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15 Jan. 2018
Wintersemester 2017/18

Foundations of Quantum Mechanics

In-class problems for the exercise class

Problem 13: True or false?

a) $R^\dagger R$ is always a positive operator.

b) If E is a positive operator, then so is $R^\dagger E R$.

c) The bounded positive operators form a subspace of the space of bounded operators.

d) The sum of two projections is positive only if they commute.

e) e^{At} is a positive operator for every self-adjoint A and $t \in \mathbb{R}$.

f) A multiplication operator $\psi \mapsto f\psi$ is positive iff the set $\{x : f(x) < 0\}$ has measure zero.

Problem 14: According to the spectral theorem, every self-adjoint operator A in $\mathscr{H}$ is unitarily equivalent to a multiplication operator M in $L^2(\Omega)$. Let $U : \mathscr{H} \to L^2(\Omega)$ denote that unitary operator, so $A = U^{-1}MU$. Give an explicit expression for the spectral PVM of A in terms of U and M.

Problem 15: Consecutive quantum measurements
Let $A_1, \ldots, A_n$ be self-adjoint operators in $\mathscr{H}$ whose spectra $\sigma(A_k)$ are purely discrete (i.e., countable), so that

$$A_k = \sum_{\alpha \in \sigma(A_k)} \alpha\, P_{k,\alpha}$$

with $P_{k,\alpha}$ the projection to the eigenspace of A_k with eigenvalue α. Consider a quantum system with initial wave function $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$ at time t_0. At times $t_1 < t_2 < \ldots < t_n$, ideal quantum measurements of $A_1, \ldots, A_n$ (respectively) are carried out with outcomes $Z_1, \ldots, Z_n \in \mathbb{R}$ ($t_0 < t_1$). Show that there is a POVM E on $\mathbb{R}^n$ such that

$$\mathbb{P}\Big((Z_1, \ldots, Z_n) \in B\Big) = \langle \psi_0 | E(B) | \psi_0 \rangle$$

and give an explicit expression for $E(B)$.

Problem 16: Let E be a POVM on the finite set $\mathscr{Z}$ acting on $\mathscr{H}$. Show that if the E_z commute pairwise, then there are a unitary $U : \mathscr{H} \to L^2(\Omega)$ and functions $f_z : \Omega \to [0,1]$ such that

$$\sum_z f_z(\omega) = 1 \quad \forall \omega \in \Omega$$

and $E_z = U^{-1} f_z U$.

Problem 17: a) Suppose E_1 and E_2 are POVMs on $\mathscr{Z}_1$ and $\mathscr{Z}_2$, respectively, both acting on $\mathscr{H}$; let $q_1, q_2 \in [0,1]$ with $q_1 + q_2 = 1$. Show that $E(B) := q_1 E_1(B \cap \mathscr{Z}_1) + q_2 E_2(B \cap \mathscr{Z}_2)$ defines a POVM on $\mathscr{Z}_1 \cup \mathscr{Z}_2$.

b) Suppose experiment $\mathscr{E}_1$ has distribution of outcomes $\langle \psi | E_1(\cdot) | \psi \rangle$, and $\mathscr{E}_2$ has distribution of outcomes $\langle \psi | E_2(\cdot) | \psi \rangle$. Describe an experiment with distribution of outcomes $\langle \psi | E(\cdot) | \psi \rangle$.

c) Give an example of a POVM for which the E_z do not pairwise commute. *Suggestion*: Choose $E_1(z)$ that does not commute with $E_2(z')$ for $\mathscr{Z}_1 \cap \mathscr{Z}_2 = \emptyset$.