

## FOUNDATIONS OF QUANTUM MECHANICS

Written homework due Wednesday November 8, 2017

**Exercise 5: Essay question:** What is surprising about the double-slit experiment?

**Exercise 6: Unitary operators.**

(a) For any vector  $\mathbf{a} \in \mathbb{R}^3$ , the *translation operator*  $T_{\mathbf{a}}$  is defined on  $L^2(\mathbb{R}^{3N})$  by

$$(T_{\mathbf{a}}\psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi(\mathbf{x}_1 - \mathbf{a}, \dots, \mathbf{x}_N - \mathbf{a}). \quad (1)$$

It shifts the wave function by  $\mathbf{a}$  in every  $\mathbf{x}_i$ . Show that  $T_{\mathbf{a}}$  is unitary.

(b) For any rotation matrix  $R \in SO(3)$ , the *rotation operator*  $U_R$  is defined on  $L^2(\mathbb{R}^{3N})$  by

$$(U_R\psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi(R^{-1}\mathbf{x}_1, \dots, R^{-1}\mathbf{x}_N). \quad (2)$$

It rotates the wave function according to  $R$  in every  $\mathbf{x}_i$ . Show that  $U_R$  is unitary.

**Exercise 7: Orthonormal system.**

For  $n \in \mathbb{Z}$ , let the function  $\varphi_n : [-\pi, \pi] \rightarrow \mathbb{C}$  be defined by

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}. \quad (3)$$

Show that they form an *orthonormal system* in  $L^2([-\pi, \pi])$ , i.e., that

$$\langle \varphi_n | \varphi_m \rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \quad (4)$$

with

$$\langle \psi | \phi \rangle = \int_{-\pi}^{\pi} \psi(x)^* \phi(x) dx. \quad (5)$$

**Exercise 8: Dense subspace.**

The space  $\ell^2 = \{(x_1, x_2, \dots) : x_n \in \mathbb{C}, \sum |x_n|^2 < \infty\}$  of all square-summable sequences is a Hilbert space with inner product

$$\langle x | y \rangle = \sum_{n=1}^{\infty} x_n^* y_n. \quad (6)$$

A subset  $S$  of a Hilbert space  $\mathcal{H}$  is called *dense* if for every  $\psi \in \mathcal{H}$  and  $\varepsilon > 0$  there is  $\phi \in S$  with  $\|\psi - \phi\| < \varepsilon$ . (For example,  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .) Let  $S$  be the subspace of  $\ell^2$  consisting of all sequences  $(x_1, x_2, \dots)$  with only finitely many nonzero entries.

(a) Show that  $S$  is dense in  $\ell^2$ .

- (b) Show that in a finite-dimensional Hilbert space (that is, without loss of generality, in  $\mathbb{C}^n$ ), the only dense subspace is the full space  $\mathbb{C}^n$ .
- 

**Reading assignment** due Friday November 3, 2017: J. Bell: De Broglie–Bohm, delayed-choice double-slit experiment, and density matrix. *International Journal of Quantum Chemistry* **14**: 155–159 (1980)

---

**Announcement:** You may be interested in a talk in the mathematical physics research seminar by Lukas Nickel on *Multi-time wave functions* on Thursday Nov. 2, 2017, at 16:15 in N14.

---

**Note:** No lectures on Friday October 27 (instructor is traveling) and on Wednesday November 1 (public holiday). The exercise class on Monday October 30, 2017, takes place as scheduled.