On the spectral gaps of long-range interacting quantum spin systems

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Abstract

Quantum phase transitions are accompanied by a closing of the spectral gap above the ground state. One of the universal critical exponents characterizing the phase transition is the *dynamical critical exponent* $z \geq 0$, which gives the asymptotic closing rate of the spectral gap γ_L with growing linear system size L, namely $\gamma_L \sim L^{-z}$. A number of works have studied z for finite-range frustration-free quantum spin systems and established the sharp general bound $z \geq 2$ at increasing levels of generality. However, the existing methods break down without the assumption of finite interaction range.

We initiate the study of spectral gap scaling for quantum spin systems with power-law-decaying interactions. We introduce the concept of *dominant interaction scale (DIS)* as a new tool to analyze long-range spectral problems and use it to prove a conditional finite-size criterion. As a consequence, we prove that a interaction of sufficiently fast power-law decay must satisfy a critical gap exponent $z \geq 1$ for either periodic or open boundary conditions. This is the first lower bound on the dynamical critical exponent for long-range interactions. Although the bound is likely not sharp, it establishes a first step toward a systematic understanding of dynamical critical exponents in long-range interacting quantum systems and highlights the problem of determining the optimal values of z.

1 Introduction

The size of the spectral gap above the ground-state energy is one of the central quantities in the analysis of quantum systems. One place where the spectral gap plays a particularly pivotal role is in the description of quantum phase transitions. Indeed, a uniformly positive gap in the thermodynamic limit is associated with a stable quantum phase [HW05, S. 11] and has strong consequences for the ground state complexity such as exponential clustering [HK06, NS06] and an area law[1] for the entanglement entropy [Has07, AKLV13, AAG22]. Conversely, a quantum phase transition is therefore necessarily accompanied by a closing of the spectral gap in the thermodynamic limit where the system size grows to infinity. The rate of the gap closing is usually expected to be polynomial in the system size, in which case it is encoded in the dynamical critical exponent z, which is supposed to capture the finite-volume scaling as

$$\gamma_L \sim L^{-z},$$

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[1] The area law for gapped ground states is currently proved only for general one-dimensional and frustration-free two-dimensional systems.

where γ_L denotes the spectral gap of a system of linear size L (e.g., a discrete box of side length L). Physically, the value of z is expected to be universal (i.e., robust to suitable symmetry-preserving) perturbations and it is indicative of the kind of continuum theory that should arise at the critical point. Indeed, note that L^{-z} corresponds to a finite-size gap that would arise from evaluating a momentum-space dispersion relation $E(k) \sim |k|^z$ at discrete momenta $|k| \sim \frac{1}{L}$. E.g., a quadratic dispersion law leads to a dynamical critical exponent $z = 2$.

In this paper, we are interested in deriving mathematically rigorous lower bounds on the possible values of z for general frustration-free quantum spin systems with long-range interactions. In words, a lower bound $z \geq z_0$ says that, if the gap closes, then it must do so at least as fast as L^{-z_0}, i.e., the gap cannot close to slowly. A mathematical lower bound on z then corresponds to a kind of no-go result, limiting what kind of continuum dispersion laws $E(k) \sim |k|^z$ can possibly arise at the critical point.

There exist by now many such results for finite-range interactions. The first such bound goes back to [LM19] which proved $z \geq 3/2$ for a class of frustration-free quantum spin Hamiltonians with finite-range interactions; see also [GM16]. A number of works have followed in the past years, which have improved the bound to $z \geq 2$ and/or extended the underlying technique to frustration-free finite-range Hamiltonians on more general graphs and with more general boundary conditions [Lem19a, Ans20, LX22, LL25, MSW25a, MSW25b, HJL25]. However, all of these works require finite interaction range.

The technical reason for this limitation goes beyond bounds on z and thus deeper than one may at first realize: Most of the direct techniques for deriving lower bounds on spectral gaps (also uniform-in-system-size lower bounds) *require a finite interaction range.* The two main classes of these techniques are as follows:

- Many of the aforementioned works on bound in z use *finite-size criteria* of Knabe-type [Kna88] and their improvements. This technique underlies many of the aforementioned lower bounds on z as well as numerous proofs of a uniform spectral gap in [LN19, Lem19b, LSW20, HHJ21, JL22, RKE$^+$26]. The basic principle behind finite-size criteria is to relate the spectral gap of a large system to the gaps of sufficiently large subsystems. Since the finite system necessarily only knows about interactions up to a finite range, the finite-range assumption is baked into the method from the outset.
- The martingale method [Nac96] and later variants, such as finite-size criteria based on ground state projectors [FNW92] and the divide-and-conquer method [KL18] link the spectral gap problem to the calculation of the angle between local ground spaces. These approaches are effective in models with an explicit or approximate description of these ground spaces; see, e.g., [FNW92, SS03, BN14, BHNY15, ARLL$^+$20, PW20, ALM22]. These methods are also highly local and require a finite interaction range.

An exception is a recently proposed direct spectral gap method based on induction-in-particle number [LNWY26]. In principle, this method avoids the long-range assumption because it performs an induction in Fock space instead of position space. It remains to be seen if it can be used to derive uniform lower bounds on spectral gaps in models with long-range interactions. However, the method is by design insensitive to system size and thus not equipped to study possible finite-size gap scaling with system size, which is our interest here.

These are all what we call “direct” methods, which we contrast with the “indirect” way to establish a spectral gap by a stability argument, where one perturbs around a model that is known to be gapped. Such stability arguments constitute a highly active field of research in their own right; see e.g. [BHM10, MZ13, DRS19, FP20, DVFPR21, NSY22, NSY24, DVFPR25, YNJ26]. Such stability results typically require a comparatively much weaker form of locality, expressed by Lieb-Robinson bounds. These are known to hold also for long-range interactions [HK06, NS06, TCE$^+$20, KS20]. Of course, any gap stability argument needs as starting point a gapped Hamiltonian to perturb around and for these one requires a "direct" proof of a gap. The point is that such a direct proof by one of the direct methods mentioned currently requires the much more stringent locality assumption of finite-range interactions. In other words, the current methodological toolkit only allows to treat perturbatively weak long-range interactions. Frustration-freeness plays a similarly foundational role in the existing direct spectral gap methods. This point is exemplified by the fact that Haldane 1983 conjecture, which asserts that the (frustrated) antiferromagnetic nearest-neighbor Heisenberg chain is gapped for any integer spin, remains wide open even for spin-1. In the stability results, the unperturbed system needs to be frustration-free, but the perturbation can break it. Removing the need for frustration-freeness is a much more profound challenge and we will restrict to frustration-free systems throughout this article.

To summarize the preceding paragraphs: All direct methods of deriving spectral gaps share the limitation that they require finite-range interactions. This state is clearly unsatisfying from a mathematical-methodological perspective. It is also physically problematic, since non-perturbative long-range interactions are native to experimental quantum platforms and underpin fundamental phenomena such as the fractional quantum Hall effect.

It is therefore an important open problem to develop direct approaches to spectral gap problems for frustration-free quantum lattice systems with long-ranged interactions.

The purpose of this paper is to initiate the analysis of spectral gap scaling for frustration-free quantum spin systems with long-range decaying interactions. We make a few simplifying assumptions: We assume that they are translation-invariant two-body interactions which are rescaled projectors. These assumptions allow us to focus on the conceptual novelties. They can be relaxed at the expense of a notationally heavier proof, which we leave to future work. We consider quantum spin systems on a discrete box in $\mathbb{Z}^D$ of side length $L \geq 2$; call this Λ_L. On the Hilbert space $\mathcal{H}_{\Lambda_L} = \bigotimes_{j\in\Lambda_L} \mathbb{C}^d$, we consider Hamiltonians

$$H_L = \sum_{j,k\in\Lambda_L} h_{j,k}$$

with interactions of the form

$$h_{j,k} = f(d(j,k))P_{j,k}$$

where $P_{j,k}$ is a projection acting non-trivially only on the jth and kth tensor copies in $\mathcal{H}_{\Lambda_L}$. We assume that the interaction between two-sites is non-negative and, say, decays polynomially with $0 \leq f(r) \leq cr^{-\beta}$ for a sufficiently large β. Our *main result* then says, roughly speaking that $z \geq 1$. The precise setup and statements are postponed to the next section.

To obtain this result, we devise a new tool to analyze long-range spectral problems which we call *dominant interaction scale*. It is defined for a finite-volume Hamiltonian and a fixed eigenvector corresponding to the spectral gap. Among all interaction lengths, we identify a scale $S = S(L)$ at which the weighted contribution of the interaction to the gap is maximal. The weight is chosen so as to compensate for the decay of the interaction and for the summability estimates used in the proof. This dominant scale separates the analysis into two regimes as follows. If the dominant interaction scale is large, then the gap eigenvector has its main energy contribution at a genuinely nonlocal distance. Since the interaction strength at such a distance is small, translation invariance and summability force the total gap itself to be small. In this regime one obtains an upper bound on the finite-volume gap directly from the largeness of $S(L)$.

If, on the other hand, the dominant interaction scale is small, then one can choose subsystems whose size is large enough to contain this scale. The remaining longer-range terms form a tail that can be controlled using the decay of f. In this case, we prove a *conditional finite-size criterion*, the condition being that the subsystem scale dominates the dominant interaction scale. It says the global gap is bounded from below in terms of the gap of suitable subsystems, up to an explicit error term depending on the decay exponent and the chosen subsystem size.

Combining these two alternatives yields our principal conclusions. For periodic boundary conditions in arbitrary spatial dimension, we prove that a gapless long-range frustration-free system must either have a rapidly closing periodic gap or else the corresponding open-boundary gaps close at least at rate $L^{-1+\delta}$, for arbitrary $\delta > 0$ under the stated hypotheses on the decay exponent.

The exponent 1 should be compared with the finite-range frustration-free bound $z \geq 2$. The difference reflects a genuine feature of long-range systems: low-energy excitations may be controlled by interactions occurring at mesoscopic or macroscopic distances, a mechanism absent in finite-range models. There exist examples of long-range frustration-free systems with gap scaling of order $1/L$ which indicate that this behavior is not merely an artifact of the proof. The present results therefore suggest a broader classification problem: determine the sharp universal lower bounds on z as a function of the decay of the interaction.

2 Setup and main results

The setup and results are decomposed into two cases. The first one considers a D-dimensional spin system with periodic boundary conditions. In the second part we focus on a 1-dimensional spin system with open boundary conditions. Handling open boundary conditions in higher dimensions requires additional arguments, which we leave to future work. We will now describe the setup and notations for both cases.

2.1 The setup

Consider a D dimensional spin system. Let $L \in \mathbb{N}$ be an even number and define

$$\Lambda_L := ([0, L] \cap \mathbb{Z})^D$$

the box in $\mathbb{Z}^D$. At every site there is a quantum spin of a local dimension $d = 2s + 1$ with the spin number s. The Hilbert space is given by

$$\mathcal{H}_{\Lambda_L} := \bigotimes_{j \in \Lambda_L} \mathbb{C}^d.$$

We will consider systems with open and closed boundary conditions and define the distance functions between two sites $j, k \in \Lambda_L$ by

$$d(j,k) := |k - j| \qquad\qquad d^{per}(j,k) := \min\{|k - j|, |k - j \mod L|\} \tag{2.1}$$

where the modulus is taken componentwise. With this notation we define for two different sites $j, k \in \Lambda_L$ the interactions

$$h_{j,k} := f(d(j,k)) P_{j,k} \otimes \mathrm{Id}_{\Lambda_L \setminus \{j,k\}} \qquad\qquad h^{per}_{j,k} := f(d^{per}(j,k)) P_{j,k} \otimes \mathrm{Id}_{\Lambda_L \setminus \{j,k\}}$$

with a continuous monotonically decaying function $f : \mathbb{R} \to \mathbb{R}$ that describes the interaction strength between two sites.

Assumption 2.1. *We assume that there is a $\beta > 0$ and either:*
(i) f exhibits exponential decay with rate β, i.e.,

$$0 \leq f(r) \leq ce^{-\beta r}, \qquad r > 0$$

(ii) f exhibits power-law decay with rate β, i.e.,

$$0 \leq f(r) \leq cr^{-\beta}, \qquad r > 0.$$

By rescaling the Hamiltonian, we can set $c = 1$ without loss of generality. For a site $j \in \Lambda_L$, we abbreviate $f(j) = f(d(j, 0))$.

We order all sites in the following lexicographic way. For two different sites $j, k \in \Lambda_L$ we set $j < k$, iff there exists a $m \in \{1, \cdots, D\}$ such that for all $i = 1, \cdots, m-1$ holds

$$j_i = k_i \qquad \text{and} \qquad j_m < k_m.$$

With this strict partial ordering we count all pairs $(j, k) \in \Lambda_L^2$ exactly one time. For example in one dimension we do not need to consider the pairs $(0, 1)$ and $(1, 0)$ separately. The Hamiltonians with open and closed boundary conditions are defined by

$$H_L := \sum_{0 \leq j < k \leq L} h_{j,k} \qquad H_L^{per} := \sum_{0 \leq j < k \leq L} h_{j,k}^{per}$$

with corresponding spectral gaps γ_L and γ_L^{per}.

The fact that all interactions are non-negative allows for a simple formulation of our standing assumption of frustration-freeness.

Assumption 2.2 (Frustration-freeness)**.** *We assume that for any $L \geq 2$, H_L^{per} is frustration-free, i.e., its ground state energy is zero.*

Here we use the notation 0 for the zero vector and L for the vector with all entries equal to L. Notice here the difference between the Hamiltonians H_L and H_L^{per} is only the distance function from Definition 2.1. Hence the eigenvalues, and especially the spectral gaps, can be different.

2.2 Main theorem

We consider throughout this paper always translation invariant and frustration free systems. In the following we state the main theorem which gives a result on the spectral gap scaling for periodic and open boundary conditions.

Theorem 2.3 (Spectral gap scaling for periodic b.c.)**.** *Let $L \geq 3$ be the system size, $D \geq 1$ the dimension and $\lambda > 0, \delta \in (0, 1)$, $p > 2D + 1$ constants. Assume that either*

- *a) the interaction strength is bounded by $0 \leq f(x) \leq ce^{-\beta x}$ for a $\beta > 0$ and the Hamiltonian H_L^{per} is gapless with $\lim_{L\to\infty} \ln(L)\gamma_L^{per} = 0$, or*
- *b) the interaction strength is bounded by $0 \leq f(x) \leq cx^{-\beta}$ for a $\beta > p + \frac{(D+\lambda)(D+p)}{\delta}$ and the Hamiltonian H_L^{per} is gapless with $\lim_{L\to\infty} L^{\frac{D(D+\lambda)}{\beta-p}} \gamma_L^{per} = 0$*

Then, it holds that either

$$\gamma_L^{per} \in O(L^{-\lambda}) \qquad \textit{or} \qquad \gamma_L \in O(L^{\delta-1}).$$

Fixing $\lambda = 1$ and $p > 2D + 1$, and taking β sufficiently large shows that at least one of the two gaps must close with exponent $z \geq 1 - \delta$. This holds for any $\delta > 0$ and in this sense, we prove $z \geq 1$.

Using techniques from [LM19], it is possible to prove a result about the open system gap only and we

2.3 Dominant interaction Scale

In this Section we introduce the concept of the dominant interaction scale. With this concept we can state more general versions of the main theorems. The dominant interaction scale is the interaction, that has the biggest contribution to the spectral gap. We consider a translation invariant Hamiltonian, that means the translation operators T_i for every dimension $i = 1, \cdots, D$ and the Hamiltonian H_L^{per} mutually commute. The translation operators T_i satisfy for all $j, k \in \Lambda_L$

$$T_i^\dagger h_{j,k} T_i = h_{j+e_i,k+e_i}.$$

Therefore we find a common eigenvector for the operators $T_1, \cdots, T_D, H_L^{per}$ in the eigenspace of γ_L^{per} and denote it by ϕ. For ϕ holds

$$\gamma_L^{per} = \langle \phi, H_L^{per} \phi \rangle \tag{2.2}$$

and for simplicity we write

$$\langle \phi, P_{j,k} \otimes \mathrm{Id}_{\Lambda_L \setminus \{j,k\}} \phi \rangle = \langle P_{j,k} \rangle$$

for every $j, k \in \Lambda_L$ and omit the dependency on ϕ. Let M be the maximum possible distance in one coordinate between two sites $j, k \in \Lambda_L$, i.e.

$$M = \max_{j,k \in \Lambda_L} d(j_1, k_1), \tag{2.3}$$

where we write j_1, k_1 for the first coordinates of j, k. Note that it is sufficient to take only the first coordinate of the sites, because we consider a lattice with equal side length. Analogously, we can define M in the case of periodic boundary conditions. For the rest of this paper we will only write M and use d or d^{per} in the definition accordingly.

Proposition 2.4 (Translation to zero)**.** *For every two sites $j, k \in \Lambda_L$ with $j < k$ there exists a site $t \in ([-M, M] \cap \mathbb{Z})^D \setminus \{0\}$ such that $\langle h_{j,k} \rangle = \langle h_{0,t} \rangle$ holds. In particular, let m be the unique index of the ordering $j < k$, then we can choose t from the set*

$$\{0\}^m \times ([1, M] \cap \mathbb{Z}) \times ([-M, M] \cap \mathbb{Z})^{D-m}.$$

For the proof of this proposition see Section 4.1. This proposition allows us to compare the different interactions. Note that the interactions are only translation invariant and not rotation invariant, so for example in general it holds

$$h_{(0,0),(1,-2)} \neq h_{(0,0),(-2,1)}.$$

Now we define for every $j \in ([-M, M] \cap \mathbb{Z})^D \setminus \{0\}$ the weight $\alpha_j := |j|^p$ for some fixed $p > 0$.

Definition 2.5 (Dominant interaction scale — DIS)**.** For a translation invariant Hamiltonian H_L we call the site $S \in \left(\left[-\frac{L}{2}, \frac{L}{2}\right] \cap \mathbb{Z}\right)^D \setminus \{0\}$ such that for all $j \in \left(\left[-\frac{L}{2}, \frac{L}{2}\right] \cap \mathbb{Z}\right)^D \setminus \{0\}$

$$\alpha_S f(S) \langle P_{0,S} \rangle \geq \alpha_j f(j) \langle P_{0,j} \rangle \tag{2.4}$$

holds, the dominant interaction scale. If there are multiple sites with property (2.4), then one of the sites with biggest absolute value is taken to be the dominant interaction scale.

Remark 2.6. The dominant interaction scale (DIS) is dependent on the system size, i.e. $S = S(L)$. Further the dominant interaction scale always exists, but it is not always unique for a fixed system size L.

2.4 Refined main theorems

With the introduction of the concept of the dominant interaction scale (DIS) we can now state more general versions of Theorem 2.3. There are two kinds of statements: the first statement one considers the case of "large" DIS and shows rapid closing of the spectral gap and the second statement considers the case of "small" DIS and derives a conditional finite-size criterion.

Let us consider the D-dimensional case with periodic boundary conditions first.

Proposition 2.7 (Large DIS gives small spectral gap — periodic b.c.). *Let the Hamiltonian H_L^{per} be gapless and let $\lambda > 0$ be a constant. Let $p > 2D + 1$ and*

a) $\beta > 0$ for an interaction strength $f(x) \leq ce^{-\beta x}$ with a dominant interaction scale S satisfying

$$|S| - \frac{p}{\beta}\ln(|S|) \geq \frac{D+\lambda}{\beta}\ln(L), \tag{2.5}$$

b) or $\beta > p + D + \lambda$ for an interaction strength $0 \leq f(x) \leq cx^{-\beta}$ with a dominant interaction scale S satisfying

$$|S| \geq c(D, p, \beta)L^{\frac{D+\lambda}{\beta-p}} \tag{2.6}$$

for a suitable constant $c(D, p, \beta) > 1$.

Then the spectral gap is upper bounded by $\gamma_L^{per} \in O\big(L^{-\lambda}\big)$.

This proposition handles the case of "large" dominant interaction scale, more precisely the inequalities (2.5) and (2.6) give a lower bound for the dominant interaction scale dependent on the system size. From now on we refer to (2.5) and (2.6) if we talk about "large" dominant interaction scale.

On the other hand for a "small" dominant interaction scale, i.e. it does not satisfy inequality (2.5), resp. (2.6), we can find a box including S. The subsystems are given by a translated box

$$\mathcal{B}_n := ([-n, n] \cap \mathbb{Z})^D \subset \Lambda_L$$

for a suitable n with the corresponding Hamiltonian

$$H_{\mathcal{B}_n} := \sum_{j,k\in\mathcal{B}_n} h_{j,k}$$

and the spectral gap $\gamma_{\mathcal{B}_n}$. The next theorem gives a finite size criterion for a lower bound on γ_L^{per} with the condition of a "small" dominant interaction scale.

Theorem 2.8 (Small DIS gives finite size criterion — periodic b.c.). *Let be $D \geq 1$ the dimension, $\lambda > 0, \delta \in (0, 1)$ two constants $L \geq 3$ the system size and choose the constant $p > 2D + 1$. Further let be*

a) $\beta > 0$ for an interaction strength $0 \leq f(x) \leq ce^{-\beta x}$ with a dominant interaction scale S satisfying

$$|S| - \frac{p}{\beta}\ln(|S|) < \frac{D+\lambda}{\beta}\ln(L),$$

b) *and* $\beta \geq p + \frac{(D+\lambda)(D+p)}{\delta}$ *for an interaction strength* $0 \leq f(x) \leq cx^{-\beta}$ *with a dominant interaction scale* S *satisfying*

$$|S| < c(D, p\beta) L^{\frac{D+\lambda}{\beta - p}}$$

for a suitable constant $c(D, p, \beta) > 1$.

Then we can choose the subsystem size $2|S(L)| < n < \frac{L}{2}$ *large enough, such that it holds a finite size criterion for the spectral gap given by*

$$|S|^D \gamma_L^{per} \geq c_1 \left(\gamma_n - c_2(\beta, p) n^{\delta - 1} \right).$$

For the proof see Section 3.2.

Corollary 2.9. *If* H_L^{per} *is in addition gapless with* $|S(L)|^D \gamma_L^{per} \to 0$ *for* $L \to \infty$, *then*

$$\gamma_L \in O(L^{\delta-1}).$$

2.5 Concluding the main theorem from the refined version

In this section we will prove Theorem 2.3. Throughout the proofs we will use the convention that we write $c = c(D) > 0$ for a constant only dependent on the dimension D. This constant can have different values from line to line and is getting updated in the calculations. The main theorems follow almost directly from the more general theorems and propositions in Section 2. For more details on the proofs for 2.7 and 2.8 for periodic boundary conditions see Section 3.

The procedure is to consider the cases “large” and “small” dominant interaction scale separately. We start by using Proposition 2.7 to show that in the “large” case the spectral gap γ_L^{per} has a certain closing rate. Afterwards we use Theorem 2.8 to cover the “small” dominant interaction scale case.

Proof of Theorem 2.3. We start with part a), that is a gapless Hamiltonian H_L^{per} with $\lim_{L\to\infty} \ln(L)\gamma_L^{per} = 0$. Let $\lambda > 0$ and $\delta \in (0,1)$ be constants. In the case where the dominant interaction scale S satisfies

$$|S| - \frac{p}{\beta}\ln(S) \geq \frac{D+\lambda}{\beta}\ln(L),$$

Proposition 2.7 yields a spectral gap scaling given by $\gamma_L^{per} \in O\big(L^{-\lambda}\big)$. Analogously for part b) we have $\beta > p + \frac{(D+\lambda)(D+p)}{\delta} > p + D + \lambda$ and hence Proposition 2.7 yields also a spectral gap scaling given by $\gamma_L^{per} \in O\big(L^{-\lambda}\big)$ and the first part of the claim follows.

Consider now for a) the case

$$|S| - \frac{p}{\beta}\ln(S) < \frac{D+\lambda}{\beta}\ln(L)$$

and for b) the case

$$|S| < c(D, p, \beta) L^{\frac{D+\lambda}{\beta - p}}.$$

Here we can use Theorem 2.8 to get a finite size criterion

$$|S^D| \gamma_L^{per} \geq c\left(\gamma_n - c n^{\delta-1} \right).$$

We required $\ln(L)\gamma_L^{per} \to 0$ and hence it holds

$$\lim_{L\to\infty} \ln(L)\gamma_L^{per} \geq \lim_{L\to\infty} \left(\frac{D+\lambda}{\beta}\ln(L) + \frac{p}{\beta}\ln\left(\frac{L}{2}\right)\right)\gamma_L^{per} \geq \lim_{L\to\infty} |S|^D \gamma_L^{per} = 0.$$

We choose the subsystem size n such that for some $\delta \in (0,1)$ holds $n^\delta > |S|$. The right hand side of the conditional finite size criterion can then be approximated by

$$c\left(\gamma_n - c\frac{|S|}{n}\right) \geq c\left(\gamma_n - \frac{c}{n^{1-\delta}}\right).$$

So it holds for $L \to \infty$

$$c|S^D|\gamma_L^{per} \geq \gamma_n - \frac{c}{n^{1-\delta}} \to 0$$

and hence $\gamma_L \in o(L^{\delta-1})$ by changing the indices. $\square$

3 Proof of refined results

In this section we will prove the more general results containing the concept of the dominant interaction scale, that is Proposition 2.7 and Theorem 2.8. The proof of Proposition 2.7 consists of short calculations and the application of some inequalities. On the other hand Theorem 2.8 needs more work and intertwines known finite size criteria, with the new concept of the dominant interaction scale.

3.1 Case of rapidly closing spectral gap

Recall that we have given a system with "large" dominant interaction scale, more precisely, (2.5) or (2.6) hold. The idea is to use translation invariance and inequality (2.4) of the dominant interaction scale. As last step we use the required inequality for the dominant interaction scale to get the claim.

Proof of Proposition 2.7. In this proof we will keep track of the value of the constants, so we can approximate them later. The spectral gap γ_L^{per} is the first non-zero eigenvalue of H_L^{per} with the eigenvector ϕ from equation (2.2), hence

$$\gamma_L^{per} = \langle H_L^{per}\rangle = \sum_{j,k\in\Lambda_L} \langle h_{j,k}\rangle.$$

Using translation invariance in Proposition 2.4 and inequality (2.4) we can simplify this to

$$\begin{aligned}\gamma_L^{per} &= (L+1)^D \sum_{t\in\left([-\frac{L}{2},\frac{L}{2}]\cap\mathbb{Z}\right)^D\setminus\{0\}} \langle h_{0,t}\rangle \\ &\leq |S|^p f(S)\langle P_{0,S}\rangle (L+1)^D \sum_{t\in\left([-\frac{L}{2},\frac{L}{2}]\cap\mathbb{Z}\right)^D\setminus\{0\}} \frac{1}{|t|^p}.\end{aligned}$$

To get an approximation for this sum we use the following proposition.

Lemma 3.1 (Integral Approximation). *Let $L \in \mathbb{N}$ be an even number and consider the lattice $\Omega_L := ([-\frac{L}{2}, \frac{L}{2}] \cap \mathbb{Z})^D$ for a dimension $D \in \mathbb{N}$. Let $g : \mathbb{R}^D \to \mathbb{R}$ be a positive monotonically decaying function, i.e. it holds for all $x, y \in \mathbb{R}^D$*

$$|x| \geq |y| \to g(x) \leq g(y).$$

Further let g be upper bounded by $g(x) \leq x^{-s}$ for some $s \in \mathbb{R}$. Then for all $a \leq b \in [1,\infty)$ the inequality

$$\sum_{\substack{k\in\Omega_L\\ a\leq|k|\leq b}} g(k) \leq ca^D g(a) + c\begin{cases}\frac{a^D g(a)}{s-D} & ,s>D\\ \ln(b) & ,s=D\\ \frac{b^D g(b)}{D-s} & ,s<D\end{cases}$$

holds for a suitable constant $c > 0$.

The proof of this Lemma is in the Appendix 5.1. Using Lemma 3.1 with $p > 2D$ and $a = 1$ we can approximate this sum by a real constant $c > 1$ and it holds

$$(L+1)^D \sum_{t \in \left([-\frac{L}{2}, \frac{L}{2}] \cap \mathbb{Z}\right)^D \setminus \{0\}} \frac{1}{|t|^p} \leq cL^D.$$

From now on we use c for a constant with value $c > 1$ in the rest of this proof. The scalar product of the projection $P_{0,S}$ can be roughly approximated by $\langle P_{0,S} \rangle \leq 1$. Hence for the spectral gap holds the claim $\gamma_L^{per} \leq L^{-\lambda}$ if the inequality

$$cL^D |S|^p f(S) \leq L^{-\lambda}$$

is true. For the cases a) this is equivalent to

$$|S| - \frac{p}{\beta} \ln(|S|) \geq \frac{D + \lambda}{\beta} \ln(L) + \frac{\ln(c)}{\beta}$$

and for b) this is equivalent to

$$|S| \geq c^{\frac{1}{\beta - p}} L^{\frac{D+\lambda}{\beta - p}}.$$

So the spectral gap γ_L^{per} is upper bounded by $L^{-\lambda}$ and the claim follows. □

3.2 Conditional finite size criterion

In Theorem 2.8 we require a "small" dominant interaction scale. This allows us to choose subsystems so big, such that they contain the dominant interaction scale. Therefore we can approximate the whole Hamiltonian H_L^{per} by subsystems. In contrast to existing finite size criteria we cannot build it out of subsystems.

Nevertheless we can roughly follow the structure of existing finite size proofs where we use the concept of the dominant interaction scale, that is for example the same strategy as in the proofs of [GM16] or [Lem19a].

Proof of Theorem 2.8. The proof consists of three parts. In the first part we get an expression for $\langle \left(H_L^{per}\right)^2 \rangle$. In the second step we introduce an auxiliary operator constructed out of size-n subsystems and compute an upper and lower bound dependent on the whole Hamiltonian for it. In the third and last step we put all results together to get a lower bound for γ_L^{per}.

3.2.1 Squaring the Hamiltonian

For two different pairs $(j,k), (j',k') \in \Lambda_L \times \Lambda_L$ with $j < k$ and $j' < k'$ we write $(j,k) \sim (j',k')$ if they have one common site. Analog we write $(j,k) \not\sim (j',k')$ if they have no common site. Consider the square of the Hamiltonian H_L^{per}. We get terms of pairs with two common sites, one common site and pairs with no common site

$$\left(H_L^{per}\right)^2 = \sum_{j<k \in \Lambda_L} h_{j,k}^2 + Q_L + R_L \tag{3.1}$$

with $Q_L = \sum_{(j,k) \sim (j',k')} \{h_{j,k}, h_{j',k'}\}$ and $R_L = \sum_{(j,k) \not\sim (j',k')} \{h_{j,k}, h_{j',k'}\}$. Taking the scalar product with respect to ϕ from equation (2.2), the first summand can be rewritten as

$$\sum_{j<k \in \Lambda_L} \langle h_{j,k}^2 \rangle = (L+1)^D \sum_{t \in ([-M,M] \cap \mathbb{Z})^D \setminus \{0\}} f(t)^2 \langle P_{0,t} \rangle$$

by using Proposition 2.4. Note there are $(L+1)^D$ pairs $j,k \in \Lambda_L$ which have the same $t \in ([-M,M] \cap \mathbb{Z})^D \setminus \{0\}$ such that $\langle P_{j,k} \rangle = \langle P_{0,t} \rangle$ holds. This leads to the factor $(L+1)^D$ in the equation. Note in addition that f is upper bounded by a decaying function, hence f^2 gets small a lot faster. Later we will only take those interactions into account which are reasonably large, that is up to the dominant interaction scale.

3.2.2 Subsystems and auxiliary operator

In this section we introduce the subsystem Hamiltonians $H_{\mathcal{B}^l}$ and the auxiliary operator A. For every site $l \in \Lambda_L$ we define the shifted box by

$$\mathcal{B}^l := \mathcal{B}_n + l := \{k \in \Lambda_L : k - l \in \mathcal{B}_n\}. \tag{3.2}$$

Recall the definition $\mathcal{B}_n := ([-n,n] \cap \mathbb{Z})^D$. The shifted subsystem Hamiltonian is defined by

$$H_{\mathcal{B}^l} := \sum_{j<k \in \mathcal{B}^l} h_{j,k}$$

with the corresponding spectral gap $\gamma_{\mathcal{B}^l}$. Now we can define the auxiliary operator by

$$A := \sum_{l \in \Lambda_L} (H_{\mathcal{B}^l})^2. \tag{3.3}$$

The aim is to find a relation between A and H_L^{per}. To compare the auxiliary operator with the Hamiltonian we us the scalar product with respect to ϕ from equation (2.2). In the next proposition we get a statement on an upper and a lower bound for the auxiliary operator A.

Lemma 3.2 (Upper and lower bound for the auxiliary operator)**.** *Let be $n > 2|S|$, $p > 2D+1$ and $\beta > 0$ for case a) and $\beta > p + \frac{(D+\lambda)(D+p)}{\delta}$ in case b). The auxiliary operator A defined by 3.3 has a lower bound given by*

$$\langle A \rangle \geq \gamma_n c \frac{n^D}{|S|^D} \langle H_L^{per} \rangle \tag{3.4}$$

and an upper bound for $D \geq 2$ given by

$$\begin{aligned}\langle A \rangle \leq & (L+1)^D \sum_{j \in ([-n,n]\cap\mathbb{Z})^D \setminus \{0\}} \prod_{m=1}^{D} (n+1-|j_m|) f(j)^2 \langle P_{0,j} \rangle \\ & + n^2 (n+1)^{D-2} \langle (Q_L + R_L) \rangle + |S|^p c(\beta,p) n^{D-1} \langle H_L^{per} \rangle.\end{aligned} \tag{3.5}$$

For $D = 1$ we get the upper bound

$$\begin{aligned}\langle A \rangle \leq & (L+1) \sum_{j \in ([-n,n]\cap\mathbb{Z}) \setminus \{0\}} (n+1-|j|) f(j)^2 \langle P_{0,j} \rangle \\ & + (n-1) \langle (Q_L + R_L) \rangle + |S|^p c(\beta,p) \langle H_L^{per} \rangle.\end{aligned}$$

The proof of this lemma is in Section 4.2.

3.2.3 Conclusion

Now we use the previous results to finish the proof. Starting with equation (3.1) and using the scalar product with respect to the eigenvector ϕ from (2.2) it holds

$$\langle (H_L^{per})^2 \rangle = (L+1)^D \sum_{t \in ([-M,M]\cap\mathbb{Z})^D \setminus\{0\}} f(t)^2 \langle P_{0,t} \rangle + \langle Q_L \rangle + \langle R_L \rangle.$$

With Lemma 3.2 the operators Q_L and R_L can be replaced by using the upper bound of the auxiliary operator given by inequality (3.5), hence

$$\begin{aligned}\langle (H_L^{per})^2 \rangle \geq{}& (L+1)^D \sum_{t \in ([-M,M]\cap\mathbb{Z})^D \setminus\{0\}} f(t)^2 \langle P_{0,t} \rangle \\ &- (L+1)^D \sum_{j \in ([-n,n]\cap\mathbb{Z})^D \setminus\{0\}} \frac{\prod_{m=1}^D (n+1-|j_m|)}{n^2(n+1)^{D-2}} f(j)^2 \langle P_{0,j} \rangle \\ &+ \frac{\langle A \rangle}{n^2(n+1)^{D-2}} - c(\beta,p)\frac{|S|^p n^{D-1}}{n^2(n+1)^{D-2}} \langle H_L^{per} \rangle.\end{aligned}$$

We will consider the first and second line of this expression together and the third line separately for better reading purposes:

- For the first two lines in the above equation it holds up to a factor of $(L+1)^D$

  $$\begin{aligned}&\sum_{t \in ([-M,M]\cap\mathbb{Z})^D \setminus\{0\}} f(t)^2 \langle P_{0,t} \rangle - \sum_{j \in ([-n,n]\cap\mathbb{Z})^D \setminus\{0\}} \frac{\prod_{m=1}^D (n+1-|j_m|)}{n^2(n+1)^{D-2}} f(j)^2 \langle P_{0,j} \rangle \\ \geq{}& \sum_{j \in ([-n,n]\cap\mathbb{Z})^D \setminus\{0\}} \left(1 - \frac{\prod_{m=1}^D (n+1-|j_m|)}{n^2(n+1)^{D-2}}\right) f(j)^2 \langle P_{0,j} \rangle.\end{aligned}$$

  If this bracket is positive, we approximate it by 0. The only cases in which this bracket is negative is exactly for $j \in \{e_1, \cdots, e_D\}$, i.e. for the unit vectors. Because of Proposition 2.4 there is at least one positive coordinate in j and hence $-e_1, \cdots, -e_D$ are not valid. Therefore the first two lines can be approximated by

  $$-\frac{1}{n}(L+1)^D \sum_{j \in \{e_1,\cdots,e_D\}} f(j)^2 \langle P_{0,j} \rangle \geq -\frac{1}{n} \langle H_L^{per} \rangle.$$

- For the second line we use the lower bound of the auxiliary operator (3.4)

  $$\frac{\langle A \rangle}{n^2(n+1)^{D-2}} - c(\beta,p)\frac{|S|^p}{n}\langle H_L^{per} \rangle \geq \gamma_n \frac{c}{|S|^D} \langle H_L^{per} \rangle - c(\beta,p)\frac{|S|^p}{n}\langle H_L^{per} \rangle.$$

In total we get

$$\begin{aligned}\langle (H_L^{per})^2 \rangle &\geq -\frac{1}{n}\langle H_L^{per} \rangle + \gamma_n \frac{c}{|S|^D}\langle H_L^{per} \rangle - c(\beta,p)\frac{|S|^p}{n}\langle H_L^{per} \rangle \\ &\geq \frac{c}{|S|^D}\left(\gamma_n - c(\beta,p)\frac{|S|^{D+p}}{n}\right)\langle H_L^{per} \rangle.\end{aligned}$$

With the approximation

$$|S|^{D+p} < \left(\frac{D+\lambda+p}{\beta}\ln L\right)^{D+p} < \frac{L}{2}$$

for a L large enough, we can choose our subsystem size such that it holds $|S|^{D+p} < n^\delta$. For case b) we need to use the requirement $\beta - p > \frac{(D+\lambda)(D+p)}{\delta}$ and get the same result. By frustration-freeness and the spectral theorem we get the claimed inequality

$$\gamma_L^{per} \geq \frac{c_1}{|S|^D}\left(\gamma_n - c_2(\beta,p)n^{\delta-1}\right)$$

and this proves Theorem 2.8. □

4 Auxiliary results

In this section we prove some important technical results. An important organizing tool for the combinatorial part will be played by the notion of scope, introduced in Definition 4.2 below. First we prove Proposition 2.4, which utilizes translation invariance. Afterwards we have a closer look on the bounds for the auxiliary operator in the periodic boundary case. The main task here is to prove Lemma 4.1, where we introduce the concept of the scope.

4.1 Preliminary reduction: Translation to zero

The idea of the proof is, that the scalar product is independent of the translation, if the eigenvector is an eigenvector of the translation operator too. So we need to show the independency and the existence of such an eigenvector.

Proof of Proposition 2.4. Let $j < k \in \Lambda_L$ be two different sites. The unitary translation operators T_i satisfy

$$T_i^\dagger h_{j,k} T_i = h_{j+e_i,k+e_i}$$

for every dimension $i = 1, \cdots, D$. Because $[H_L^{per}, T_i] = 0$ and $[T_i, T_j] = 0$ for all $i \neq j = 1, \cdots, D$ holds, we can simultaneously diagonalize them. So a common eigenvector exists and we denoted it by ϕ. The energy of the simultaneous eigenvector ϕ of H_L^{per} and T_i is distributed evenly across the terms

$$\langle h_{j,k}\rangle = \langle T_i^\dagger h_{j,k} T_i\rangle = \langle h_{j+e_i,k+e_i}\rangle$$

for all dimensions. By repeatedly using this equation we find $\langle h_{j,k}\rangle = \langle h_{0,k-j}\rangle$. Recall the definition of $j < k$: there is a $m \in \{1, \cdots, D\}$ such that for all $1 \leq i < m$ holds $j_i = k_i$ and $j_m < k_m$. The coordinates $(k-j)_i$ satisfy therefore

$$(k-j)_i \in \begin{cases} \{0\} & i < m \\ \{1, \cdots, M\} & i = m \\ \{-M, \cdots, M\} & i > m. \end{cases}$$

Now we set $t_i := (k-j)_i$ for all $i = 1, \cdots, D$ and get the claim.

Especially if we consider two different edges $j, k \in \mathcal{B}^l$ in a box from Definition 3.2 there exists a $t \in ([-n,n] \cap \mathbb{Z})^D$ such that

$$\langle h_{j,k}\rangle = \langle h_{0,t}\rangle$$

holds. □

Example: In figure 1 and 2 it is shown how we translate $h_{a,b}$ and $h_{c,d}$ to $h_{0,b-a}$ and $h_{0,d-c}$. Note that the first coordinate of t can never be negative.

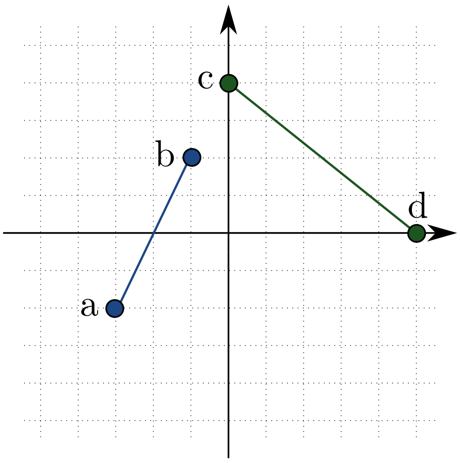

Figure 1: The blue line represents the interaction $h_{a,b}$ and the green line the interaction $h_{c,d}$.

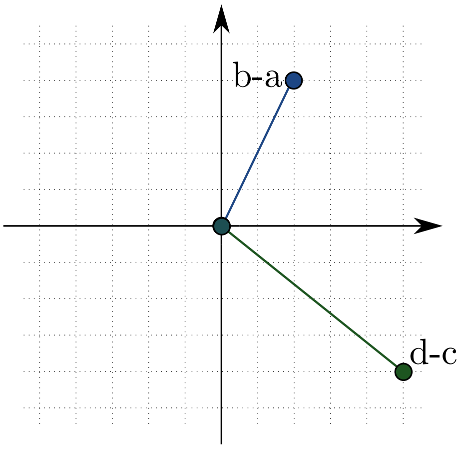

Figure 2: After the translation we have the interactions $h_{0,b-a}$ for blue and $h_{0,d-c}$ for green.

4.2 Bounds on the auxiliary operator

This section hast two parts. The first one gives a proof for the lower bound on the auxiliary operator. The key here is to use translation invariance and the dominant interaction scale to approximate the Hamiltonian. In the second part we prove the upper bound for the auxiliary operator. The auxiliary operator consists of squared interactions, that means we get three terms, some with two common sites, some with one common site and some with no common site. The main work here is to calculate how often we count each of them in the boxes. For some this is easier than for others.

Proof of Lemma 3.2 Part 1. Fix a site $l \in \Lambda_L$, then it holds by the spectral theorem and frustration-freeness

$$H_{\mathcal{B}^l}^2 \geq \gamma_{\mathcal{B}^l} H_{\mathcal{B}^l} = \gamma_n H_{\mathcal{B}^l}$$

while using translation invariance for the second step. With Proposition 2.4 it holds further

$$\langle H_{\mathcal{B}^l}^2 \rangle \geq \gamma_n \sum_{j<k \in \mathcal{B}^l} \langle h_{j,k} \rangle \geq \gamma_n \sum_{t \in ([-n,n] \cap \mathbb{Z})^D \setminus \{0\}} \langle h_{0,t} \rangle \prod_{m=1}^{D} (n+1-|t_m|).$$

Every summand $\langle h_{j,k} \rangle$ is not negative and every box B^l contains at least one S-term since the box size is $n > 2|S|$. So we neglect all $t \in ([-n,n] \cap \mathbb{Z})^D \setminus \{0\}$ with $|t| > |S|$ and get the approximation

$$\langle H_{\mathcal{B}^l}^2 \rangle \geq \gamma_n \sum_{\substack{t \in ([-n,n] \cap \mathbb{Z})^D \setminus \{0\} \\ |t| \leq |S|}} \langle h_{0,t} \rangle \prod_{m=1}^{D} (n+1-|t_m|). \tag{4.1}$$

This product can be approximated by

$$\prod_{m=1}^{D} (n+1-|t_m|) \geq \prod_{m=1}^{D} (n+1-|S|) \geq \left(n+1-\frac{L}{2}\right)^D \geq \frac{n^D}{2^D},$$

where we used the assumption $n > 2|S|$ in the last step. The equation (4.1) is independent of the site l, so the auxiliary operator has the same expression up to a factor of $(L+1)^D$, hence

$$\langle A\rangle = \langle \sum_{l\in\Lambda_L} H_{\mathcal{B}^l}^2\rangle \geq \gamma_n (L+1)^D \frac{n^D}{2^D} \sum_{\substack{t\in([-n,n]\cap\mathbb{Z})^D\setminus\{0\}\\ |t|\leq|S|}} \langle h_{0,t}\rangle.$$

The last step we are missing is a connection to H_L^{per}. The key difference here to existing finite size proofs, is that the subsystems $H_{\mathcal{B}^l}$ and the Hamiltonian H_L^{per} do not share the same summands, since

$$\langle H_L^{per}\rangle = (L+1)^D \sum_{t\in([-M,M]\cap\mathbb{Z})^D\setminus\{0\}} \langle h_{0,t}\rangle.$$

Especially the long range interactions are not contained in the subsystems. To be more clear, exactly the set

$$\tilde{W} := \{t\in([-M,M]\cap\mathbb{Z})^D\setminus\{0\} : |t| > |S|\}$$

is missing. Define $W := \tilde{W}\cup\{S\}$, then we can use the interaction of dominant interaction scale to approximate all the terms outside of the subsystems. By using property (2.4) we have

$$\langle h_{0,S}\rangle = \frac{\sum_{t\in W}\frac{1}{\alpha_t}}{\sum_{t\in W}\frac{1}{\alpha_t}}\langle h_{0,S}\rangle \geq \frac{1}{\alpha_S\sum_{t\in W}\frac{1}{\alpha_t}}\sum_{t\in W}\langle h_{0,t}\rangle.$$

Hence for the auxiliary operator holds

$$\langle A\rangle \geq \gamma_n (L+1)^D \frac{n^D}{2^D\alpha_S\sum_{t\in W}\frac{1}{\alpha_t}} \sum_{t\in([-M,M]\cap\mathbb{Z})^D\setminus\{0\}} \langle h_{0,t}\rangle = c\gamma_n n^D\langle H_L^{per}\rangle \frac{1}{\alpha_S\sum_{t\in W}\frac{1}{\alpha_t}}.$$

With Lemma 3.1 and $p > D$ we can approximate the remaining sum by

$$\alpha_S\sum_{t\in W}\frac{1}{\alpha_t} = 1 + |S|^p \sum_{\substack{t\in\tilde{W}\\ |t|>|S|}} \left(\frac{1}{|t|}\right)^p \leq c|S|^D$$

and the claim

$$\langle A\rangle \geq \gamma_n c\frac{n^D}{|S|^D}\langle H_L^{per}\rangle$$

follows. □

We recall now the second part of proposition of the auxiliary operator inequality.

(Auxiliary operator inequalities - Part 2). *With the eigenvector $\phi \in Eig(\gamma_L^{per})$ from equation* (2.2) *holds for $D\geq 2$*

$$\begin{aligned}\langle A\rangle \leq &(L+1)^D \sum_{j\in([-n,n]\cap\mathbb{Z})^D\setminus\{0\}} \prod_{m=1}^{D} (n+1-|j_m|) f(j)^2 \langle P_{0,j}\rangle\\ &+ n^2(n+1)^{D-2}\langle (Q_L+R_L)\rangle + |S|^p c(\beta,p) n^{D-1}\langle H_L^{per}\rangle.\end{aligned}$$

For $D=1$ we get the upper bound

$$\begin{aligned}\langle A\rangle \leq &(L+1) \sum_{j\in([-n,n]\cap\mathbb{Z})\setminus\{0\}} (n+1-|j|) f(j)^2 \langle P_{0,j}\rangle\\ &+ (n-1)\langle (Q_L+R_L)\rangle + |S|^p c(\beta,p)\langle H_L^{per}\rangle.\end{aligned}$$

Proof of Lemma 3.2 Part 2. Starting with the auxiliary operator we square the subsystem hamiltonians $H_{\mathcal{B}^l}$ and sort the terms the following way

$$\langle A\rangle = \sum_{l\in\Lambda_L}\sum_{j<k\in\mathcal{B}^l}(f(j-k))^2\langle P_{j,k}\rangle + \sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle + \sum_{l\in\Lambda_L}\langle R_{\mathcal{B}^l}\rangle$$

where we define

$$Q_{\mathcal{B}^l} := \sum_{(j,k)\sim_{\mathcal{B}^l}(j',k')}\{h_{j,k}, h_{j',k'}\} \qquad \text{and} \qquad R_{\mathcal{B}^l} := \sum_{(j,k)\not\sim_{\mathcal{B}^l}(j',k')}\{h_{j,k}, h_{j',k'}\}.$$

Now we consider every summand on its own:

- In the first one we consider pairs $(j,k)\in\Lambda_L\times\Lambda_L$ with $j<k$ and $|j_i-k_i|\leq n$ for all $i=1,\cdots,D$. Every pair (j,k) appears in $\prod_{m=1}^{D}(n+1-(|j_m-k_m|))$ different boxes and hence

  $$\begin{aligned}\sum_{l\in\Lambda_L}\sum_{j<k\in\mathcal{B}^l}(f(j-k))^2\langle P_{j,k}\rangle &= \sum_{\substack{j<k\in\Lambda_L\\ |j_i-k_i|\leq n}}\prod_{m=1}^{D}(n+1-|j_m-k_m|)(f(j-k))^2\langle P_{j,k}\rangle\\ &= (L+1)^D\sum_{t\in([-n,n]\cap\mathbb{Z})^D\setminus\{0\}}\prod_{m=1}^{D}(n+1-|t_m|)f(t)^2\langle P_{0,t}\rangle\end{aligned}$$

  where we used Proposition 2.4 in the second step.

- For the third term it holds

  $$\sum_{l\in\Lambda_L}\langle R_{\mathcal{B}_l}\rangle \leq \begin{cases}(n-2)\langle R_L\rangle & ,D=1\\ n^2(n+1)^{D-2}\langle R_L\rangle & ,D\geq 2\end{cases}$$

  because all terms are not negative and every $(j,k)\not\sim(j',k')$ gets counted at most $(n-2)\leq(n-1)$ times for $D=1$ and $n^2(n+1)^{D-2}$ times in the case $D\geq 2$.

- The next lemma handles the second term.

  Lemma 4.1 (Site-sharing terms)**.** *Let $p>2D+1$ and let further be in case a) $\beta>0$ and $\beta>p+\frac{(D+\lambda)(D+p)}{\delta}$ in case b), then the approximations*

  $$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &\leq (n-1)\langle Q_L\rangle + \alpha_S c(\beta,p)\langle H_L^{per}\rangle &&,D=1\\ \sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &\leq n^2(n+1)^{D-2}\langle Q_L\rangle + \alpha_S c(\beta,p)n^{D-1}\langle H_L^{per}\rangle &&,D\geq 2\end{aligned}$$

  with a suitable constants $c(\beta,p)>0$ hold.

Together this proves the upper bound given by (3.5) for the auxiliary operator. □

4.3 Treating the site-sharing terms

This proposition deals with all terms containing one common site. We want to represent $\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle$ in terms of $\langle Q_L\rangle$. The idea is that the remainders are small enough such that they can be approximated suitably. It turns out that we need to count the terms quite accurately and therefore we introduce the concept of the *scope*. Geometrically, the scope of two edges is the size of the minimal axis-aligned box containing the three sites.

Definition 4.2 (Scope)**.** Let $(j,k) \sim (j',k')$ be two different pairs of sites in Λ_L^2 with one common site, i.e. $\{j,k,j',k'\} = \{j,k,t\}$ for some $t \in \{j',k'\}$. Let without loss of generality be k the common site. Then the scope $r \in \Lambda_L$ of the sites is defined for every coordinate $i = 1, \cdots, D$ by

$$r_i := \begin{cases} |j_i - k_i| + |k_i - t_i| & j_i \leq k_i \leq t_i \vee t_i \leq k_i \leq j_i \\ \max\{|j_i - k_i|, |k_i - t_i|\} & else \end{cases}$$

If j or t is the common edge we switch k_i and j_i or t_i in the definition respectively.

The coordinates of the scope indicate the length of the smallest interval, such that every point j_i, k_i, t_i is in that interval. For an intuition see figure 3. It shows how to get from three points in two dimensions to their scope.

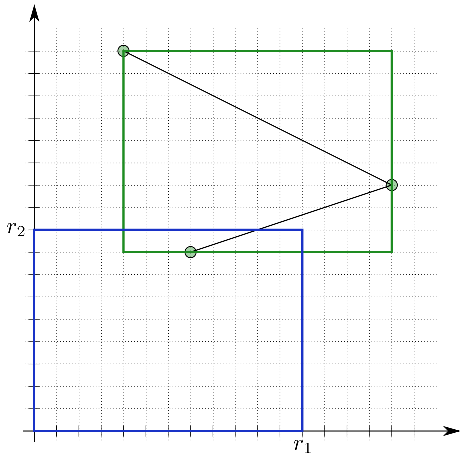

Figure 3: Scope of three different points. The green rectangle is the smallest rectangle including all three points. The blue rectangle is a shifted green rectangle and the diagonal of the blue rectangle represents the scope.

That means the scope can be described as the diagonal of the smallest box around three points. We will see that how often we count the terms is only dependent on the scope. To approximate the remainders we also use the operator chauchy-schwarz inequality and approximate sums by an integral.

Proof of Lemma 4.1. We consider a box $\mathcal{B}^l$ for a fixed $l \in \Lambda_L$. Then every two pairs $(j,k) \sim_{\mathcal{B}^l} (j',k')$ have a scope with values in $\mathcal{B}_n = ([0,n] \cap \mathbb{Z})^D$.

On the other hand note that each term $(j,k) \sim (j',k')$ with pairs in Λ_L^2 and scope $r \in \mathcal{B}_n$ gets counted in $\prod_{m=1}^{D}(n+1-r_m)$ different boxes. We define now four disjoint sets

$$\begin{aligned} E_0 &:= \{r = e_i + e_j : i,j \in \{1, \cdots, D\}\} \\ E_1 &:= \{r \in \mathcal{B}_n : 1 < |r| \leq 8 + 4\sqrt{D}\} \setminus E_0 \\ E_2 &:= \{r \in \mathcal{B}_n : |r| > 8 + 4\sqrt{D}\} \\ E_3 &:= \{r \in \Lambda_L \setminus \mathcal{B}_n\}. \end{aligned}$$

The set E_0 contains the scopes of nearest neighbor interactions. The set E_1 and E_2 contain the remaining scope values of the interactions in the subsystems. If the scope value is in the set E_3 we do not count it in any box, but they occur in the Q_L term. We can rewrite the left side of Lemma 4.1 to

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &= \sum_{l\in\Lambda_L}\sum_{(j,k)\sim_{\mathcal{B}^l}(j',k')}\langle\{h_{j,k},h_{j',k'}\}\rangle\\ &= \sum_{l\in\Lambda_L}\sum_{\substack{r\in\mathcal{B}_n\\|r|>1}}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope(j,k,j',k')=r}}\langle\{h_{j,k},h_{j',k'}\}\rangle\\ &= \sum_{\substack{r\in\mathcal{B}_n\\|r|>1}}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope(j,k,j',k')=r}}\prod_{m=1}^{D}(n+1-r_m)\langle\{h_{j,k},h_{j',k'}\}\rangle. \qquad (4.2)\end{aligned}$$

Here we split up the sum into terms with the same scope values, because this value decides how often we count the terms. So the nearest neighbor terms, that is $r\in E_0$, get counted the most and hence we want to replace the nearest neighbor terms later with $\langle Q_L\rangle$.

4.3.1 Nearest neighbor terms

The nearest neighbor terms, denoted by Q_{NN} and defined by

$$Q_{NN} := \sum_{r\in E_0}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope=r}}\{h_{j,k},h_{j',k'}\}$$

consist of the terms with scope $r=e_i+e_j$ with $i,j=1,\cdots,D$. For $i=j$ we count the term in $(n-1)(n+1)^{D-1}$ different boxes and for $i\neq j$ we count it in $n^2(n+1)^{D-2}$ boxes. For $D=1$ only the case $i=j$ appears and hence we count the term $n-1$ times.

We denote the $i=j$ terms as *line* and the $i\neq j$ terms as *edge* terms. With $Q_{NN}=Q_{line}+Q_{edge}$ we get for $D\geq 2$

$$\begin{aligned}\sum_{l\in\Lambda_L}\sum_{r\in E_0}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope=r}}\langle\{h_{j,k},h_{j',k'}\}\rangle &= n^2(n+1)^{D-2}\langle Q_{edge}\rangle+(n-1)(n+1)^{D-1}\langle Q_{line}\rangle\\ &= n^2(n+1)^{D-2}\langle Q_{NN}\rangle-(n+1)^{D-2}\langle Q_{line}\rangle.\end{aligned}$$

The line terms can be handled using the operator chauchy-schwarz inequality and Proposition 2.4 the following way

$$\begin{aligned}-\langle Q_{line}\rangle &= -\sum_{i=1}^{D}\sum_{j\in\Lambda_L}\langle\{h_{j,j+e_i},h_{j+e_i,j+2e_i}\}\rangle\\ &\leq 2\sum_{i=1}^{D}\sum_{j\in\Lambda_L}\langle h_{j,j+e_i}\rangle\leq 2\langle H_L^{per}\rangle.\end{aligned}$$

For $D=1$ this simplifies to

$$\sum_{l\in\Lambda_L}\sum_{r\in E_0}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope=r}}\langle\{h_{j,k},h_{j',k'}\}\rangle=(n-1)\langle Q_{NN}\rangle.$$

Returning to equation (4.2) we can rewrite it for $D\geq 2$ as

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &= n^2(n+1)^{D-2}\langle Q_{NN}\rangle+2(n+1)^{D-2}\langle H_L^{per}\rangle\\ &\quad+\sum_{r\in E_1\cup E_2}\sum_{\substack{(j,k)\sim_{\mathcal{B}^l}(j',k')\\ scope(j,k,j',k')=r}}\prod_{m=1}^{D}(n+1-r_m)\langle\{h_{j,k},h_{j',k'}\}\rangle. \qquad (4.3)\end{aligned}$$

We introduce now the $\langle Q_L\rangle$ term from equation (3.1) and split the sum and group the terms with same scope. Hence we have

$$\begin{aligned}\langle Q_L\rangle &= \sum_{\substack{r\in\Lambda_L\\ |r|>1}} \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{k,t}\}\rangle\\ &= \langle Q_{NN}\rangle + \left(\sum_{r\in E_1\cup E_2} + \sum_{r\in E_3}\right) \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{k,t}\}\rangle.\end{aligned}$$

We replace the nearest neighbor term Q_{NN} in equation (4.3) with the Q_L term. All the terms that get counted less than $n^2(n+1)^{D-2}$ times need to be subtracted and we get

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle \le{}& n^2(n+1)^{D-2}\langle Q_L\rangle + 2(n+1)^{D-2}\langle H_L\rangle \qquad (4.4)\\ &- \sum_{r\in E_1\cup E_2}\left(n^2(n+1)^{D-2} - \prod_{m=1}^{D}(n+1-r_m)\right) \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{j',k'}\}\rangle\\ &- n^2(n+1)^{D-2}\sum_{r\in E_3} \sum_{\substack{(j,k)\sim(k,t)\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{j',k'}\}\rangle.\end{aligned}$$

Analogously for $D=1$ this simplifies to

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle \le{}& (n-1)\langle Q_L\rangle \qquad (4.5)\\ &- \sum_{r\in E_1\cup E_2}((n-1)-(n+1-r)) \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{j',k'}\}\rangle\\ &- (n-1)\sum_{r\in E_3} \sum_{\substack{(j,k)\sim(k,t)\\ scope(j,k,j',k')=r}} \langle\{h_{j,k},h_{j',k'}\}\rangle.\end{aligned}$$

So far, we have an expression for $\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle$ in terms of $\langle Q_L\rangle$ and some left over terms. Now we have to approximate the remaining terms and express them by $\langle H_L^{per}\rangle$.

4.3.2 Reduction to scope dependency

In this subsection we focus on the remainders of inequality (4.4) and (4.5). Our goal is to find an expression, which is only dependent on the scope r and contains $\langle H_L^{per}\rangle$. In the first step we use the operator cauchy-schwarz inequality

$$-\langle\{P_{j,k},P_{j',k'}\}\rangle \le \langle P_{j,k}\rangle + \langle P_{j',k'}\rangle$$

for every $j,k,j',k'\in\Lambda_L$ and get rid of the minus signs.

By Proposition 2.4 we can translate all the $\langle P_{j,k}\rangle$ terms by j and the $\langle P_{j',k'}\rangle$ terms by j'. Further by definition (2.4) we can approximate the terms with the dominant interaction scale and hence it holds

$$\begin{aligned}&\sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} f(j-k)f(j'-k')\left(\langle P_{j,k}\rangle + \langle P_{j',k'}\rangle\right)\\ \le{}&\alpha_S f(S)\langle P_{0,S}\rangle \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right).\end{aligned}$$

Note that now all factors are positive. The next proposition is used to simplify some expressions, especially the sum over the sites.

Proposition 4.3 (Combinatorial results). *a) Let $(j,k) \sim (j',k')$ be two different pairs of sites in Λ_L^2 with one common site. Consider $(j,k) \sim (j',k')$ with scope $r \in \Lambda_L$ and $|r| > 1$. Then it holds*

$$\sum_{\substack{(j,k)\sim_{\mathcal{B}_r}(j',k') \\ scope(j,k,j',k')=r}} 1 \leq c|r|^D. \tag{4.6}$$

b) Let $\mathcal{B}_n$ be a box in dimension D with side length n. Let further be $r \in \mathcal{B}_n$ with $|r| > 1$, then it holds

$$n^2(n+1)^{D-2} - \prod_{m=1}^{D}(n+1-r_m) \leq c\sum_{m=0}^{D-1} n^m|r|^{D-m}. \tag{4.7}$$

We used the notation r_m for the coordinates of r.

Using inequality (4.7) we can approximate $\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle$ by

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &\leq n^2(n+1)^{D-2}\langle Q_L\rangle + 2(n+1)^{D-2}\langle H_L\rangle \\ &\quad + c\alpha_S f(S)\langle P_{0,S}\rangle \sum_{m=0}^{D-1} n^m \sum_{r\in E_1\cup E_2} |r|^{D-m} \sum_{\substack{(j,k)\sim(j',k') \\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right) \\ &\quad + c\alpha_S f(S)\langle P_{0,S}\rangle n^2(n+1)^{D-2} \sum_{r\in E_3} \sum_{\substack{(j,k)\sim(j',k') \\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right).\end{aligned}$$

On the other hand for the case $D = 1$ we get the expression

$$\begin{aligned}\sum_{l\in\Lambda_L}\langle Q_{\mathcal{B}^l}\rangle &\leq (n-1)\langle Q_L\rangle \\ &\quad + c\alpha_S f(S)\langle P_{0,S}\rangle \sum_{r\in E_1\cup E_2} |r| \sum_{\substack{(j,k)\sim(j',k') \\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right) \\ &\quad + c\alpha_S f(S)\langle P_{0,S}\rangle (n-1) \sum_{r\in E_3} \sum_{\substack{(j,k)\sim(j',k') \\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right).\end{aligned}$$

Now we are left with the sums dependent on j,k,j',k'. To find an expression only dependent on the scope we consider the sets E_1, E_2, E_3 separately.

For all $r \in E_1$ it holds either $|j-k| \geq \sqrt{2}$ or $|j'-k'| \geq \sqrt{2}$. By symmetry in half of the cases holds the first one and in the other half the second one and we can approximate it by some constant

$$\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}} \leq \frac{f(\sqrt{2})}{\alpha_1} + \frac{f(1)}{\alpha_{\sqrt{2}}} = c(\beta,p)$$

for all j,k,j',k' with scope $r \in E_1$. The next proposition gives a relation between the sites $(j,k) \sim (j',k')$ and their scope r, which we use for scopes $|r| > 4$.

Proposition 4.4 (minimum distance)**.** *Let $j, k, t \in \Lambda_L$ be different sites. Consider the two pairs $(j,k) \sim (k,t)$ with scope r. Then it holds $|j-k| \geq \frac{|r|}{4}$ or $|k-t| \geq \frac{|r|}{4}$.*

This Proposition 4.4 holds especially for scopes in E_2 and E_3. Using the translation invariance we can shift every term $(j,k) \sim (j',k')$ with scope r into the box $\mathcal{B}_r := \prod_{m=1}^{D}([0,r_m] \cap \mathbb{Z})$ leading to a factor of $(L+1)^D$ and hence

$$\sum_{\substack{(j,k)\sim(k,t)\\ scope=r}} 1 = \sum_{\substack{(j,k)\sim_{\mathcal{B}_r}(k,t)\\ scope=r}} (L+1)^D \leq (L+1)^D c|r|^D$$

holds, where we used inequality (4.6). Using this approximation we get for $r \in E_1$

$$\sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} \left(\frac{f(j'-k')}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{j'-k'}}\right) \leq c(\beta,p) \sum_{\substack{(j,k)\sim(j',k')\\ scope(j,k,j',k')=r}} 1 \leq c(\beta,p)(L+1)^D|r|^D.$$

For $|r| > 4$ and especially for $r \in E_2 \cup E_3$ it holds with Proposition 4.4

$$\sum_{\substack{(j,k)\sim(k,t)\\ scope=r}} \left(\frac{f(k-t)}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{t-k}}\right) \leq \left(\frac{f\left(\frac{r}{4}\right)}{\alpha_1} + \frac{f(1)}{\alpha_{\frac{r}{4}}}\right) \sum_{\substack{(j,k)\sim(k,t)\\ scope=r}} 1$$
$$\leq c(\beta,p)(L+1)^D|r|^D \left(e^{-\beta|r|} + |r|^{-p}\right)$$

where we used $f(1) = 1 = \alpha_1$, $f(r/4) \leq c(\beta)e^{-\beta|r|}$ for case a) and $\alpha_{r/4} = c(p)|r|^{-p}$. In case b) we get $f(r/4) \leq c(\beta)|r|^{-\beta}$ and hence the inequality

$$\sum_{\substack{(j,k)\sim(k,t)\\ scope=r}} \left(\frac{f(k-t)}{\alpha_{k-j}} + \frac{f(k-j)}{\alpha_{t-k}}\right) \leq c(\beta,p)(L+1)^D|r|^D \left(|r|^{-\beta} + |r|^{-p}\right).$$

Recall that every term $\langle P_{0,j}\rangle$ is not negative and we can approximate $(L+1)^D f(S)\langle P_{0,S}\rangle$ by $\langle H_L^{per}\rangle$ and hence

$$(L+1)^D \alpha_S f(S)\langle P_{0,S}\rangle \leq \alpha_S \langle H_L^{per}\rangle.$$

Putting this together while using Proposition 4.3 we have for $D \geq 2$ the expression

$$\begin{aligned}\sum_{l\in\Lambda_L} \langle Q_{\mathcal{B}^l}\rangle \leq{}& n^2(n+1)^{D-2}\langle Q_L\rangle + 2(n+1)^{D-2}\langle H_L^{per}\rangle \\ &+ c(\beta,p)\alpha_S\langle H_L^{per}\rangle \sum_{m=0}^{D-1} n^m \sum_{r\in E_1} |r|^{2D-m} \\ &+ c(\beta,p)\alpha_S\langle H_L^{per}\rangle \sum_{m=0}^{D-1} n^m \sum_{r\in E_2} |r|^{2D-m}\left(e^{-\beta|r|} + |r|^{-p}\right) \\ &+ c(\beta,p)\alpha_S\langle H_L^{per}\rangle n^2(n+1)^{D-2} \sum_{r\in E_3} |r|^D \left(e^{-\beta|r|} + |r|^{-p}\right).\end{aligned}$$

This calculation is for case a) and for case b) we can replace $e^{-\beta|r|}$ with $|r|^{-\beta}$. For $D = 1$ the same approximations are true and a similar expression can be obtained. All of these sums converge and the value can be approximated by using an integral approximation. The calculations for the approximation can be found in the appendix 5.2. Obviously since it holds $\beta > p$ in case b) it suffices to show that the sum with $|r|^{D-p}$ converges to get the convergence of $|r|^{D-\beta}$. In conclusion for general dimension $D \geq 2$ it holds

$$\sum_{l \in \Lambda_L} \langle Q_{\mathcal{B}_l} \rangle \leq n^2 (n+1)^{D-2} \langle Q_L \rangle + 2(n+1)^{D-2} \langle H_L^{per} \rangle$$
$$+ c(\beta, p) \alpha_S \langle H_L^{per} \rangle n^{D-1}.$$

For $D = 1$ the approximations in the appendix can be done analogously and hence it holds

$$\sum_{l \in \Lambda_L} \langle Q_{\mathcal{B}_l} \rangle \leq (n-1) \langle Q_L \rangle + c(\beta, p) \alpha_S \langle H_L^{per} \rangle.$$

This proves Lemma 4.1.

$\square$

References

[AAG22] Anurag Anshu, Itai Arad, and David Gosset. An area law for 2d frustration-free spin systems. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 12–18, 2022.

[AKLV13] Itai Arad, Alexei Kitaev, Zeph Landau, and Umesh Vazirani. An area law and sub-exponential algorithm for 1d systems, 2013.

[ALM22] Radu Andrei, Marius Lemm, and Ramis Movassagh. The spin-one motzkin chain is gapped for any area weight $t < 1$. *arXiv preprint arXiv:2204.04517*, 2022.

[Ans20] Anurag Anshu. Improved local spectral gap thresholds for lattices of finite size. *Physical Review B*, 101(16):165104, 2020.

[ARLL+20] Houssam Abdul-Rahman, Marius Lemm, Angelo Lucia, Bruno Nachtergaele, and Amanda Young. A class of two-dimensional aklt models with a gap. *Contemporary Mathematics*, 741:1–21, 2020.

[BHM10] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis. Topological quantum order: stability under local perturbations. *Journal of mathematical physics*, 51(9), 2010.

[BHNY15] Sven Bachmann, Eman Hamza, Bruno Nachtergaele, and Amanda Young. Product vacua and boundary state models in d-dimensions. *Journal of Statistical Physics*, 160(3):636–658, 2015.

[BN14] Sven Bachmann and Bruno Nachtergaele. Product vacua with boundary states and the classification of gapped phases. *Communications in Mathematical Physics*, 329(2):509–544, 2014.

[DRS19] Wojciech De Roeck and Manfred Salmhofer. Persistence of exponential decay and spectral gaps for interacting fermions. *Communications in Mathematical Physics*, 365(2):773–796, 2019.

[DVFPR21] Simone Del Vecchio, Jürg Fröhlich, Alessandro Pizzo, and Stefano Rossi. Lie–schwinger block-diagonalization and gapped quantum chains with unbounded interactions. *Communications in Mathematical Physics*, 381(3):1115–1152, 2021.

[DVFPR25] Simone Del Vecchio, Jürg Fröhlich, Alessandro Pizzo, and Alessio Ranallo. Boundary effects and the stability of the low energy spectrum of the aklt model. *Journal of Functional Analysis*, page 111269, 2025.

[FNW92] M. Fannes, B. Nachtergaele, and R. F. Werner. Finitely correlated states on quantum spin chains. *Communications in Mathematical Physics*, 144(3):443–490, 3 1992.

[FP20] Juerg Froehlich and Alessandro Pizzo. Lie–schwinger block-diagonalization and gapped quantum chains. *Communications in Mathematical Physics*, 375(3):2039–2069, 2020.

[GM16] David Gosset and Evgeny Mozgunov. Local gap threshold for frustration-free spin systems. *Journal of Mathematical Physics*, 57(9), September 2016.

[Has07] M B Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08024–P08024, August 2007.

[HHJ21] Jonas Haferkamp and Nicholas Hunter-Jones. Improved spectral gaps for random quantum circuits: Large local dimensions and all-to-all interactions. *Physical Review A*, 104(2):022417, 2021.

[HJL25] Nicholas Hunter-Jones and Marius Lemm. Two classes of quantum spin systems that are gapped on any bounded-degree graph. *arXiv preprint arXiv:2509.22438*, 2025.

[HK06] Matthew B Hastings and Tohru Koma. Spectral gap and exponential decay of correlations. *Communications in mathematical physics*, 265(3):781–804, 2006.

[HW05] Matthew B Hastings and Xiao-Gang Wen. Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance. *Physical Review B—Condensed Matter and Materials Physics*, 72(4):045141, 2005.

[JL22] Ian Jauslin and Marius Lemm. Random translation-invariant hamiltonians and their spectral gaps. *Quantum*, 6:790, 2022.

[KL18] Michael J Kastoryano and Angelo Lucia. Divide and conquer method for proving gaps of frustration free hamiltonians. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(3):033105, 2018.

[Kna88] Stefan Knabe. Energy gaps and elementary excitations for certain vbs-quantum antiferromagnets. *Journal of Statistical Physics*, 52(3):627–638, 8 1988.

[KS20] Tomotaka Kuwahara and Keiji Saito. Strictly linear light cones in long-range interacting systems of arbitrary dimensions. *Physical Review X*, 10(3):031010, 2020.

[Lem19a] Marius Lemm. Finite-size criteria for spectral gaps in d-dimensional quantum spin systems, 2019.

[Lem19b] Marius Lemm. Gaplessness is not generic for translation-invariant spin chains. *Physical Review B*, 100(3):035113, 2019.

[LL25] Marius Lemm and Angelo Lucia. On the critical finite-size gap scaling for frustration-free hamiltonians. *Reviews in Mathematical Physics*, June 2025.

[LM19] Marius Lemm and Evgeny Mozgunov. Spectral gaps of frustration-free spin systems with boundary. *Journal of Mathematical Physics*, 60(5), May 2019.

[LN19] Marius Lemm and Bruno Nachtergaele. Gapped pvbs models for all species numbers and dimensions. *Reviews in Mathematical Physics*, 31(09):1950028, 2019.

[LNWY26] Marius Lemm, Bruno Nachtergaele, Simone Warzel, and Amanda Young. Recursive spectral relations and the charge vs neutral gap in fractional quantum hall systems. *Journal of Mathematical Physics*, 67(1), 2026.

[LSW20] Marius Lemm, Anders W Sandvik, and Ling Wang. Existence of a spectral gap in the affleck-kennedy-lieb-tasaki model on the hexagonal lattice. *Physical review letters*, 124(17):177204, 2020.

[LX22] Marius Lemm and David Xiang. Quantitatively improved finite-size criteria for spectral gaps. *Journal of Physics A: Mathematical and Theoretical*, 55(29):295203, July 2022.

[MSW25a] Rintaro Masaoka, Tomohiro Soejima, and Haruki Watanabe. Rigorous lower bound of the dynamical critical exponent of the ising model: M. rintaro et al. *Journal of Statistical Physics*, 192(6):76, 2025.

[MSW25b] Rintaro Masaoka, Tomohiro Soejima, and Haruki Watanabe. Rigorous lower bound on dynamical exponents in gapless frustration-free systems. *Physical Review X*, 15(4):041050, 2025.

[MZ13] Spyridon Michalakis and Justyna P Zwolak. Stability of frustration-free hamiltonians. *Communications in Mathematical Physics*, 322(2):277–302, 2013.

[Nac96] Bruno Nachtergaele. The spectral gap for some spin chains with discrete symmetry breaking. *Communications in Mathematical Physics*, 175(3):565–606, February 1996.

[NS06] Bruno Nachtergaele and Robert Sims. Lieb-robinson bounds and the exponential clustering theorem. *Communications in mathematical physics*, 265(1):119–130, 2006.

[NSY22] Bruno Nachtergaele, Robert Sims, and Amanda Young. Quasi-locality bounds for quantum lattice systems. part ii. perturbations of frustration-free spin models with gapped ground states. In *Annales Henri Poincaré*, volume 23, pages 393–511. Springer, 2022.

[NSY24] Bruno Nachtergaele, Robert Sims, and Amanda Young. Stability of the bulk gap for frustration-free topologically ordered quantum lattice systems. *Letters in Mathematical Physics*, 114(1):24, 2024.

[PW20] Nicholas Pomata and Tzu-Chieh Wei. Demonstrating the affleck-kennedy-lieb-tasaki spectral gap on 2d degree-3 lattices. *Physical review letters*, 124(17):177203, 2020.

[RKE$^+$26] Kshiti Sneh Rai, Ilya Kull, Patrick Emonts, Jordi Tura, Norbert Schuch, and Flavio Baccari. A hierarchy of spectral gap certificates for frustration-free spin systems. *Quantum*, 10:2065, 2026.

[S. 11] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims. Automorphic equivalence within gapped phases of quantum lattice systems. *arXiv:1102.0842v2*, 2011.

[SS03] Wolfgang L Spitzer and Shannon Starr. Improved bounds on the spectral gap above frustration-free ground states of quantum spin chains. *Letters in Mathematical Physics*, 63(2):165–177, 2003.

[TCE$^+$20] Minh C Tran, Chi-Fang Chen, Adam Ehrenberg, Andrew Y Guo, Abhinav Deshpande, Yifan Hong, Zhe-Xuan Gong, Alexey V Gorshkov, and Andrew Lucas. Hierarchy of linear light cones with long-range interactions. *Physical Review X*, 10(3):031009, 2020.

[YNJ26] Amanda Young, Bruno Nachtergaele, and Andrew Jackson. Local topological quantum order and spectral gap stability for the aklt models on the hexagonal and lieb lattices. *arXiv preprint arXiv:2605.12184*, 2026.

5 Appendix

5.1 Proof of Lemma 3.1 by comparing sums with integrals

We first prove Lemma 3.1.

Lemma 3.1. Let $a \leq b \in [1, \infty)$, then it holds

$$\begin{aligned}\sum_{\substack{k \in \Omega_L \\ a \leq |k| \leq b}} g(k) &\leq \sum_{\substack{k \in \Omega_L \\ a \leq |k| \leq a+\sqrt{D}}} g(k) + \sum_{\substack{k \in \Omega_L \\ a+\sqrt{D} \leq |k| \leq b}} \prod_{m=1}^{D} \int_{k_m-1}^{k_m} g(k) dx_m \\ &\leq (a+\sqrt{D})^D g(a) + \sum_{\substack{k \in \Omega_L \\ a+\sqrt{D} \leq |k| \leq b}} \prod_{m=1}^{D} \int_{k_m-1}^{k_m} g(|x|-\sqrt{D}) dx_m\end{aligned}$$

where we used $|x| - \sqrt{D} \leq |k|$ for all $x \in [k_1 - 1, k_1] \times \cdots \times [k_D - 1, k_D]$. Now we use spherical coordinates to compute the second summand. The angle integrals can be computed and the result only depends on the dimension D, because g only depends on the absolute value.

For the second summand we hence get

$$\begin{aligned}\sum_{\substack{k \in \Omega_L \\ a+\sqrt{D} \leq |k| \leq b}} \prod_{m=1}^{D} \int_{k_m-1}^{k_m} g(|x|-\sqrt{D}) dx_m &\leq c \int_a^{b+\sqrt{D}} \rho^{D-1} g(\rho - \sqrt{D}) d\rho \\ &\leq c \begin{cases} \frac{a^{D-s}}{s-D} & , s > D \\ \ln(b) & , s = D \\ \frac{b^{D-s}}{D-s} & , s < D \end{cases}\end{aligned}$$

where we used the approximations $g(|x|-\sqrt{D}) \leq cx^{-s}$. Now we use $a \geq 1$ and get the claim

$$\sum_{\substack{k\in\Omega_L\\ a\leq|k|\leq b}} g(k) \leq ca^D g(a) + c\begin{cases} \frac{a^{D-s}}{s-D} & ,s>D \\ \ln(b) & ,s=D \\ \frac{b^{D-s}}{D-s} & ,s<D. \end{cases}$$

□

5.2 Proof of Lemma 4.1 by further estimates on sums by integrals

In this section we get an approximation for some sums used in the proof of Lemma 4.1. Explicitly the claim is that we can approximate

$$\sum_{m=0}^{D-1} n^m \left(\sum_{r\in E_1} |r|^{2D-m} + \sum_{r\in E_2} |r|^{2D-m}\left(e^{-\beta|r|}+|r|^{-p}\right)\right) \leq c(\beta,p)n^{D-1}$$

$$\sum_{r\in E_3} |r|^D \left(e^{-\beta|r|}+|r|^{-p}\right) \leq c(\beta,p)\frac{1}{n}.$$

We will consider the sets E_1, E_2, E_3 seperately.

Proof. For all scopes $r\in E_1$ we can the approximation $|r| \leq 8+4\sqrt{D} \leq c$ and it holds

$$\sum_{m=0}^{D-1} n^m \sum_{r\in E_1} |r|^{2D-m} \leq cn^{D-1}.$$

For scopes $r \in E_2 \cup E_3$ we approximate the sums by integrals using Lemma 3.1. We will now proof $\sum_{r\in E_2} |r|^{2D-m-p} \leq \frac{c(p)}{n}$ and $\sum_{r\in E_2} |r|^{2D-m} e^{-\beta|r|} \leq \frac{c(\beta)}{n}$ follows immediately for all $\beta > 0$. Now let us distinguish three different cases for m.

- The first set consist of all $m \in \{0,\cdots,D-1\}$ with $m > 3D-p$.
- The second set consist of all $m \in \{0,\cdots,D-1\}$ with $m = 3D-p$.
- The third set consist of all $m \in \{0,\cdots,D-1\}$ with $m < 3D-p$.

Note that the second and third sets can be empty.

With $\{r\in E_2\} \subset \{k \in \Lambda_L : 4 \leq |k| \leq \sqrt{D}n\}$ we use now Lemma 3.1 to approximate the sum

$$\sum_{r\in E_2} |r|^{-(p+m-2D)} \leq c(p) + \begin{cases} c(p) & ,m>3D-p \\ c\ln(n) & ,m=3D-p \\ c(p)n^{3D-p-m} & ,m<3D-p. \end{cases}$$

Therefore it holds with $3D-p < D-1$

$$\sum_{m=0}^{D-1} n^m \sum_{r\in E_2} |r|^{-(p+m-2D)} \leq c(p)n^{D-1}.$$

Hence with $\beta > 0$ we also have

$$\sum_{m=0}^{D-1} n^m \sum_{r\in E_2} |r|^{2D-m} e^{-\beta|r|} \leq c(\beta)n^{D-1}.$$

Consider now scopes $r \in E_3$. We can use the first case in Lemma 3.1, because it holds $p > 2D+1$. Hence we get

$$\sum_{r \in E_3} |r|^{D-p} \leq cn^D n^{D-p} + \frac{c}{p-2D} n^{p-2D} \leq c(p)\frac{1}{n}$$

and it follows

$$\sum_{r \in E_3} |r|^D e^{-\beta|r|} \leq c(\beta)\frac{1}{n}.$$

So the claimed approximations hold. $\square$

5.3 Proof of Proposition 4.3 on combinatorics at fixed scope

In this section we prove some combinatorial results from Proposition 4.3.

For part a) we consider two pairs with one common site and scope r in the box $\mathcal{B}_r$. We give an upper bound for the number of possible combinations of such pairs only dependent on the scope.

In part b) we use induction to prove an identity and the claim follows immediately.

Proof of Proposition 4.3 part a). Let $(j,k) \sim (j',k')$ be two pairs in Λ_L^2 with one common site and scope r. If we allow $j = k$ and $j' = k'$ we add more combinations and hence the approximation

$$\sum_{\substack{(j,k)\sim_{\mathcal{B}_r}(j',k') \\ scope(j,k,j',k')=r}} 1 \leq \prod_{m=1}^{D} \sum_{\substack{(j_m,k_m)\sim_{\mathcal{B}_{r_m}}(j'_m,k'_m) \\ scope(j_m,k_m,j'_m,k'_m)=r_m}} 1$$

holds, where we use on the right side, that the sites can be equal. Note that on the left side we have $\mathcal{B}_r := \prod_{m=1}^{D}([0,r_m] \cap \mathbb{Z})$ and on the right side we have $\mathcal{B}_{r_m} := [0,r_m] \cap \mathbb{Z}$.

We claim that in every coordinate there are at most $6(r_m+1)$ combinations. If $r_m = 0$ then also $j_m = k_m = j'_m = k'_m = 0$ holds and there is only one combination for the three points. If $r_m \neq 0$, then at least two points are at the interval borders and the third point does not have any restrictions. Hence with permutation of the points we have $6(r_m+1)$ possible combinations for every coordinate. The approximation of the claim follows with

$$\sum_{\substack{(j,k)\sim_{\mathcal{B}_r}(j',k') \\ scope(j,k,j',k')=r}} 1 \leq \prod_{m=1}^{D} 6(r_m+1) \leq c|r|^D$$

where we used $r_m + 1 \leq c|r|$. $\square$

Proof of Proposition 4.3 part b). Let $3 < n \in \mathbb{N}$ and let $r \in \mathcal{B}_n$ with $|r| > 1$. We claim that the equation

$$\prod_{m=1}^{D}(n+1-r_m) = \sum_{m=0}^{D}(n+1)^m \sum_{\substack{E \subset \mathcal{P}(\{1,\cdots,D\}) \\ |E|=D-m}} \prod_{i \in E}(-r_i) \tag{5.1}$$

is true, where we define $\prod_{i\in\emptyset} = 1$. We proof this equation by induction. For $D = 1$ we have on the left side $(n+1-r_1)$ and on the right side $-r_1 + n + 1$.

For the induction step $D \mapsto D+1$ we get the following calculation

$$\prod_{m=1}^{D+1}(n+1-r_m) = \left(\sum_{m=0}^{D}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D\})\\ |E|=D-m}} \prod_{i\in E}(-r_i)\right)(n+1-r_{D+1})$$

$$= \sum_{m=0}^{D}(n+1)^{m+1} \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D\})\\ |E|=D-m}} \prod_{i\in E}(-r_i) + \sum_{m=0}^{D}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D\})\\ |E|=D-m}} \prod_{i\in E}(-r_i)(-r_{D+1}).$$

In the first step we used the induction hypopaper. Nest we have a closer look on the indices and calculate

$$= \sum_{m=1}^{D+1}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D+1\})\\ D+1\notin E, |E|=D+1-m}} \prod_{i\in E}(-r_i) + \sum_{m=0}^{D}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D+1\})\\ D+1\in E, |E|=D+1-m}} \prod_{i\in E}(-r_i)$$

$$= \sum_{m=0}^{D+1}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D+1\})\\ |E|=D+1-m}} \prod_{i\in E}(-r_i).$$

Therefore equation (5.1) holds. By using this equation and $r_i \leq |r|$ we have

$$\begin{aligned} n^2(n+1)^{D-2} - \prod_{m=1}^{D}(n+1-r_m) &= n^2(n+1)^{D-2} - \sum_{m=0}^{D}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D\})\\ |E|=D-m}} \prod_{i\in E}(-r_i) \\ &\leq \sum_{m=0}^{D-1}(n+1)^m \sum_{\substack{E\subset\mathcal{P}(\{1,\cdots,D\})\\ |E|=D-m}} |r|^{|E|} \\ &= c\sum_{m=0}^{D-1}(n+1)^m|r|^{D-m}. \end{aligned}$$

This proves Proposition 4.3. □

5.4 Proof of Proposition 4.4 about minimal distance

In this section the aim is to get a lower bound on the minimum distance between sites. We consider two pairs with one common site and the claim is that at least one of the pairs has a certain distance between the sites. The proof uses mainly the scope definition. First we prove a statement in one dimension and extend it to D dimensions.

Proof of Proposition 4.4. Let $(j,k) \sim (j',k')$ be two pairs in Λ_L^2 with one common site and scope r. Recall the definition of the scope 4.2 and fix a dimension $i \in \{0,\cdots,D\}$. Consider the first case of the scope definition, that is let $j_i \leq k_i \leq t_i$ or $t_i \leq k_i \leq j_i$ be true. Then by definition of the scope we have

$$r_i = |j_i - k_i| + |k_i - t_i|.$$

Hence $\frac{r_i}{2} \leq |j_i - k_i|$ or $\frac{r_i}{2} \leq |k_i - t_i|$ holds.

In the other case we have by definition

$$r_i = \max\{|j_i - k_i|, |k_i - t_i|\}.$$

So clearly $\frac{r_i}{2} \leq |j_i - k_i|$ or $\frac{r_i}{2} \leq |k_i - t_i|$ holds in this case too and therefore we proveed the claim in one dimension. Define now the two sets

$$\Omega_1 := \{i \in \{1, \cdots, D\} : \frac{r_i}{2} \leq |j_i - k_i|\}$$
$$\Omega_2 := \{i \in \{1, \cdots, D\} : \frac{r_i}{2} \leq |k_i - t_i|\}.$$

Note that $\Omega_1 \cup \Omega_2 = \{1, \cdots, D\}$ holds. Hence we have

$$\begin{aligned} |j-k| + |k-t| &\geq \left(\sum_{i \in \Omega_1} |j_i - k_i|^2 \right)^{1/2} + \left(\sum_{i \in \Omega_2} |k_i - t_i|^2 \right)^{1/2} \\ &\geq \left(\sum_{i \in \Omega_1 \cup \Omega_2} \left(\frac{r_i}{2} \right)^2 \right)^{1/2} = \frac{|r|}{2} \end{aligned}$$

and therefore $|j-k| \geq \frac{|r|}{4}$ or $|k-t| \geq \frac{|r|}{4}$ is true. □