Canonical Typicality For Other Ensembles Than Micro-Canonical

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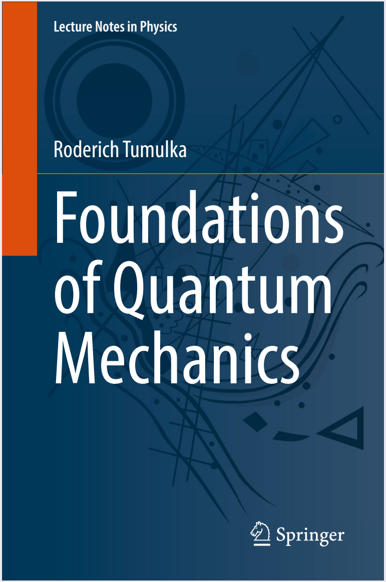

Overview

Overview 1: canonical typicality

$\mathbb{S}(\mathscr{H}) := \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ unit sphere

"Micro-canonical ensemble" in QM

Choose micro-canonical energy interval $[E - \Delta E, E]$. The corresponding spectral subspace of $H = \sum_n E_n |n\rangle\langle n|$ is

$$\mathscr{H}_{\mathrm{mc}} = \mathrm{span}\{|n\rangle : E_n \in [E - \Delta E, E]\}$$

"micro-canonical ensemble" $= u_{\mathrm{mc}} =$ uniform prob. distrib. on $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$

Reduced density matrix

Systems $a, b, S = a \cup b$, $\mathscr{H}_S = \mathscr{H}_a \otimes \mathscr{H}_b$, b large. $\rho_a^\psi := \mathrm{tr}_b\, |\psi\rangle\langle\psi|$.

Canonical typicality [≈ 2006], roughly speaking

Suppose $H \approx H_a \otimes I_b + I_a \otimes H_b$.

Then "most ψ" have $\rho_a^\psi \approx \rho_{\mathrm{can}} := Z^{-1} e^{-\beta H_a}$ with suitable β.

Overview 2: GAP measure

GAP measure [2006]

Given a density matrix ρ, $\mathrm{GAP}(\rho)$ is the most spread-out measure on $\mathbb{S}(\mathscr{H})$ with density matrix ρ (def. later). For $\rho \propto P_{\mathscr{K}}$, $\mathrm{GAP}(\rho) = u_{\mathbb{S}(\mathscr{K})}$.

"Canonical ensemble" in QM [2006]

$\mathrm{GAP}(\rho_{\mathrm{can}})$ is the distribution of wave functions in thermal equilibrium with a heat bath at temperature $1/\beta$.

Result of ours [2023]

Canonical typicality holds not only for the micro-canonical but also for the canonical ensemble. More generally, also for GAP measures but not not in general for other measures.

Overview 3: conditional wave function

Concept of conditional wave function [Dürr, Goldstein, Zanghì 1992]

In Bohmian mechanics, $\psi_a(x) := \mathcal{N}\,\Psi_S(x, Y)$.

For a given ONB $B = (\varphi_j)_j$ of $\mathscr{H}_b$, $\psi_a := \mathcal{N}\,\langle\varphi_J|\Psi_S\rangle_b$ with Born-distributed J, i.e., $\mathbb{P}(J = j) = \|\langle\varphi_J|\Psi_S\rangle_b\|_a^2$.

The distribution of ψ_a will be denoted by $\text{Born}_a^{\Psi,B}$.

As if quantum measurement with eigenbasis B and outcome J, which leads to collapsed $\psi_a \otimes \varphi_J$. A basic fact: $\mathbb{E}|\psi_a\rangle\langle\psi_a| = \rho_a^{\Psi}$.

Distribution of ψ_a in thermal equil. [Goldstein, Lebowitz, Tumulka, Zanghì 2006]

For u_{mc}-most Ψ and u_{ONB}-most B, $\text{Born}_a^{\Psi,B} \approx \text{GAP}(\rho_{\mathrm{can}})$.

New result

If d_b is large and ρ has small eigenvalues, then for $\text{GAP}(\rho)$-most Ψ and u_{ONB}-most B, $\text{Born}_a^{\Psi,b} \approx \text{GAP}(\mathrm{tr}_b\,\rho)$.

Overview 4: dynamical typicality

Dynamical typicality [Bartsch, Gemmer 2009; Müller, Gross, Eisert 2011]

Let A, B be observables on $\mathscr{H}$ with large $\dim \mathscr{H}$, not too large eigenvalues, and $\alpha \in \mathbb{R}$. For most $\psi \in \mathbb{S}(\mathscr{H})$ with $\langle\psi|A|\psi\rangle = \alpha$, $\langle\psi|B|\psi\rangle$ is nearly the same.

Macro spaces [von Neumann 1929]

Macro states ν correspond to high-dimensional, mutually orthogonal subspaces $\mathscr{H}_\nu$ such that $\mathscr{H} = \oplus_\nu \mathscr{H}_\nu$. They are the joint eigenspaces of the macro observables.

Variant of dynamical typicality [Balz, Gemmer et al. 2019; Teufel, Tumulka, Vogel 2022]

For u_{ν_0}-most $\psi_0 \in \mathbb{S}(\mathscr{H}_{\nu_0})$, $t \mapsto \|P_\nu \psi_t\|^2$ is nearly the same.

New result

If b is large and ρ has small eigenvalues, then for GAP(ρ)-most ψ and most $t \in [0, T]$, $\rho_a^{\psi_t} \approx \mathrm{tr}_b\, \rho_t$.

Overview 5: of MITE and MATE

Macroscopic thermal equilibrium (MATE)

A quantum system in state $\psi \in \mathscr{H}$ is in MATE when $\|P_\nu \psi\|^2 \geq 1 - \varepsilon$ and ν is a thermal equilibrium state. (Usually, there is 1 such $\nu = \nu_E^{\mathrm{eq}}$ in each $\mathscr{H}_{\mathrm{mc}}$, and $\dim \mathscr{H}_{\nu_E^{\mathrm{eq}}} / \dim \mathscr{H}_{\mathrm{mc}} \approx 1$. Most $\psi \in \mathbb{S}(\mathscr{H}_{\mathrm{mc}})$ are in MATE.)

For generic macroscopic systems, most ψ have a stronger property:

Microscopic thermal equilibrium (MITE)

A quantum system in state $\psi \in \mathscr{H}$ is in MITE when all micro observables (i.e., those referring only to a small subsystem a) have a probability distribution in ψ that coincides with their thermal probability distribution. (That is equivalent to $\rho_a^\psi \approx \rho_{\mathrm{can}}$.)

MITE arises from canonical typicality.

Some Orientation

Canonical typicality, abstractly formulated

Let $\mathscr{H}_R$ be a high-dimensional subspace of $\mathscr{H}_S = \mathscr{H}_a \otimes \mathscr{H}_b$, and b sufficiently large. Let $\rho_R = P_R/d_R$ (the normalized projection to $\mathscr{H}_R$) with $d_R := \dim \mathscr{H}_R$. Then for most $\psi \in \mathbb{S}(\mathscr{H}_R)$,

$$\rho_a^\psi \approx \mathrm{tr}_b\, \rho_R,$$

where "most ψ" refers to the uniform distribution u_R over $\mathbb{S}(\mathscr{H}_R)$.

This phenomenon was discovered by several groups independently [Lloyd 1988; Gemmer, Mahler 2003; Goldstein, Lebowitz, Tumulka, Zanghì 2006; Popescu, Short, Winter 2006] preliminary [Schrödinger 1952]

If $\mathscr{H}_R = \mathscr{H}_{\mathrm{mc}}$, b is large, and $H \approx H_a \otimes I_b + I_a \otimes H_b$, then $\mathrm{tr}_b\, \rho_{\mathrm{mc}} \approx \rho_{\mathrm{can}}$.

Motivation

- Measures can serve as probability measures (for actual ensembles) or typicality measures (for hypothetical ensembles representing "most" cases). Theorems of measure theory apply to both.
- In an actual ensemble, the distribution of ψ might not be uniform. In thermal equilibrium, it will be GAP.
- In a hypothetical ensemble, still relevant: u_{mc} corresponds to cutting off energy coefficients of ψ outside $[E - \Delta E, E]$ abruptly. The intuitive idea that canonical typicality applies to "any old" ψ is confirmed by our results, just with corrections (i.e., deviations of $\mathrm{tr}_b\, \rho$ from $\mathrm{tr}_b\, \rho_{\mathrm{mc}}$).
- Robustness towards changes in the underlying measure.
- Equivalence of ensembles: we can start from the "micro-canonical ensemble" u_{mc} or the "canonical ensemble" $\mathrm{GAP}(\rho_{\mathrm{can}})$ of wave functions.
- thermodynamic limit: with regions $A_1 \subset A_2 \subset \ldots \subset \mathbb{R}^3$, we want (conditional) wave functions $\psi_{A_1}, \psi_{A_2}, \ldots$ such that each is the conditional wave function of the next, and each is GAP distributed.
- Our results also illustrate that GAP measures are natural.

Not every measure does what GAP(ρ) does

Canonical typicality is not true for all measures.

Counter-example

Let $\rho = \sum_n p_n |n\rangle\langle n|$ and

$$\mu = \sum_{n=1}^{D} p_n \, \delta_{|n\rangle}$$

(the narrowest, most concentrated measure with density matrix ρ). $\psi \sim \mu$ is a random eigenvector. Suppose $|n\rangle = |\ell\rangle_a \otimes |m\rangle_b$; then $\rho_a^{|n\rangle} = \mathrm{tr}_b \, |n\rangle\langle n| = |\ell\rangle_a\langle\ell|$, so $\rho_a^{|n\rangle}$ is always pure and thus far away from $\mathrm{tr}_b \, \rho = \sum_{\ell,m} p_{\ell m} |\ell\rangle_a\langle\ell|$ (highly mixed).

(If instead of a product basis, $\{|n\rangle\}_n$ were purely random, then $\rho_a^{|n\rangle} \approx d_a^{-1} I_a$ and thus also $\mathrm{tr}_b \, \rho$ is close to $d_a^{-1} I_a$, so $\rho_a^{\psi} \approx \mathrm{tr}_b \, \rho$ for μ-most ψ.)

Previously Known Bounds

Canonical Typicality: Polynomial Bounds

Theorem 1 [Sugita 2007, Goldstein et al. 2017]

Let $\mathscr{H}_a$ and $\mathscr{H}_b$ be Hilbert spaces of respective dimensions $d_a, d_b \in \mathbb{N}$, $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$, and $\mathscr{H}_R$ any subspace of $\mathscr{H}$ of dimension d_R. Then for every $\varepsilon > 0$,

$$u_R\Big\{\psi \in \mathbb{S}(\mathscr{H}_R) : \big\|\rho_a^\psi - \mathrm{tr}_b\, \rho_R\big\|_{\mathrm{tr}} < \varepsilon\Big\} \geq 1 - \frac{d_a^4}{\varepsilon^2 d_R}.$$

trace norm $\|A\|_{\mathrm{tr}} = \mathrm{tr}\sqrt{A^\dagger A}$
$= \sum_n |a_n|$ for $A = \sum_n a_n |n\rangle\langle n|$.

Theorem 2 [Popescu, Short, Winter 2006]

With the notation and hypotheses as in Theorem 1, for every $\varepsilon > 0$,

$$u_R\Big\{\psi \in \mathbb{S}(\mathscr{H}_R) : \big\|\rho_a - \mathrm{tr}_b\,\rho_R\big\|_{\mathrm{tr}} < \varepsilon\Big\} \geq 1 - 4\exp\Big(-\frac{d_R\varepsilon^2}{18\pi^3}\Big).$$

Lévy's Lemma (Concentration of Measure)

Theorem 3 (Lévy's lemma) [P. Lévy 1922, V. Milman 1986]

Let $\mathscr{H}$ be a Hilbert space with $D := \dim \mathscr{H} < \infty$, let $f : \mathbb{S}(\mathscr{H}) \to \mathbb{R}$ be a function with Lipschitz constant η, and let $\varepsilon > 0$. Then

$$u\Big\{\psi \in \mathbb{S}(\mathscr{H}) : \big|f(\psi) - u(f)\big| < \varepsilon\Big\} \geq 1 - 4\exp\Big(-\frac{\tilde{C}D\varepsilon^2}{\eta^2}\Big),$$

where $\tilde{C} = 2/9\pi^3$ and

$$u(f) := \int_{\mathbb{S}(\mathscr{H})} f(\psi)u(d\psi).$$

"On a high-dimensional sphere, functions with high regularity are nearly constant."

$\eta = \sup|\nabla f|$

GAP Measures

Measure and density matrix

- For any probability measure μ on $\mathbb{S}(\mathscr{H})$, its density matrix is

$$\rho_\mu = \int_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, |\psi\rangle\langle\psi|\,.$$

  Then for any experiment with random outcome Z,

$$\mathbb{P}(Z = z) = \mathbb{E}_{\psi\sim\mu}\langle\psi|E(z)|\psi\rangle = \mathrm{tr}(\rho_\mu\, E(z))$$

  for the appropriate POVM $E(\cdot)$.
- If μ has mean zero, then ρ_μ is the covariance matrix of μ.
- Many-to-one: $\rho_\mu = \rho_{\mu'} \not\Rightarrow \mu = \mu'$
- For $\mu = \mathrm{GAP}(\rho)$ we have that $\rho_\mu = \rho$.

GAP Measures: Definition in 3 Steps

G AUSSIAN

A DJUSTED

P ROJECTED

GAP Measures: Definition in 3 Steps

GAUSSIAN: Start with
$G(\rho)$ = Gaussian measure on $\mathscr{H}$ with covariance ρ,
i.e., $\mathbb{E}_{G(\rho)} \langle \phi|\psi\rangle\langle\psi|\chi\rangle = \langle\phi|\rho|\chi\rangle \quad \forall \phi, \chi \in \mathscr{H}$.

Construction

If $\rho = \sum_n p_n |n\rangle\langle n|$ spectral decomposition
then let $\mathrm{Re}\, Z_n$, $\mathrm{Im}\, Z_n$ be independent Gaussian random variables with mean 0 and variance $p_n/2$; set $\psi = \sum_n Z_n |n\rangle$.

Ex $\mathscr{H} = \mathbb{C}^k$: $\dfrac{dG(\rho)}{d\lambda}(\psi) = \dfrac{1}{\pi^k \det \rho} e^{-\langle\psi|\rho^{-1}|\psi\rangle}$

GAP Measures: Definition in 3 Steps

GAUSSIAN: Start with
$G(\rho)$ = Gaussian measure on $\mathscr{H}$ with covariance ρ

ADJUSTED: To obtain the measure $GA(\rho)$ on $\mathscr{H}$, multiply by a density function $\psi \mapsto \|\psi\|^2$:
$GA(\rho)(d\psi) = \|\psi\|^2 \, G(\rho)(d\psi)$

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PROJECTED to the unit sphere $\mathbb{S}(\mathscr{H})$: $\psi^{GAP} = \dfrac{\psi^{GA}}{\|\psi^{GA}\|}$
or $GAP(\rho)(B) = GA(\rho)(\mathbb{R}^+ B)$ for $B \subseteq \mathbb{S}(\mathscr{H})$.

The adjustment factor compensates the change in covariance due to projection to $\mathbb{S}(\mathscr{H})$, thus $\rho_{GAP(\rho)} = \rho$.

GAP Measures: Properties

- covariant

$$U_* \, GAP(\rho) = GAP(U\rho U^{-1})$$

for every unitary U on $\mathscr{H}$
$\Rightarrow$ stationary under every unitary evolution that preserves ρ

- hereditary
"If a system has temperature $1/\beta$ then also every subsystem"
"GAP of a product density matrix has GAP marginal"
If $\Psi \in \mathbb{S}(\mathscr{H}_a \otimes \mathscr{H}_b)$ has distribution $GAP(\rho_a \otimes \rho_b)$ then, for any ONB $\{\varphi_j\}$ of $\mathscr{H}_b$, the conditional wave function ψ_a has marginal distribution $\int_{\mathbb{S}(\mathscr{H})} \mathrm{GAP}(\rho_a \otimes \rho_b)(d\Psi) \, \mathrm{Born}_a^{\Psi,B} = \mathrm{GAP}(\rho_1)$.

New Results

Generalized Canonical Typicality: Polynomial Bounds

Theorem 4 [Teufel, Tumulka, Vogel 2023]

Suppose $d_a = \dim \mathscr{H}_a < \infty$ and $\mathscr{H}_b$ is separable (i.e., has finite or countable dimension). Let ρ be a density matrix on $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$ with $\|\rho\| < 1/4$. Then for every $\varepsilon > 0$,

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathscr{H}) : \big\|\rho_a - \mathrm{tr}_b\,\rho\big\|_{\mathrm{tr}} < \varepsilon\Big\} \geq 1 - \frac{28 d_a^5\,\mathrm{tr}\,\rho^2}{\varepsilon^2}\,.$$

$\|\rho\|$ = largest eigenvalue; $\mathrm{tr}\,\rho^2$ = "purity" = average eigenvalue.

For the proof of Theorem 4, we need the following Proposition:

Proposition 5 [Reimann 2008; Teufel, Tumulka, Vogel 2023]

Let ρ be a density matrix on a separable Hilbert space $\mathscr{H}$ with positive eigenvalues such that $\|\rho\| < 1/4$, and let $\dim \mathscr{H} \geq 4$. For any bounded operator $A : \mathscr{H} \to \mathscr{H}$,

$$\mathrm{Var}\langle\psi|A|\psi\rangle \leq 28\|A\|^2\,\mathrm{tr}\,\rho^2\,.$$

Generalized Canonical Typicality: Exponential Bounds

Theorem 6 [Teufel, Tumulka, Vogel 2023]

Suppose $d_a = \dim \mathscr{H}_a < \infty$ and $\mathscr{H}_b$ is separable. Let ρ be a density matrix on $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$. Then for every $\varepsilon > 0$,

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathscr{H}) : \big\|\rho_a - \mathrm{tr}_b\,\rho\big\|_{\mathrm{tr}} < \varepsilon\Big\} \geq 1 - 6d_a^2 \exp\Big(-\frac{\tilde{C}\varepsilon^2}{d_a^2\|\rho\|}\Big),$$

where $\tilde{C} = 1/512\pi^2$.

For the proof of Theorem 6, we need a version of Lévy's lemma for GAP measures.

Lévy's Lemma for GAP Measures

Theorem 7 [Teufel, Tumulka, Vogel 2023]

Let $\mathscr{H}$ be a separable Hilbert space, let $f : \mathbb{S}(\mathscr{H}) \to \mathbb{R}$ be a function with Lipschitz constant η, let ρ be a density matrix on $\mathscr{H}$, and let $\varepsilon > 0$. Then

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathscr{H}) : \big|f(\psi) - \mathrm{GAP}(\rho)(f)\big| < \varepsilon\Big\} \geq 1 - 6\exp\Big(-\frac{C\varepsilon^2}{\eta^2\|\rho\|}\Big),$$

where $C = 1/128\pi^2$.

"If the eigenvalues of ρ are small, then $\mathrm{GAP}(\rho)$ behaves like a very spread-out measure."

Remarks

- If $\rho = \rho_R$, then $\|\rho\| = 1/d_R$, $\mathrm{GAP}(\rho) = u_R$, and we recover the canonical typicality results up to worse constants and additional factors of d_a.
- In both our generalized canonical typicality theorems one needs that the eigenvalues of ρ are small.
- Equivalence of ensembles: If a and b interact weakly, then both ρ_{mc} and ρ_{can} in $\mathscr{H}_S = \mathscr{H}_a \otimes \mathscr{H}_b$ lead to reduced density matrices close to a canonical density matrix for a, $\mathrm{tr}_b\, \rho_{\mathrm{mc}} \approx \rho_{a,\mathrm{can}} \approx \mathrm{tr}_b\, \rho_{\mathrm{can}}$; we can start from either u_{mc} or $\mathrm{GAP}(\rho_{\mathrm{can}})$ and obtain for both ensembles of ψ that ρ_a^{ψ} is nearly constant and nearly canonical.

Conditional wave function for GAP-typical Ψ

Theorem 9 [Teufel, Tumulka, Vogel 2023]

Let $\varepsilon, \delta \in (0,1)$, let $f : \mathbb{S}(\mathscr{H}_a) \to \mathbb{R}$ be any continuous (test) function, and let $d_b \geq \max\{4, d_a, 32\|f\|_\infty^2/\varepsilon^2\delta\}$. Then there is $p > 0$ such that for every density matrix ρ on $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$ with $\operatorname{tr}\rho^2 < p$,

$$\mathsf{GAP}(\rho) \times u_{\mathrm{ONB}}\Big\{(\Psi, B) \in \mathbb{S}(\mathscr{H}) \times \mathrm{ONB}(\mathscr{H}_b) : \Big|\mathrm{Born}_a^{\Psi,B}(f) - \mathsf{GAP}(\operatorname{tr}_b \rho)(f)\Big| < \varepsilon\Big\} \geq 1 - \delta\,.$$

Remark: Weak convergence of measures $P_n \Rightarrow P$ is equivalent to $P_n(f) \to P(f)$ for every bounded continuous (test) function f. Here, f is automatically bounded because $\mathbb{S}(\mathscr{H}_a)$ is compact.

Dynamical typicality: Polynomial bounds

Theorem 8 [Teufel, Tumulka, Vogel 2023]

Suppose $d_a = \dim \mathscr{H}_a < \infty$ and $\mathscr{H}_b$ is separable. Let ρ be a density matrix on $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b$ with $\|\rho\| < 1/4$. Then for every $\varepsilon, T > 0$,

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathscr{H}) : \frac{1}{T}\int_0^T \left\|\rho_a^{\psi_t} - \mathrm{tr}_b\, \rho_t\right\|_{\mathrm{tr}}^2 dt < \varepsilon\right\} \geq 1 - \frac{28 d_a^3 \,\mathrm{tr}\, \rho^2}{\varepsilon}.$$

Thank you for your attention