Prof. Dr. Roderich Tumulka
Mathematisches Institut, Universität Tübingen

Assignment 5
Wintersemester 2017/18

Foundations of Quantum Mechanics

Written homework due Wednesday November 29, 2017

Exercise 18: Essay question. Describe the delayed-choice double-slit experiment. Why does it seem paradoxical? How does the paradox get resolved in Bohmian mechanics?

Exercise 19: Spinors
Verify that $|\boldsymbol{\omega}(\phi)| = \|\phi\|_S^2 = \phi^*\phi$. Proceed as follows: By (9.6), $\boldsymbol{\omega}(z\phi) = |z|^2\boldsymbol{\omega}(\phi)$, it suffices to show that unit spinors are associated with unit vectors. By (9.6) again, it suffices to consider ϕ with $\phi_1 \in \mathbb{R}$ (else replace ϕ by $e^{i\theta}\phi$ with appropriate θ). So we can assume, without loss of generality, $\phi = (\cos\alpha, e^{i\beta}\sin\alpha)$ with $\alpha, \beta \in \mathbb{R}$. Evaluate $\phi^*\boldsymbol{\sigma}\phi$ explicitly in terms of α and β, using the explicit formulas (9.3) for $\boldsymbol{\sigma}$. Then check that it is a unit vector.

Exercise 20: Iterated Stern-Gerlach experiment
Consider the following experiment on a single electron. Suppose it has a wave function of the product form $\psi_s(\boldsymbol{x}) = \phi_s\,\chi(\boldsymbol{x})$, and we focus only on the spinor. The initial spinor is $\phi = (1, 0)$.

(a) A Stern–Gerlach experiment in the y-direction (or σ_2-measurement) is carried out, then a Stern–Gerlach experiment in the z-direction (or σ_3-measurement). Both measurements taken together have four possible outcomes: up-up, up-down, down-up, down-down. Find the probabilities of the four outcomes.

(b) As in (a), but now the z-experiment comes first and the y-experiment afterwards. (The fact that the answer to (b) is different from that to (a) is often expressed by saying that the observables σ_2 and σ_3 "cannot be measured simultaneously.")

(c) More generally, if the two self-adjoint matrices A and B have spectral decomposition $A = \sum_\alpha \alpha P_\alpha$ and $B = \sum_\beta \beta Q_\beta$, what is the joint distribution of the outcomes if A is measured first and B right afterwards? What if B is measured first and A right afterwards? Show that the two agree iff every P_α commutes with every Q_β. (This happens iff $AB = BA$.)

Exercise 21: Lie algebras of $SO(3)$ and $SU(2)$
A *Lie group* G, named after Sophus Lie (1842–1899), is a group that is also a manifold such that the group multiplication and inversion are smooth mappings. Examples of Lie groups include $GL(n), SO(n), U(n), SU(n)$. The elements infinitesimally close to 1 in G form the *Lie algebra* g of G; more precisely, g is the tangent space of 1, which is here the set

$$\left\{ \frac{dA}{dt}(t=0) \middle| A : (-1,1) \to G \text{ smooth}, A(0) = 1 \right\}.$$

(a) Determine the Lie algebras $so(3)$ and $su(2)$ as subspaces of the space of all real 3×3 (complex 2×2) matrices.

(b) The *exponential mapping* $\exp : g \to G$ can be heuristically understood as follows: For $X \in g$, a corresponding group element infinitesimally close to 1 can be written as $1 + X/n$ with n a large natural number (so $1/n$ serves as an infinitesimal dt). Hence, roughly speaking, $(1 + X/n) \in G$, hence $(1 + X/n)^n \in G$; take the limit $n \to \infty$ to obtain $\exp(X) =: e^X$. Verify that the matrix exponential (defined by the exponential series) actually maps $so(3)$ to $SO(3)$ and $su(2)$ to $SU(2)$. (Hint: diagonalize $X \in g$.)

(c) For $X, Y \in g$, what does group multiplication of e^X and e^Y look like? We know that the solution Z of $e^Z = e^X e^Y$ is $Z = X + Y$ if X and Y commute, but not in general. A version of the *Baker–Campbell–Hausdorff formula* says that

$$\text{the solution of } e^Z = e^{-tX} e^{-tY} e^{t(X+Y)} \text{ is } Z = \tfrac{1}{2}t^2[X,Y] + O(t^3)$$

as $t \to 0$, with $[X,Y] = XY - YX$ the *commutator* or *Lie bracket*. The Lie bracket is an operation on g that encodes how the group multiplication deviates from addition in g. Thus, one defines a *Lie algebra* in general as a vector space together with a bracket $[\cdot,\cdot] : g \times g \to g$ that is anti-symmetric, bilinear, and satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\,.$$

Verify that $so(3)$ and $su(2)$ (with commutators as Lie brackets) are isomorphic Lie algebras. (Hint: The Pauli matrices have something to do with rotations about the x-, y-, and z-axis.)

Reading assignment due Friday December 1, 2017: J. Bell, *Are There Quantum Jumps?*, chapter 22 in *Speakable and Unspeakable in Quantum Mechanics*, sections 1-3 and 5