Foundations of Quantum Mechanics

Written homework due Wednesday January 24, 2018

Exercise 35: Essay question. Explain why Schrödinger's theory Sm has a many-worlds character.

Exercise 36: Allcock's paradox[1]

Allcock considered a "soft detector," i.e., one for which the particle may fly through the detector volume for a while before being detected. An as effective description, Allcock proposed an imaginary potential. For example, for a single particle in 1d and a detector in the right half axis, he considered the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - iv1_{x\geq 0}\psi \tag{1}$$

with $v > 0$ a constant. The time evolution then is not unitary.

(a) Derive from (1) the continuity equation

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}\, j - \tfrac{2v}{\hbar}1_{x\geq 0}\rho \tag{2}$$

for $\rho = |\psi|^2$ and $j = \frac{\hbar}{m}\mathrm{Im}[\psi^*\partial\psi/\partial x]$. Eq. (2) is the evolution of the probability density of a particle that moves along Bohmian trajectories and disappears spontaneously (stochastically) at rate $2v/\hbar$ whenever it stays in the region $x \geq 0$. $\|\psi_t\|^2$ is a decreasing function of t and represents the probability that the particle has not been detected (and disappeared from the model) yet.

(b) To obain an effective description of a hard detector (i.e., one that will detect the particle as soon as it reaches the region $x \geq 0$), Allcock assumed that ψ_0 is concentrated in $\{x < 0\}$ and took the limit $v \to \infty$, but found that the particle never gets detected ($\|\psi_t\|^2 = \text{const.}$)! That is parallel to the quantum Zeno effect.

Prove the following simplified version: In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1, 0)$ evolve with the (non-self-adjoint) Hamiltonian

$$H_v = \begin{pmatrix} 0 & 1 \\ 1 & -iv \end{pmatrix}. \tag{3}$$

Then for every $t > 0$, $\psi_t = e^{-iH_vt/\hbar}\psi_0 \to \psi_0$ as $v \to \infty$.

Exercise 37: Boundary conditions

On the half axis $(-\infty, 0]$, consider the Schrödinger equation $i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2$ with boundary condition

$$\alpha\frac{\partial\psi}{\partial x}(x = 0) + \beta\psi(x = 0) = 0 \tag{4}$$

with constants $\alpha, \beta \in \mathbb{C}$. For $\alpha = 0, \beta = 1$ this is called a *Dirichlet boundary condition*, for $\alpha = 1$ and $\beta = 0$ a *Neumann boundary condition*. [This is Carl Neumann (1832–1925; in Tübingen 1865–1868), not John von Neumann (1903–1957).] For general $(\alpha, \beta) \neq (0, 0)$ it is called a *Robin boundary condition*. Which choices of (α, β) imply that $j(x = 0) = 0$? (They are reflecting boundary conditions and lead to a unitary time evolution.) Which imply that $j(x = 0) > 0$ whenever $\psi(x = 0) \neq 0$? (They are absorbing boundary conditions and lead to loss of probability.)

[1] G.R. Allcock: The time of arrival in quantum mechanics II. The individual measurement. *Annals of Physics* **53**: 286–310 (1969)