

Probability Distribution of the Time at Which an Ideal Detector Clicks

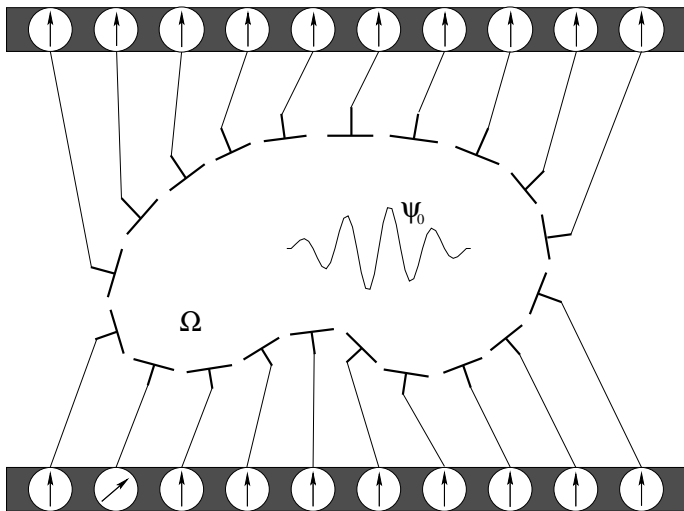
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Joint work, in part, with Abhishek Dhar (Bangalore)
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Problem of detection time and place



$T \in [0, \infty), \mathbf{X} \in \partial\Omega, Z = (T, \mathbf{X})$

Picture: redrawn after Detlef Dürr



Predicted detection time distribution

- Although QM does not provide a self-adjoint time operator, it makes an unambiguous prediction for the distribution of Z (though in an un-orthodox way): Solve the Schrödinger equation of the big system formed by “the” particle, all detectors, a clock, and a recording device, constructed so as to keep a record of which detector clicked when. At a late time t , make a quantum measurement of the record.
- It follows that the distribution of Z is given by a POVM,

$$\text{Prob}_{\psi_0}(Z \in \Delta) = \langle \psi_0 | E(\Delta) | \psi_0 \rangle.$$

POVM (positive-operator-valued measure) on \mathcal{Z}

Def: For every (measurable) set $\Delta \subseteq \mathcal{Z}$, $E(\Delta)$ is a positive operator.

$E(\mathcal{Z}) = I$, and $E(\Delta_1 \cup \Delta_2 \cup \dots) = E(\Delta_1) + E(\Delta_2) + \dots$

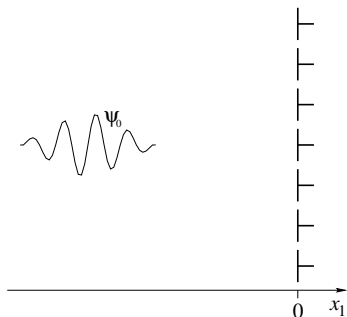
if $\Delta_1, \Delta_2, \dots$ are mutually disjoint.

Special case: PVM (projection-valued measure)

- Is there a practical way of computing $E(\cdot)$, at least approximately?
Without solving a Schrödinger equation for $> 10^{23}$ particles?

The ideal detector hypothesis

- While the correct POVM $E(\cdot)$ will depend on all details of the detectors, including their quantum states at time 0, the hope (“ideal detector hypothesis”) is that there is a particular POVM E_0 (or maybe E_κ depending on one or few parameters κ) in the cloud of E 's that is a good approximation and can be expressed by some simple rule.
- The hope is nourished by two facts:
 - In practice, detection probabilities do not seem to depend much on the detailed states of the detectors (except that different types of detectors are sensitive to different particle species and at different energy ranges).
 - For detection at a single time t , the distribution of \mathbf{X} is $|\psi|^2$, independently of the details of the detector.
- In this talk, I am proposing a rule defining a POVM E_κ for an **ideal detector**.



- Say, $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega = \text{plane } \{x_1 = 0\}$.
 - Make an instantaneous quantum measurement of the event $x_1 > 0$ (the projection operator $\mathbf{1}_{x_1 > 0}$) at regular time intervals $\tau > 0$.
 - Consider the limit $\tau \rightarrow 0$.
 - Result: In the limit, the probability of ever finding $x_1 > 0$ becomes 0.
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- That seems to make any concept of ideal detector impossible. (“A watched pot never boils.”)
 - —or at least, any concept of a “hard” detector that detects the particle as soon as it arrives at $\partial\Omega$; a “soft” detector that takes a while to notice the particle still seems possible.
 - Yet, I will show that even a hard detector is possible.

Allcock's [1969] difficulty

- Again, $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega = \text{plane } \{x_1 = 0\}$.
- Model of soft detector:
- Consider Schrödinger equation in \mathbb{R}^3 with complex potential

$$V(\mathbf{x}) = \begin{cases} -iv & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0, \end{cases}$$

where $v > 0$ is a constant.

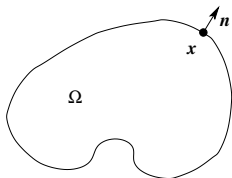
- This means that in the right half space the particle has rate $2v/\hbar$ of being absorbed (loss of $\|\psi\|^2$). $\text{Prob}(\mathbf{X} \in d^3\mathbf{x} | T) = |\psi_T(\mathbf{x})|^2 d^3\mathbf{x}$. Average lifetime in the detector volume = $\hbar/2v$.
- Difficulty: In the **hard limit** $v \rightarrow \infty$, $\psi_t(\mathbf{x}) = 0$ for $x_1 > 0$ and all $t > 0$, so the particle never gets detected.
- Again, a hard ideal detector seems impossible.

Proposed solution: The “absorbing boundary rule”

- Solve the 1-particle Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$ with “absorbing boundary condition” (ABC)

$$\mathbf{n}(\mathbf{x}) \cdot \nabla \psi(\mathbf{x}) = i\kappa \psi(\mathbf{x})$$

at every $\mathbf{x} \in \partial\Omega$, where $\mathbf{n}(\mathbf{x}) =$ outward unit normal vector to $\partial\Omega$ at \mathbf{x} , and $\kappa > 0$ a constant.



- ABC implies that the probability current $\mathbf{j}^\psi = \frac{\hbar}{m} \text{Im}[\psi^* \nabla \psi]$ points outward at $\partial\Omega$:

$$\mathbf{n} \cdot \mathbf{j} = \frac{\hbar}{m} \text{Im}[\psi^* \mathbf{n} \cdot \nabla \psi] = \frac{\hbar}{m} \text{Im}[\psi^* i\kappa \psi] = \frac{\hbar}{m} \kappa |\psi|^2 \geq 0.$$

- $\text{Prob}_{\psi_0} (T \in dt, \mathbf{X} \in d^2\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{j}^{\psi_t}(\mathbf{x}) dt d^2\mathbf{x}$ assuming $\|\psi_0\| = 1$.
- If the experiments get interrupted at time t before detection, the collapsed wave function is $\psi_t / \|\psi_t\|$.

Properties

- $\|\psi_t\|^2 = \text{Prob}_{\psi_0}(T > t)$ “survival probability,” decreasing in t
- The time evolution of ψ is not unitary (Hamiltonian not self-adjoint) due to loss at $\partial\Omega$, but well defined by the Hille-Yosida theorem:
 $\psi_t = W_t\psi_0$ with $W_t = e^{-iHt/\hbar}$ a contraction semigroup ($t \geq 0$) on $L^2(\Omega)$.
- skew-adjoint part(H) is a negative operator, i.e., $\text{Im}\langle\psi|H\psi\rangle \leq 0$.
- H is not necessarily diagonalizable; if it is, then spectrum $\subseteq \{x + iy \in \mathbb{C} : y \leq 0\}$ = lower half plane.
- In Bohmian mechanics, the particle with $|\psi_0|^2$ -distributed initial condition $\mathbf{X}(0)$ moves according to the equation of motion

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{j}^{\psi_t}(\mathbf{X}(t))}{|\psi_t(\mathbf{X}(t))|^2}$$

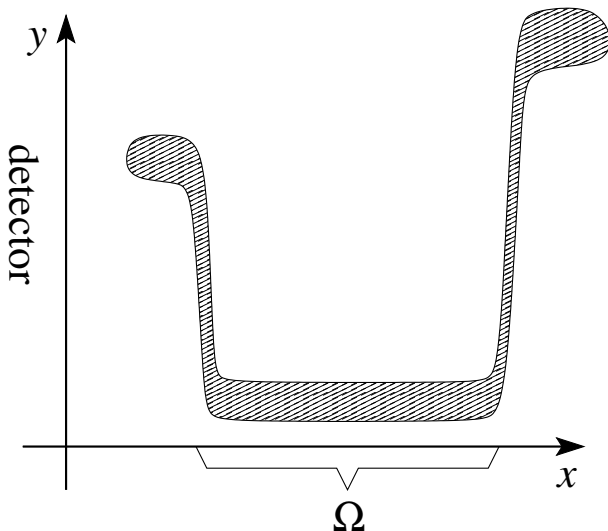
until it hits $\partial\Omega$ at time T and place $\mathbf{X} = \mathbf{X}(T)$, and gets absorbed.
 $\text{Prob}_{\psi_0}(\mathbf{X}(t) \in d^3\mathbf{x}) = |\psi_t(\mathbf{x})|^2 d^3\mathbf{x}$.

- energy-time uncertainty relation $\Delta E \Delta T \geq \hbar/2$
with E referring to $-\frac{\hbar^2}{2m}\nabla^2$ on $L^2(\mathbb{R}^3)$

- $E_\kappa(dt \times d^2\mathbf{x}) = \frac{\hbar\kappa}{m} W_t^\dagger |\mathbf{x}\rangle\langle\mathbf{x}| W_t dt d^2\mathbf{x},$
 $E_\kappa(T = \infty) = \lim_{t \rightarrow \infty} W_t^\dagger W_t$
- on $\mathcal{L} = [0, \infty) \times \partial\Omega \cup \{\infty\}$, acting on $L^2(\Omega)$
- not PVM \Rightarrow no “eigenstates of detection time”

Why to expect an absorbing boundary

configuration space:



- The ABC was considered by Werner in 1987 [J. Math. Phys.], indeed for detection time distribution (“any contraction semigroup determines a natural arrival time observable”).
- Afterward [1988], Werner studied less compelling approaches to the detection time distribution.
- The ABC received almost no attention. In an 86-pages review paper [Muga and Leavens, Phys. Rep. 2000], the ABC was mentioned in passing but not even written down.
- The ABC was mentioned by Fevens and Jiang in 1999 [SIAM J. Sci. Comput.] for numerical simulation of the Schrödinger eq. on \mathbb{R} with finitely many lattice points, but dropped in favor of a higher-order BC that absorbs more of the wave.

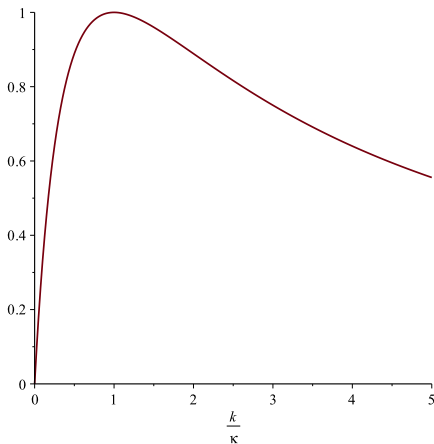
Reflection from $\partial\Omega$

While the Bohmian **particle** always gets absorbed when it hits $\partial\Omega$, the **wave function** gets partly reflected and partly absorbed.

Absorption coeff. $A_k = 1 - R_k$,
reflection coefficient $R_k = |c_k|^2$
for eigenfct $\psi(x) = e^{ikx} + c_k e^{-ikx}$
satisfying ABC $\psi'(0) = i\kappa\psi(0)$
in 1D.


κ = wave number of maximal absorption

Corollary: The presence of the detectors changes the Bohmian trajectories even **before** they reach $\partial\Omega$.



Plot of A_k

Lattice version (e.g., in 1D)

- lattice of mesh width $\varepsilon > 0$: $\Lambda = \{\varepsilon, 2\varepsilon, 3\varepsilon, \dots, N\varepsilon\}$
- $\mathcal{H} = L^2(\Lambda) = \mathbb{C}^N$ 
- Hamiltonian = discrete Laplacian, $H = -(\hbar^2/2m\varepsilon^2) \times$

$$\begin{bmatrix} -1 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -1 & \end{bmatrix} \text{ or } \begin{bmatrix} i\kappa\varepsilon - 1 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & i\kappa\varepsilon - 1 & \end{bmatrix}$$

Neumann b.c.

absorbing b.c.

- H not self-adjoint, $W_t = e^{-iHt/\hbar}$ contraction semigroup ($t \geq 0$)
- $\text{Prob}_{\psi_0}(T \in dt, \mathbf{X} = N\varepsilon) = \frac{\hbar\kappa}{m\varepsilon} |\psi_t(N\varepsilon)|^2 dt$

Avoiding the quantum Zeno effect

- Again, lattice of mesh width ε , $\Lambda = \{\varepsilon, 2\varepsilon, 3\varepsilon, \dots, N\varepsilon\}$
- Neumann b.c.
- quantum measurement of $P = |N\varepsilon\rangle\langle N\varepsilon|$ at times $\tau, 2\tau, 3\tau, \dots$
- quantum Zeno effect occurs in the limit $\tau \rightarrow 0$, $\varepsilon = \text{const.}$,
 $N = \text{const.}$
- The limit $\tau \rightarrow 0$, $\varepsilon \rightarrow 0$, $N \rightarrow \infty$, $N\varepsilon \rightarrow L$, $\tau/\varepsilon^3 \rightarrow 4m\kappa/\hbar$ leads to the absorbing boundary rule (no quantum Zeno effect!).
- Thus, a non-trivial limit is possible.

Avoiding Allcock's difficulty

- Again, $\Omega = \{x_1 \leq 0\}$ and $\partial\Omega = \text{plane } \{x_1 = 0\}$.
- Different model of soft detector:
- Consider Schrödinger equation in $\{x_1 \leq L\}$ with complex potential

$$V(\mathbf{x}) = \begin{cases} -iv & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

and Neumann boundary condition

$$\frac{\partial\psi}{\partial x_1}(L, x_2, x_3) = 0.$$

- The **hard limit** $v \rightarrow \infty$, $L \rightarrow 0$, $vL \rightarrow \frac{\hbar^2 \kappa}{2m} > 0$ leads to the absorbing boundary rule.
- Thus, a non-trivial hard limit is possible.

Further developments

- continuum limit of the lattice version reproduces the continuum version
- rigorous existence of ψ_t, Z, E_κ
- version of the rule for moving detectors
- version of the rule for several particles, in particular how to collapse ψ after the first detection
- version of the rule for particles with spin
- one may measure a spin component simultaneously with the detection
- version of the rule for the Dirac equation
- non-relativistic limit of the Dirac equation with ABC
- version of the rule in curved space-time
- boundary may be partly spacelike and partly timelike
- formulation in terms of multi-time wave functions for n particles
- ...so the absorbing boundary rule is very robust!

Thank you for your attention