Positron Position Operators

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Overview

- In non-rel quantum theories, the position operators form a POVM (positive-operator-valued measure, def to come) on configuration space $\mathcal{Q}$ acting on Hilbert space $\mathscr{H}$.
- I consider the free standard quantized Dirac field (def to come).
- Many physicists don't consider a position POVM for the quantized Dirac field, only field operators.
- I think we should have a position POVM.
- There is an obvious choice P_{obv} of POVM for the quantized Dirac field.
- I think P_{obv} is physically wrong. Today I propose a different POVM P_{nat}, which I think is physically correct.
- Considerations here are non-rigorous. Even the existence of P_{nat} depends on a mathematical conjecture which I can neither prove nor disprove.
- Malament's theorem seems to exclude position operators in relativistic theories. I will explain why it doesn't apply to P_{nat}.

Reminder: PVM and POVM

Definition: PVM

A PVM (projection-valued measure) on a measurable space $\mathcal{Q}$ acting on a Hilbert space $\mathscr{H}$ associates with every subset $B \subseteq \mathcal{Q}$ a projection $P(B)$ such that $P(\mathcal{Q}) = I$ and $P(B_1) + P(B_2) + \ldots = P(B_1 \cup B_2 \cup \ldots)$ ("σ-additive") whenever $B_i \cap B_j = \emptyset \ \forall i \neq j$.

Ex: $\mathscr{H} = L^2(\mathcal{Q}, \mathbb{C}^d)$, $\quad P(B)\psi(q) = 1_{q \in B}\psi(q)$.

Ex: The position operators on $\mathbb{R}^n$, $\quad X_i\psi(x_1 \ldots x_n) = x_i\psi(x_1 \ldots x_n)$ jointly correspond to the PVM $P(B) = 1_{q \in B}$ on $\mathcal{Q} = \mathbb{R}^n$, $\mathscr{H} = L^2(\mathbb{R}^n)$.

Definition: POVM

The same, just $P(B)$ doesn't have to be a projection, it can be a positive operator. (So every PVM is a POVM.)

Ex: $Q(B) = P_{\mathscr{K}} P(B) P_{\mathscr{K}}$ acting on subspace $\mathscr{K} \subseteq \mathscr{H}$ with P a PVM

A POVM P and a unit vector ψ always define a prob distribution on $\mathcal{Q}$, $\mathbb{P}(B) = \langle \psi | P(B) | \psi \rangle$.

What a position POVM is good for

- characterize the distribution of results of an ideal detector
- define the distribution of Bohmian particles
- define the probabilities of macroscopic configurations of pointers or cats [e.g., C. Beck 2021]

You can often do without a position POVM (and use only a momentum POVM, which is easier) if you only consider $t = -\infty$ and $t = \infty$.

The quantized Dirac field (1)

[following Thaller's 1992 book]

- Let $c = 1 = \hbar$, $\mathscr{H}_1 = L^2(\mathbb{R}^3, \mathbb{C}^4)$, $H_1 = -i\boldsymbol{\alpha} \cdot \nabla + m\beta$ (Dirac), $\mathscr{H}_{1\pm}$ = pos. (neg.) spectral subspace of H_1, $P_{1\pm}$ = proj to $\mathscr{H}_{1\pm}$
- $C\psi(\boldsymbol{x}) = i\alpha_2\psi^*(\boldsymbol{x})$ charge conjugation operator (anti-unitary), $C(\mathscr{H}_{1\mp}) = \mathscr{H}_{1\pm}$, $CH_1C^{-1} = -H_1$.
- fermionic Fock space $\mathscr{F}(\mathscr{K}_1) = \bigoplus_{n=0}^{\infty} \mathrm{Anti}\, \mathscr{K}_1^{\otimes n}$
- $\mathscr{H} := \mathscr{F}(\mathscr{H}_{1+}) \otimes \mathscr{F}(\overline{\mathscr{H}_{1-}}) \cong \mathscr{F}(\mathscr{H}_{1+}) \otimes \mathscr{F}(\mathscr{H}_{1+})$
- $\mathscr{H} \ni \psi = \psi^{n,\bar{n}}_{s_1 \ldots s_n \bar{s}_1 \ldots \bar{s}_{\bar{n}}}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n, \bar{\boldsymbol{x}}_1 \ldots \bar{\boldsymbol{x}}_{\bar{n}}) = \psi^{n\bar{n}}(x_1 \ldots x_n, \bar{x}_1 \ldots \bar{x}_{\bar{n}})$ with $\bar{n} \in \mathbb{N}_0$ and $\boldsymbol{x}_j, \bar{\boldsymbol{x}}_i \in \mathbb{R}^3$ and $s_j, \bar{s}_i \in \{1, 2, 3, 4\}$ and $x = (\boldsymbol{x}, s)$ (let me call ψ Thaller's representation)
- $|\Omega\rangle = (1, 0, 0 \ldots)$ ~~vacuum state~~ sea state
- H = lift of H_1 is a positive operator, $|\Omega\rangle$ = ground state

The quantized Dirac field (2)

- For $f \in \mathscr{H}_{1+}$ and $g \in \mathscr{H}_{1-}$, the electron annihilation/creation operators and positron annihilation/creation operators are

$$a(f)\psi^{n\overline{n}}(x_1 \dots x_n, \overline{x}_1 \dots \overline{x}_{\overline{n}}) = \sqrt{n+1} \sum_{s=1}^{4} \int d^3\boldsymbol{x}\, f(x)^* \, \psi^{n+1,\overline{n}}(x, x_1 \dots x_n, \overline{x}_1 \dots \overline{x}_{\overline{n}})$$

$$a^\dagger(f)\psi^{n\overline{n}}(x_1 \dots x_n, \overline{x}_1 \dots \overline{x}_{\overline{n}}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j+1} f(x_j)\, \psi^{n-1,\overline{n}}(x_1 \dots \widehat{x_j} \dots \overline{x}_{\overline{n}})$$

$$b(g)\psi^{n\overline{n}}(x_1 \dots x_n, \overline{x}_1 \dots \overline{x}_{\overline{n}}) = (-1)^n \sqrt{\overline{n}+1} \sum_{\overline{s}=1}^{4} \int d^3\overline{\boldsymbol{x}}\, Cg(\overline{x})^* \psi^{n,\overline{n}+1}(x_1 \dots x_n, \overline{x}, \overline{x}_1 \dots \overline{x}_{\overline{n}})$$

$$b^\dagger(g)\psi^{n\overline{n}}(x_1 \dots x_n, \overline{x}_1 \dots \overline{x}_{\overline{n}}) = \frac{(-1)^n}{\sqrt{\overline{n}}} \sum_{\overline{j}=1}^{\overline{n}} (-1)^{\overline{j}+1} Cg(\overline{x}_{\overline{j}})\, \psi^{n-1,\overline{n}}(x_1 \dots \widehat{\overline{x}_{\overline{j}}} \dots \overline{x}_{\overline{n}})$$

- For $f \in \mathscr{H}_1$, the field operator is

$$\Psi(f) = a(P_{1+}f) + b^\dagger(P_{1-}f)\,.$$

The quantized Dirac field (3)

- Anti-commutation relations for $f_1, f_2 \in \mathscr{H}_1$:

$$\{\Psi(f_1), \Psi(f_2)\} = \{\Psi^\dagger(f_1), \Psi^\dagger(f_2)\} = 0,$$
$$\{\Psi(f_1), \Psi^\dagger(f_2)\} = \langle f_1 | f_2 \rangle_{\mathscr{H}_1} \; I.$$

The obvious POVM

$$\rho^{n\overline{n}}_{\text{obv}}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n, \overline{\boldsymbol{x}}_1 \ldots \overline{\boldsymbol{x}}_{\overline{n}}) = \sum_{\substack{s_1 \ldots s_n \\ \overline{s}_1 \ldots \overline{s}_{\overline{n}}}} \Big| \psi^{n\overline{n}}(x_1 \ldots x_n, \overline{x}_1 \ldots \overline{x}_{\overline{n}}) \Big|^2$$

corresponds to $\mathbb{P}_{\text{obv}}(B) = \int_{\pi^{-1}(B)} dq \, \rho_{\text{obv}}(q) = \langle \psi | P_{\text{obv}}(B) | \psi \rangle$ with π the unordering map,

$$\pi(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n, \overline{\boldsymbol{x}}_1 \ldots \overline{\boldsymbol{x}}_{\overline{n}}) = \big(\{\boldsymbol{x}_1 \ldots \boldsymbol{x}_n\}, \{\overline{\boldsymbol{x}}_1 \ldots \overline{\boldsymbol{x}}_{\overline{n}}\}\big) \,,$$

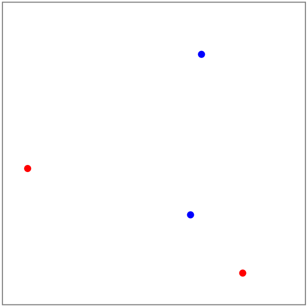
and P_{obv} a POVM on

$$\mathcal{Q} = \mathcal{Q}_{\text{el}} \times \mathcal{Q}_{\text{pos}} = \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$$

with $\Gamma(S) = \{q \subset S : \#q < \infty\}$ the set of finite subsets of S. In fact,

$$P_{\text{obv}}(B) = P_{\mathscr{H}} 1_{\pi^{-1}(B)} P_{\mathscr{H}}$$

from the PVM $1_{\pi^{-1}(B)}$ on $\mathscr{F}(\mathscr{H}_1) \otimes \mathscr{F}(\mathscr{H}_1) \subseteq L^2(\mathcal{Q}, \text{spin-space})$.



The natural POVM

- Charge operators: For $A \subseteq \mathbb{R}^3$,

$$Q(A) := \int_A d^3\boldsymbol{x}\, Q(\boldsymbol{x}) = -\int_A d^3\boldsymbol{x} \sum_{s=1}^{4} :\Psi^\dagger(x)\,\Psi(x):$$

$$= -\int_A d^3\boldsymbol{x} \sum_{s=1}^{4} \Psi^\dagger(x)\,\Psi(x) + \infty$$

satisfy $[Q(A), Q(B)] = 0$ and have spectrum $\mathbb{Z}$.

- Diagonalize all $Q(A)$ simultaneously. That defines a PVM on their joint spectrum, which is the definition of P_{nat}.

My conjecture

The joint spectrum consists (up to P_{nat}-null sets) of locally bounded functions $q(A)$.

What the conjecture means

- Consider, instead of $\mathbb{R}^3$, a box $[0, L)^3$ of side length L, and $H_1 = -i\boldsymbol{\alpha} \cdot \nabla + \beta m$ with periodic boundary conditions ("3-torus $\mathbb{T}^3$").
- When subdividing $A \subset \mathbb{T}^3$ into disjoint $A_1 \cup A_2$, then $q(A) = q(A_1) + q(A_2)$.
- Subdivide repeatedly into ever-smaller volumes.
- In the limit we might end up with finite or infinite amounts of positive and negative charge. The conjecture says it is finite in a finite volume (such as $\mathbb{T}^3$).
- As a consequence of the conjecture, P_{nat} is a PVM on

$$\mathcal{Q} = \Gamma(\mathbb{T}^3) \times \Gamma(\mathbb{T}^3) .$$

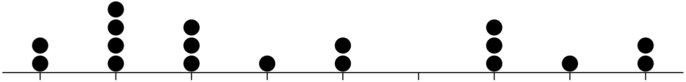
- In $\mathbb{R}^3$, P_{nat} is a PVM on $\mathcal{Q}_{\text{loc.fin}} = \Gamma_{\text{loc.fin}}(\mathbb{R}^3) \times \Gamma_{\text{loc.fin}}(\mathbb{R}^3)$ with $\Gamma_{\text{loc.fin}}(S) = \{q \subset S : \#(q \cap B_r(\boldsymbol{x})) < \infty \, \forall r > 0 \, \forall \boldsymbol{x} \in S\}$ the space of locally finite configurations.
- But I expect that in reality 3-space has finite volume (is approximately $\mathbb{S}^3$).

Another approach to P_{nat}: discretize (1)

- Consider, instead of $\mathbb{R}^3$ or $\mathbb{T}^3$, a finite lattice $\mathcal{L} := \left([0, L) \cap \frac{L}{N}\mathbb{Z}\right)^3$ with N^3 sites, and $H_1 = -i\boldsymbol{\alpha} \cdot \nabla + \beta m$ with ∇ the difference operator,
$$\nabla_j \psi(\boldsymbol{x}) = \frac{\psi(\boldsymbol{x} + \frac{L}{N}\boldsymbol{e}_j) - \psi(\boldsymbol{x})}{\frac{L}{N}},$$
and periodic boundaries.
- $\mathscr{H}_1 = L^2(\mathcal{L}, \mathbb{C}^4)$ has dim $4N^3$, $\mathscr{H} = \mathscr{F}(\mathscr{H}_1)$ has dim 16^{N^3}, $\mathscr{H} = \bigotimes_{\boldsymbol{x} \in \mathcal{L}} \mathscr{H}_x$, where $\mathscr{H}_x = \mathscr{F}(\mathbb{C}^4) = \mathscr{F}_0 \oplus \mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_4$ has dim $16 = 1 + 4 + 6 + 4 + 1$
- $\Psi_s(\boldsymbol{x})$ annihilation operator, $\Psi_s^\dagger(\boldsymbol{x})$ creation operator

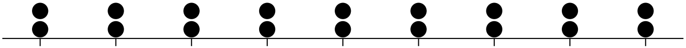
Another approach to P_{nat}: discretize (2)

- Write $\mathscr{F}(\mathscr{H}_1) \ni \psi = \psi^\ell(\boldsymbol{x}_1 \ldots \boldsymbol{x}_\ell)$, $|\psi|^2$ defines a prob distribution over the configuration space $\mathcal{Q}_N = \{0, 1, 2, 3, 4\}^{\mathcal{L}}$ (unordered configuration = occupation numbers).



I call the "particles" in this configuration pre-particles (b/c I claim they are not ontologically the real particles), the PVM P_{pre}.

- The bottom configuration has 0 pre-particles.
- $|B\rangle$ = bottom state (unique up to phase) has 0 pre-particles.
- The level configuration $q_{\mathscr{L}}$ has 2 pre-particles at each lattice site.

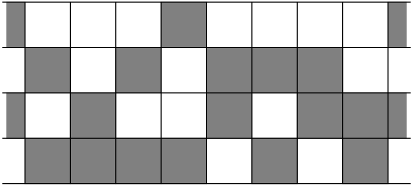


- A level state is a state concentrated on the level configuration.

Level space $\mathscr{L} = \{\text{level states}\} = \bigotimes_{\boldsymbol{x} \in \mathcal{L}} \mathscr{F}_2(\mathbb{C}^4) = \text{range } P_{\mathrm{pre}}(\{q_{\mathscr{L}}\})$ is a subspace of dim 6^{N^3}.

Another approach to P_{nat}: discretize (3)

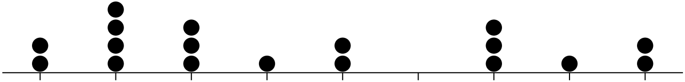
- $\{e_1, e_2, e_3, e_4\} :=$ canonical ONB of $\mathbb{C}^4$, $|ij\rangle = e_i \wedge e_j$. Then $\{|12\rangle, |13\rangle, |14\rangle, |23\rangle, |24\rangle, |34\rangle\}$ is an ONB of $\mathscr{F}_2(\mathbb{C}^4)$, and $\{\otimes_{x\in\mathcal{L}} |i_x j_x\rangle : i_x < j_x\}$ is an ONB of $\mathscr{L}$.



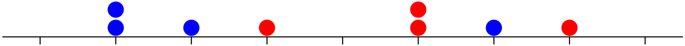
- The sea state is $|\Omega\rangle = \varphi_1 \wedge \varphi_2 \wedge \ldots$ (up to phase) with $\{\varphi_1, \varphi_2 \ldots\}$ an ONB of the negative spectral subspace of H_1.
- The sea state is not level, $|\Omega\rangle \notin \mathscr{L}$. It is an eigenstate of $\Psi^\dagger(\varphi_k)\Psi(\varphi_k)$, but not of $\Psi^\dagger(x)\Psi(x)$.

Another approach to P_{nat}: discretize (4)

- "$P_{\mathrm{nat},N}(q) = P_{\mathrm{pre}}(q + q_{\mathscr{L}})$," range $P_{\mathrm{nat},N}(\{\emptyset\}) = \mathscr{L}$



becomes



- Continuum limit $N \to \infty$ (L fixed), $\mathcal{L} \to \mathbb{T}^3$
- Consider only states that differ from $|\Omega\rangle$ by finitely many pre-particles
- Set $P_{\mathrm{nat}} = \lim_{N\to\infty} P_{\mathrm{nat},N}$.

Conjecture implies

$$\langle\Omega|P_{\mathrm{nat},N}(\{\emptyset\})|\Omega\rangle \xrightarrow{N\to\infty} \langle\Omega|P_{\mathrm{nat}}(\{\emptyset\})|\Omega\rangle \in (0,1)$$

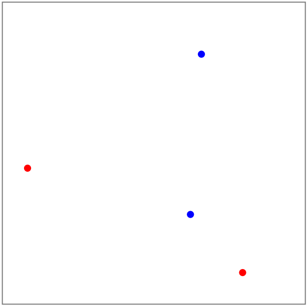
but $\langle\Omega|P_{\mathrm{nat}}(\{\emptyset\})|\Omega\rangle \xrightarrow{L\to\infty} 0$.

Ontology

I propose an ontology with a variable but finite number of electron and positron point particles, $\mathcal{Q} = \Gamma(\mathbb{T}^3) \times \Gamma(\mathbb{T}^3)$, and Bohmian trajectories (later slide).

Born rule: $\mathbb{P}(B) = \langle\psi|P_{\mathrm{nat}}(B)|\psi\rangle$

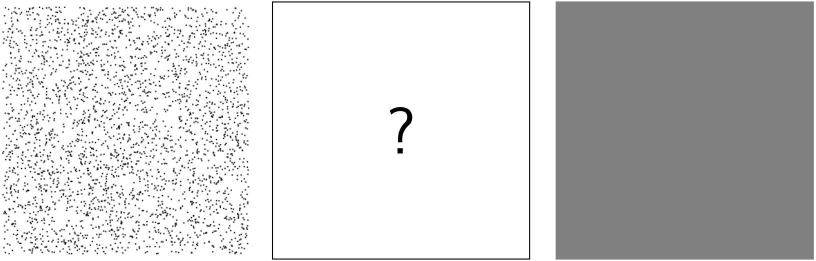
Note that range $P_{\mathrm{nat}}(\{\emptyset\})$ has dim ∞, unlike usual.



Dirac sea ontology (1)

[Bohm and Hiley 1993, Colin and Struyve 2007, Deckert, Esfeld, and Oldofredi 2019]

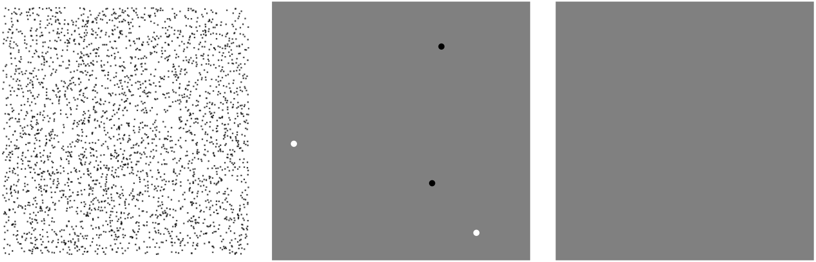
- Take what I called “pre-particles” as the ontology: an actual infinitude of particles. One kind of particles.
- What does a typical configuration for $|\Omega\rangle$ look like?



- I always pictured it like the left image, like a countable dense set.

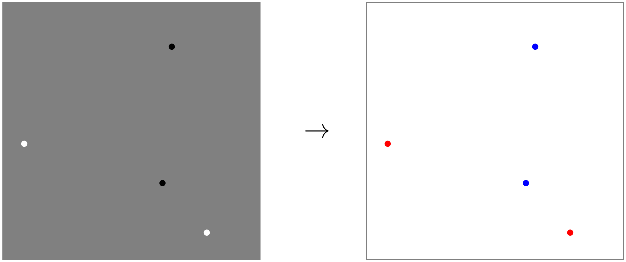
Dirac sea ontology (2)

- If my conjecture is correct, then the picture should look like the middle image:



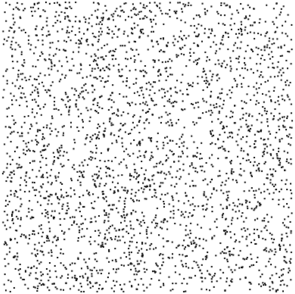
Dirac sea ontology (3)

- If my conjecture is correct, then my proposal is the continuum limit of the Dirac sea ontology except for a reinterpretation:



Dirac sea ontology (4)

- I used to worry about the Dirac sea ontology for this reason:
- Consider $|\psi\rangle = a^\dagger(f)|\Omega\rangle$.
- Where is the particle?



Another proposed ontology

[Teufel, personal communication]

Consider $\mathscr{F}(L^2(\mathbb{R}^3, \mathbb{C}^4))$: states that are "finitely many particles away" from the bottom state $|B\rangle$. $H =$ lifted H_1 (i.e., allow negative energies). "Positron" is just a name for an electron with negative energy.

Standard objection (presumably valid)

Unstable. No thermal equilibrium state exists. Interaction would lead to particles with greater and greater magnitude of energy.

Common use of P_{obv}

While it is standard in QFT to never mention a POVM, often remarks about how to interpret $\psi \in \mathscr{H}$ are inspired by P_{obv}:

- Thaller (1992) p. 277: "the probability that there are just n particles and $[\overline{n}]$ antiparticles at a given time is $[\|\psi^{n\overline{n}}\|^2]$"
- Thaller (1992) p. 277: "$[|\Omega\rangle]$ describes the possibility that there are no particles at all"
- Schweber (1961) p. 230: "The basis vectors $[a^\dagger(P_{1+}x_1)\cdots a^\dagger(P_{1+}x_n)b^\dagger(P_{1-}\overline{x}_1)\cdots b^\dagger(P_{1-}\overline{x}_{\overline{n}})|\Omega\rangle]$ span the states in which there exist $[n]$ particles and $[\overline{n}]$ antiparticles."
- Schweber (1961) p. 231: "$[\psi^{01}(x)]$ is the probability amplitude for finding the antiparticle [at $\boldsymbol{x}$]."

Against $P_{\rm obv}$ (1)

Difficulty 1

A function $f \in \mathscr{H}_{1+}$ that vanishes in an open set vanishes everywhere.
In particular, $\rho_{\rm obv}$ can never have compact support.
So how do you collapse $\psi = \psi^{10}$ after detecting a particle in the compact region $A \subset \mathbb{R}^3$? Will a particle, detected at $t = 0$ in $A = \text{closure}(B_r(\mathbf{0}))$, have positive probability at any $t > 0$ to be in $\mathbb{R}^3 \setminus B_{2r}(\mathbf{0})$?
There seems to be a conflict with propagation locality (PL).

Against P_{obv} (2)

Difficulty 2

P_{obv} seems to violate the principle of interaction locality (IL) (which says that the Hamiltonian contains no interaction terms between spacelike separated regions). Actually, it seems that no interaction would obey (IL) with P_{obv} while conserving particle number.

Illustration: Assuming otherwise, consider a 2-electron wave fct ψ^{20} at times 0 and t. Let $F_t = \exp(-iH_0 t)$ be the free time evolution. By (IL), ψ_t agrees with $F_t\psi_0$ outside the t-neighborhood of the diagonal. If both ψ_t and ψ_0 lie in $\mathscr{H}_{1+} \otimes \mathscr{H}_{1+}$, then so does $\psi_t - F_t\psi_0$. But the only function in $\mathscr{H}_{1+} \otimes \mathscr{H}_{1+}$ that vanishes outside the t-neighborhood of the diagonal is 0. So, the evolution agrees with $F_t \Rightarrow$ contradiction.

Explicit formula for ρ_{nat} on $\mathbb{T}^3$

$$\rho_{\mathrm{nat}}^{n\bar{n}}(\boldsymbol{x}_1 \ldots \boldsymbol{x}_n, \bar{\boldsymbol{x}}_1 \ldots \bar{\boldsymbol{x}}_{\bar{n}}) =$$

$$\frac{1}{3^{n+\bar{n}}} \sum_{\substack{s_1 \ldots s_n \\ \bar{s}_1 \ldots \bar{s}_{\bar{n}}}} \langle \psi | \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_n) \Psi(\bar{x}_1) \cdots \Psi(\bar{x}_{\bar{n}}) \times$$

$$\times \quad P_{\mathrm{nat}}(\{\emptyset\}) \Psi^\dagger(\bar{x}_{\bar{n}}) \cdots \Psi^\dagger(\bar{x}_1) \Psi(x_n) \cdots \Psi(x_1) | \psi \rangle,$$

provided the locations are pairwise distinct.

Spontaneous pair creation

- usually means this: In an external field $A_\mu(t, \boldsymbol{x})$ with $A_\mu(t = -\infty, \boldsymbol{x}) = 0 = A_\mu(t = \infty, \boldsymbol{x})$, $\psi(t = -\infty) = |\Omega\rangle$ may evolve to $\psi(t = \infty)$ with $\psi^{11} \neq 0$.
  [Greiner and Reinhardt 1984, Pickl and Dürr 2008]
- but here means something else: $[H, P_{\mathrm{nat}}(\mathcal{Q}^{n\bar{n}})] \neq 0$.
- That is, even if $A_\mu(t, \boldsymbol{x}) = 0$, for some states some probability gets transported to a different sector of P_{nat}-particle number.
- (This doesn't happen to $|\Omega\rangle$, though. Since $|\Omega\rangle$ is stationary and its own time reverse, all currents vanish.)

Bohmian trajectories

- $e^- e^+$ pairs spontaneously appear (emerging from the same point $\boldsymbol{x}$), i.e., configuration $(q;\overline{q}) \to (q,\boldsymbol{x};\overline{q},\boldsymbol{x})$, at rate (per $d\boldsymbol{x}$)

$$2\,\mathrm{Im}^+ \frac{\langle\psi|P_{\mathrm{nat}}(q,\boldsymbol{x};\overline{q},\boldsymbol{x})HP_{\mathrm{nat}}(q,\overline{q})|\psi\rangle}{\langle\psi|P_{\mathrm{nat}}(q,\overline{q})|\psi\rangle}$$

[Dürr et al. 2004-2006]

- Particles move at (subluminal) velocity

$$\frac{d\boldsymbol{X}_j^{\mu_j}}{dt} \propto \frac{J^{0\ldots0\mu_j0\ldots0}(X)}{J^{0\ldots0}(X)} \quad \text{with}$$

$$J^{\mu_1\ldots\mu_n\overline{\mu}_1\ldots\overline{\mu}_{\overline{n}}}(\boldsymbol{x}_1\ldots\boldsymbol{x}_n,\overline{\boldsymbol{x}}_1\ldots\overline{\boldsymbol{x}}_{\overline{n}}) = \frac{1}{3^{n+\overline{n}}}\sum_{\substack{s_1\ldots s_1'\ldots\\ \overline{s}_1\ldots\overline{s}_1'\ldots}} (\gamma^0\gamma^{\mu_1})_{s_1s_1'}\cdots(\gamma^0\gamma^{\overline{\mu}_1})_{\overline{s}_1'\overline{s}_1}\cdots\times$$

$$\times\langle\psi|\Psi^\dagger_{s_1}(\boldsymbol{x}_1)\cdots\Psi_{\overline{s}_1}(\overline{\boldsymbol{x}}_1)\cdots P_{\mathrm{nat}}(\{\emptyset\})\Psi^\dagger_{\overline{s}'_{\overline{n}}}(\overline{\boldsymbol{x}}_{\overline{n}})\cdots\Psi_{s_n'}(\boldsymbol{x}_n)\cdots|\psi\rangle.$$

- When e^- and e^+ meet, both disappear.
(All this can be obtained from the lattice approximation. [Vink 1993])

What if the conjecture is wrong?

- It is still possible to simultaneously diagonalize all $Q(A)$, and this would define a PVM on some kind of locally infinite configurations.
- It is not obvious whether a Bohmian motion could be defined for such configurations.
- And it is not obvious whether one could read off from such a configuration whether Schrödinger's cat is dead or alive.
- And this means it is not obvious whether such a PVM would even define probabilities for Schrödinger's cat being dead or alive.

Reeh-Schlieder theorem

Theorem [Reeh and Schlieder 1961]

Let $R \neq \emptyset$ be a bounded open region in Minkowski space-time. The set of all polynomials in $a_t^\dagger(P_{1+}x), a_t(P_{1+}x), b_t^\dagger(P_{1-}x), b_t(P_{1-}x)$ (Heisenberg-evolved) with $(t, \boldsymbol{x}) \in R$ applied to $|\Omega\rangle$ is dense in $\mathscr{H}$.

Sounds paradoxical:

- Starting from the vacuum, you only act with operators localized in R.
- You get any state, also with non-small particle probability at spacelike separation from R.

But the sense of paradox evaporates when you accept that $|\Omega\rangle$ does not mean vacuum at all. Think of it as an anti-symmetrized product of plane waves. Then $|\Omega\rangle$ looks like a highly entangled state.

Malament's theorem (1)

[Malament 1996, Halvorson and Clifton 2002]

- This is a family of theorems of the following type: Assume a Hilbert space $\mathscr{H}$, a Hamiltonian H bounded from below, a POVM P on particle configurations acting on $\mathscr{H}$, and a number of reasonable-sounding, relativity-inspired hypotheses about P. Then a contradiction follows.
- These theorems are often taken to exclude a particle ontology or position operators in relativistic QFT.
- So how does P_{nat} get around it?

Malament's theorem (2)

Malament's Theorem

Suppose $\mathscr{H}$ is a Hilbert space (for 1 particle), H bounded from below, P a PVM on $\mathbb{R}^3$ acting on $\mathscr{H}$, and U a unitary representation of the translation group of $\mathbb{R}^3$ on $\mathscr{H}$. Suppose further that (i) P is translation covariant w.r.t. U and (ii) propagation locality holds, $e^{-iHt}P(A)e^{iHt} \leq P(B_{|t|}(A))$ with $B_r(A) = \cup_{\boldsymbol{x}\in A} B_r(\boldsymbol{x})$. Then a contradiction follows.

- Let us leave aside that P_{nat} on $\mathbb{T}^3$ may not fall under this theorem because the theorem assumes $\mathbb{R}^3$.
- On either $\mathbb{T}^3$ or $\mathbb{R}^3$, P_{nat} violates the assumption that the 1-particle sector, range $P(\mathcal{Q}^{10})$, is invariant under the time evolution.
- On either $\mathbb{T}^3$ or $\mathbb{R}^3$, P_{nat} violates the assumption of propagation locality (next slide).
- It would be interesting to know which violation is more essential.

Locality properties

- Locality: events at $x \in \mathbb{R}^4$ can't influence events at spacelike separated $y \in \mathbb{R}^4$. (Wrong, by Bell's theorem.)
- Interaction locality (IL): there is no interaction term in the Hamiltonian between spacelike separated regions. (Sounds right.)
- Propagation locality (PL): Wave functions don't propagate faster than light. (Sounds right.)
- It is hard to separate PL from no particle creation from the vacuum (NCFV).
- A definition of PL given by [Lienert and Tumulka 2017], more or less $e^{-iHt}P(\forall(A))e^{iHt} \leq P(\forall(B_{|t|}(A)))$ with $\forall(A) = \{q \in \mathcal{Q} : q \subseteq A\}$, turns out equivalent to NCFV. P_{nat} violates this definition, but all speeds of electrons and positrons are $\leq c$.
- Definition of IL by [Lienert and Tumulka 2017]: $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_{A^c}$, $P(B \subset \forall(A))$ acts on $\mathscr{H}_A$, evolution to Cauchy surface (next slide) $\Sigma \supset A$ is of the form $I \otimes U$. That seems to be true for P_{nat} (as obtained from the Dirac sea perspective).

Curved space-time and Cauchy surfaces (1)

also [Deckert and Merkl 2014–2016]

- Cauchy surface ≈ spacelike 3-surface
- Given curved space-time $(\mathscr{M}, g)$ (globally hyperbolic) so that
- Cauchy surfaces are compact (finite 3-volume).
- It is known how to define the 1-particle Dirac eq on $(\mathscr{M}, g)$. ψ is a cross-section of a rank-4 complex vector bundle $\mathscr{D}$ over $\mathscr{M}$.
- That defines $\mathscr{H}_{1\Sigma}$ for every Cauchy surface Σ and unitary evolution $U_{1\Sigma}^{\Sigma'} : \mathscr{H}_{1\Sigma} \to \mathscr{H}_{1\Sigma'}$.
- Suppose we are given "sea spaces" $\mathscr{K}_\Sigma$ so that $U_{1\Sigma}^{\Sigma'} \mathscr{K}_\Sigma$ differs from $\mathscr{K}_{\Sigma'}$ by only finitely many dimensions ($U_{1\Sigma}^{\Sigma'} \mathscr{K}_\Sigma \cap \mathscr{K}_{\Sigma'}$ has finite codim in both).
- Then it is defined how to construct $\mathscr{H}_\Sigma = \mathscr{F}(\mathscr{K}_\Sigma^\perp) \otimes \mathscr{F}(\overline{\mathscr{K}_\Sigma})$ and the unitary time evolution $U_\Sigma^{\Sigma'} : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$ and P_{nat} on $\mathcal{Q}_\Sigma = \Gamma(\Sigma) \times \Gamma(\Sigma)$ acting on $\mathscr{H}_\Sigma$.

Curved space-time and Cauchy surfaces (2)

- What is missing: A law selecting $\mathscr{K}_\Sigma$.
- You can't use the negative spectral subspace of $H_{1\Sigma}$ b/c $H_{1\Sigma}$ depends on a choice of lapse and shift function.
- In Minkowski space-time, there is a natural choice: On a hyperplane Υ, choose $\mathscr{K}_\Upsilon = \mathscr{H}_{1\Upsilon-}$. Theorem: On every hyperplane Υ', $U_\Upsilon^{\Upsilon'} \mathscr{H}_{1\Upsilon-} = \mathscr{H}_{1\Upsilon'-}$.
- Thus, it is consistent to define $\mathscr{K}_\Sigma = U_\Upsilon^\Sigma \mathscr{H}_{1\Upsilon-}$ for any Σ. But not for general $(\mathscr{M}, g)$.

Thank you for your attention