Why Bohmian mechanics allows for different roles of density matrices

Roderich Tumulka

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Birmingham, 24 November 2020

Basic math facts (1)

Notation $\mathbb{S}(\mathscr{H}) := \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ unit sphere

Definition

A positive-operator-valued measure (POVM) is a family of positive operators (i.e., all generalized eigenvalues ≥ 0) E_z such that $\sum_z E_z = I$.

Main theorem about POVMs

Suppose the experiment $\mathscr{E}$ can be carried out on a quantum system with arbitrary wave function $\psi \in \mathbb{S}(\mathscr{H})$. Then there is a POVM E such that for every $\psi \in \mathbb{S}(\mathscr{H})$, the outcome Z of $\mathscr{E}$ on a system with ψ has probability distribution

$$\mathbb{P}(Z = z) = \langle \psi | E_z | \psi \rangle \,.$$

Basic math facts (2)

Corollary

If μ is a probability distribution over $\mathbb{S}(\mathscr{H})$ and a system has random wave function ψ with distribution μ, then Z has distribution

$$\mathbb{P}(Z=z) = \int\limits_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, \langle\psi|E_z|\psi\rangle = \mathrm{tr}(W\, E_z)$$

with $\quad W = \int\limits_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, |\psi\rangle\langle\psi| \quad$ the statistical density matrix.

Corollary

If two distributions μ, μ' have the same W, then the corresponding ensembles have the same distribution of outcomes for every experiment, and are thus empirically indistinguishable.

Basic math facts (3)

Examples

- $\mu(\{|\text{up}\rangle\}) = \frac{1}{2}$, $\mu(\{|\text{down}\rangle\}) = \frac{1}{2}$ $\quad\Rightarrow\quad W = \frac{1}{2}I.$
- $\mu'(\{|\text{left}\rangle\}) = \frac{1}{2}$, $\mu'(\{|\text{right}\rangle\}) = \frac{1}{2}$ $\quad\Rightarrow\quad W = \frac{1}{2}I.$
- $\mu'' =$ uniform over $\mathbb{S}(\mathbb{C}^2)$ $\quad\Rightarrow\quad W = \frac{1}{2}I.$
- Thus, $\mu \mapsto W$ is many-to-one.

By the way,

suppose Alice chooses one of μ, μ', μ'', prepares an ensemble accordingly, hands it over to Bob with the challenge to empirically determine her choice. It's impossible for Bob. This can be used to show that there are facts in the world that we cannot determine empirically ("limitations to knowledge").

Bohmian mechanics (1)

Definition of the theory (non-relativistic version)

Particles are material points with positions $\boldsymbol{Q}_j(t) \in \mathbb{R}^3$ $(j = 1, \ldots, N)$ at time t governed by Bohm's equation of motion

$$\frac{d\boldsymbol{Q}_j}{dt} = \frac{\hbar}{m_j} \mathrm{Im} \frac{\psi^* \nabla_j \psi}{\psi^* \psi}\Big|_{(t,\boldsymbol{Q}_1(t),\ldots,\boldsymbol{Q}_N(t))} , \tag{1}$$

where $\psi : \mathbb{R}_t \times \mathbb{R}^{3N}_Q \to \mathbb{C}^k$ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = -\sum_{j=1}^{N} \frac{\hbar^2}{2m_j} \nabla_j^2 \psi + V\psi \, .$$

At the initial (and thus any) time t, $Q(t) = \big(\boldsymbol{Q}_1(t), \ldots, \boldsymbol{Q}_N(t)\big)$ has probability density $|\psi_t(q)|^2$.

In short, $\dfrac{dQ}{dt} = v^{\psi_t}(Q_t)$ with v the velocity vector field on configuration space defined by $v_j^{\psi} = \frac{\hbar}{m_j} \mathrm{Im} \frac{\psi^* \nabla_j \psi}{\psi^* \psi}$.

Bohmian mechanics (2)

- If, in an ensemble, 50% of systems have wave function ψ_1 and 50% have ψ_2, then 50% of the systems have configurations moving with velocity v^{ψ_1} and 50% with v^{ψ_2}.
- Different μ's with the same W usually lead to different distributions over v's.
- Thus, in Bohmian mechanics, W alone does not determine the ensemble of trajectories. The latter depends on μ, not just on W.
- What is there in reality is ψ, not W.
- But we could change the defining equations...

W-BM (1)

[Bell 1980, Dürr et al. quant-ph/0311127, Maroney quant-ph/0311149]

Definition of the modified theory

Particle positions $\boldsymbol{Q}_j(t) \in \mathbb{R}^3$ ($j = 1, \ldots, N$) are governed by the eq of motion

$$\frac{d\boldsymbol{Q}_j}{dt} = \frac{\hbar}{m_j} \mathrm{Im} \frac{\nabla_{\boldsymbol{q}_j} \mathrm{tr}_{\mathbb{C}^k} W}{\mathrm{tr}_{\mathbb{C}^k} W}\Big|_{(t, q=q'=Q)}, \tag{2}$$

where $W_{ss'}(q, q') = \langle q, s | W | q', s' \rangle$ (q = position configuration in $\mathbb{R}^{3N}$, s = spin indices in $\{1, \ldots, k\}$) evolves according to the von Neumann eq

$$i\hbar \frac{\partial W}{\partial t} = [H, W] \,.$$

At the initial (and thus any) time t, $Q(t)$ has probability density $\mathrm{tr}_{\mathbb{C}^k} W_t(q, q)$.

If W is pure, $W = |\psi\rangle\langle\psi|$, then W-BM reduces to ordinary ψ-BM.

W-BM (2)

- In ψ-BM, the state is (Q, ψ); the ontology consists of particles and a wave function $\psi \in \mathscr{H}$.
- In W-BM, the state is (Q, W); the ontology consists of particles and a density matrix $W : \mathscr{H} \to \mathscr{H}$.
- In ψ-BM, the statistical density matrix encodes (incomplete) information about μ.
- In W-BM, W is an object in nature, something fundamental, something real (density matrix realism).
- If W is non-pure, then the trajectories of W-BM usually do not agree with those of ψ-BM for any ψ.

Empirical equivalence

Let μ-BM be ψ-BM with random ψ with distribution μ. Let W be the density matrix of μ; then μ-BM is empirically equivalent to W-BM.

Proof: At any time t, the probability density of $Q(t)$ according to μ-BM, $\int \mu_t(d\psi)\, |\psi(q)|^2$, agrees with that according to W-BM, $\mathrm{tr}_{\mathbb{C}^k} W(q,q)$. $\square$

Conditional wave function and conditional density matrix

Another application of eq (2) concerns the conditional density matrix.

Conditional wave function

If $k = 1$ (spin-0, complex-valued ψ), then let $\psi^{\mathrm{cond}}(x) = \mathcal{N}\,\psi(x, Y)$. ψ^{cond} does not obey a Schrödinger eq (except in special cases), but always

$$\frac{dX}{dt} = v^{\psi^{\mathrm{cond}}}(X(t)) \text{ as in (1).}$$

If $k > 1$, then $\psi(x, Y)$ has the wrong number of spin indices. Remedy:

Conditional density matrix

Let $W^{\mathrm{cond}}_{ss'}(x, x') = \mathcal{N} \sum_{r=1}^{k} \psi_{sr}(x, Y)\, \psi^*_{s'r}(x', Y)$.

W^{cond} does not obey a von Neumann eq (except in special cases), but always

$$\frac{dX}{dt} = v^{W^{\mathrm{cond}}}(X(t)) \text{ as in (2).}$$

Ghirardi-Rimini-Weber (GRW) collapse theory

- Density matrix realism is not limited to Bohmian mechanics, but is an option for most theories with local "beables" in 3d.
- E.g., GRW theory postulates stochastic deviation from the Schrödinger eq (collapse as part of the fundamental time evolution law for ψ). Simple version: $\psi_t = \mathcal{N}\, L_{j_n}(\boldsymbol{x}_n, t_n) \cdots L_{j_1}(\boldsymbol{x}_1, t_1)\, \psi_0$, where $L_j(\boldsymbol{x}, t)$ operator = collapse (as in unsharp position measurement) on particle j at $\boldsymbol{x}$ at time t; with random collapses

$$\mathbb{P}\Big(\boldsymbol{x}_1, t_1, j_1, \ldots, \boldsymbol{x}_n, t_n, j_n\Big) = \Big\| L\, \psi_0 \Big\|^2 \text{ with } L = L_{j_n}(\boldsymbol{x}_n, t_n) \cdots L_{j_1}(\boldsymbol{x}_1, t_1)\,.$$

  In the "flash" ontology, local beables $= (\boldsymbol{x}, t, j)$.
- Modified version W-GRW,

$$\mathbb{P}\Big(\boldsymbol{x}_1, t_1, j_1, \ldots, \boldsymbol{x}_n, t_n, j_n\Big) = \operatorname{tr}\Big[L\, W\, L^\dagger\Big],$$

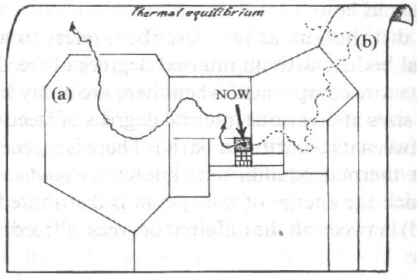
  respects density matrix realism and is empirically equivalent to ψ-GRW with random, μ-distributed ψ_0 (and W the density matrix of μ); indeed, the flashes have the same distribution.

Gao's objection

- Gao [philsci-archive.pitt.edu/18210] has argued that the Pusey-Barrett-Rudolph (PBR) theorem [arxiv.org/abs/1111.3328] excludes density matrix realism.
- Reasoning: PBR show ψ is real, so W-realism must be wrong.
- That is incorrect.
- The PBR theorem shows that if for every pure state ψ an ensemble $p^\psi(\lambda)$ of ontic states λ is given that reproduces the quantum predictions for ψ, then (under certain reasonable assumptions) for any ψ_1, ψ_2 that are not multiples of each other, p^{ψ_1} and p^{ψ_2} are disjoint.
- This fact does not conflict with the possible existence of ensembles $p^\mu(\lambda)$ or $p^W(\lambda)$ reproducing the quantum predictions for W.
- Obviously, some ontic models (such as ψ-BM) allow that two ensembles $p^\mu(\lambda), p^{\mu'}(\lambda)$ with the same W overlap, so there is no theorem analogous to PBR for density matrices.

Classically:

- Gibbs entropy $S_G(\rho) = -k_B \int_{\mathbb{R}^{6N}} dx\ \rho(x) \log \rho(x)$ [$\rho =$ prob density]
- Boltzmann entropy $S_B(X) = k_B \log \mathrm{vol}(\Gamma_\nu)$ if $X \in \Gamma_\nu$,
  $\Gamma_\nu =$ macro set for macro state ν, $\mathbb{R}^{6N} = \bigcup_\nu \Gamma_\nu$ partition
- ensemblism vs individualism [Goldstein et al. arxiv.org/abs/1903.11870]
- When is a system in thermal eq?
  Ensemblist: When $\rho = \rho_{\mathrm{eq}}$. Individualist: When $X \in \Gamma_{\mathrm{eq}}$.
- A classical system has X, but it is not obvious what ρ should be.
- Usually, macro sets have very different volumes, and phase points tend to move to larger and larger macro sets (so S_B increases).
- This is Boltzmann's explanation of the arrow of time.
- Works if: at the initial time, X_0 of the universe is a typical point in Γ_{ν_0} with very-low-entropy ν_0 ("past hypothesis").



Drawing: R. Penrose

Statistical mechanics (2)

Quantum mechanically:

- von Neumann entropy $S_{vN}(W) = -k_B \mathrm{tr}(W \log W)$
- quantum Boltzmann entropy $S_{qB}(\psi) = k_B \log \dim \mathscr{H}_\nu$ if $\psi \in \mathscr{H}_\nu$,
  $\mathscr{H}_\nu =$ macro space for macro state ν, $\mathscr{H} = \bigoplus_\nu \mathscr{H}_\nu$
- It may seem that ensemblism works better in QM because systems may in fact have a ρ or a W:
  - $|\psi|^2$ defines objective probabilities in configuration space
  - subsystems have reduced density matrix $W^{\text{red}} = \mathrm{tr}_{\text{env}} |\psi\rangle\langle\psi|$
  - W^{cond} represents the maximal possible knowledge about a subsystem
  - under density matrix realism, even the universe has a W.
- But other difficulties with ensemblism remain: $S_{vN}(W)$ measures how mixed W is, and not what the ψ in the support of W are like.
- Past hypothesis: The initial wave function ψ_0 of the universe is a typical vector in $\mathbb{S}(\mathscr{H}_{\nu_0})$ with very-low-entropy ν_0.

Thank you for your attention