, \pi]$. We then have

$$\begin{aligned}
\int\limits_{\mathbb{H}P^1} c_2(F) &= \frac{6}{\pi^2} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{\pi} \int\limits_0^{\infty} \frac{r^3 \sin^2 \chi \sin \varphi}{(1+r^2)^4} \mathrm{d}r \mathrm{d}\chi \mathrm{d}\varphi \mathrm{d}\theta \\
&= \frac{12}{\pi} \left(\int\limits_0^{\pi} \int\limits_0^{\pi} \sin^2 \chi \sin \varphi \mathrm{d}\chi \mathrm{d}\varphi \right) \left(\int\limits_0^{\infty} \frac{r^3}{(1+r^2)^4} \mathrm{d}r \right) \\
&= \frac{12}{\pi} \left(2\frac{\pi}{2} \right) \left(\frac{2}{6} \int\limits_0^{\infty} \frac{r}{(1+r^2)^3} \mathrm{d}r \right) \\
&= 4 \left[-\frac{1}{4} \frac{1}{(1+r^2)^2} \right]_{r=0}^{r=\infty} = 1
\end{aligned}$$

Thus the bundle is not trivial.

6 References

Cohen, R. L. (2002). The Topology of Fiber Bundles Lecture Notes. `http://math.stanford.edu/~ralph/fiber.pdf`, Accessed: June 5, 2018.

Dupont, J. L. (1978). *Curvature and Characteristic Classes*, Springer-Verlag Berlin Heidelberg.

Greub, W., Halperin, S. and Vanstone, R. (1973). *Connections, Curvature and Cohomology*, Vol. 2, Academinc Press New York and London.

Kobayashi, S. and Nomizu, K. (1969). *Foundations of Differetial Geometry*, Vol. 2, Interscience Publishers.

Morrison, S. (2000). Connections on Principal Fibre Bundles. `https://tqft.net/papers/ConnectionsAndBundles.pdf`, Accessed: April 6, 2018.

Naber, G. L. (1997). *Topology, Geometry, and Gauge Fields, Foundations*, Academinc Press New York and London.

Nakahara, M. (2003). *Geometry, Topology and Physics*, Second edn, Taylor & Francis Group, LLC.

Schwinger, J. (2001). *Quantum Mechanics*, Springer-Verlag Berlin Heidelberg GmbH.

Shnir, Y. M. (2005). *Magnetic Monopoles*, Springer-Verlag Berlin Heidelberg.

Westenholz, C. v. (1978). *Differential forms in mathematical physics*, North-Holland Publishing Company.