Probability of Particle Creation and Interior-Boundary Conditions for Fock Space

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Joint work with Stefan Teufel, Julian Schmidt, and Jonas Lampart

Congratulations to Guenter Ahlers, Cristina Marchetti, and Shelly Goldstein!

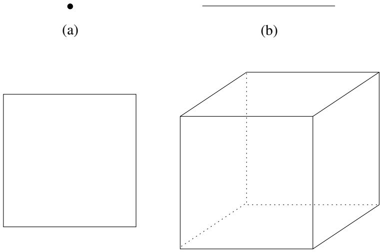
- Hamiltonians H on Fock space with particle creation and annihilation usually are UV divergent.
- To obtain a well-defined H you can introduce a UV cut-off by
    - discretizing space or
    - smearing out the creation/annihilation terms
- The novel approach of interior-boundary conditions (IBCs) can provide a well-defined H without need for a UV cut-off.
- I will present
    - rigorous results for the non-relativistic case
    - Bohmian trajectories

Particle–position representation of a Fock space vector

Configuration space of a variable number of particles:

$$\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n}$$

$$= \bigcup_{n=0}^{\infty} \mathcal{Q}^{(n)}$$



(a) (b) (c) (d)

here $d = 1$,
$n = 0, 1, 2, 3$

Fock space:

- $\mathscr{F}^{\pm} = \bigoplus_{n=0}^{\infty} S_{\pm} \mathscr{H}_1^{\otimes n}$
  with S_+ = symmetrizer, S_- = anti-symmetrizer, $\mathscr{H}_1$ = 1-particle Hilbert space = $L^2(\mathbb{R}^3, \mathbb{C}^k)$
- $\psi \in \mathscr{F} \Rightarrow \psi = \big(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots\big)$
- $\psi : \mathcal{Q} \to S$ with S = value space = $\cup_{n=0}^{\infty} (\mathbb{C}^k)^{\otimes n}$
- ψ is an (anti-)symmetric function

Example: a non-relativistic QFT

- There are two species of particles, x-particles and y-particles.
- x-particles can emit and absorb y-particles.
- configuration space $\mathcal{Q} = \bigcup_{m,n=0}^{\infty} (\mathbb{R}^3_x)^m \times (\mathbb{R}^3_y)^n$
- Hilbert space $\mathscr{H} = \mathscr{F}_x^- \otimes \mathscr{F}_y^+$
- $\psi : \mathcal{Q} \to \mathbb{C}$, $\psi = \psi(x^m, y^n)$, where x^m is any x-configuration with m particles
- As always, $i\partial_t \psi = H\psi$ Schrödinger eq

Naive original Hamiltonian (UV divergent)

$$H_{\mathrm{orig}} = H_x^{\mathrm{free}} + H_y^{\mathrm{free}} + g \int_{\mathbb{R}^3} d^3\boldsymbol{x}\; a_x^\dagger(\boldsymbol{x}) \Big(a_y(\boldsymbol{x}) + a_y^\dagger(\boldsymbol{x}) \Big) a_x(\boldsymbol{x})$$

g = coupling constant $\in \mathbb{R}$

H_{orig} in the particle-position representation

Naive original Hamiltonian (UV divergent)

$$
\begin{aligned}
(H_{\text{orig}}\psi)(x^m, y^n) = &-\frac{\hbar^2}{2m_x}\sum_{i=1}^{m}\nabla^2_{\boldsymbol{x}_i}\psi(x^m, y^n) - \frac{\hbar^2}{2m_y}\sum_{j=1}^{n}\nabla^2_{\boldsymbol{y}_j}\psi(x^m, y^n) + \\
&+ \; nE_0\psi(x^m, y^n) + \\
&+ \; g\sqrt{n+1}\sum_{i=1}^{m}\psi\big(x^m, (y^n, \boldsymbol{x}_i)\big) + \\
&+ \; \frac{g}{\sqrt{n}}\sum_{i=1}^{m}\sum_{j=1}^{n}\delta^3(\boldsymbol{x}_i - \boldsymbol{y}_j)\,\psi\big(x^m, y^n \setminus \boldsymbol{y}_j\big)\,,
\end{aligned}
$$

with E_0 = rest energy, $y^n \setminus \boldsymbol{y}_j$ = leave out $\boldsymbol{y}_j$.

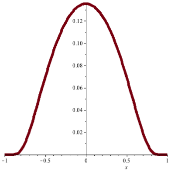
H_{orig} is ill-defined because the wave fct of the newly created y-particle, $\delta^3(\boldsymbol{x} - \boldsymbol{y})$, does not lie in $L^2(\mathbb{R}^3)$ (or, has infinite energy).

Well-defined, regularized version of H

UV cut-off $\varphi \in L^2(\mathbb{R}^3)$:

$$
\begin{aligned}
(H_{\mathrm{cutoff}}\psi)(x^m, y^n) = {} & -\frac{\hbar^2}{2m_x}\sum_{i=1}^m \nabla^2_{\boldsymbol{x}_i}\psi(x^m,y^n) - \frac{\hbar^2}{2m_y}\sum_{j=1}^n \nabla^2_{\boldsymbol{y}_j}\psi(x^m,y^n) + \\
& + \; nE_0\psi(x^m,y^n) + \\
& + \; g\sqrt{n+1}\sum_{i=1}^m \int_{\mathbb{R}^3} d^3\boldsymbol{y}\,\varphi^*(\boldsymbol{x}_i - \boldsymbol{y})\,\psi\big(x^m,(y^n,\boldsymbol{y})\big) + \\
& + \; \frac{g}{\sqrt{n}}\sum_{i=1}^m\sum_{j=1}^n \varphi(\boldsymbol{x}_i - \boldsymbol{y}_j)\,\psi\big(x^m, y^n \setminus \boldsymbol{y}_j\big)
\end{aligned}
$$

"smearing out" the x-particle
with "charge distribution" $\varphi(\cdot - \boldsymbol{x})$



Novel idea: Interior–boundary condition

Interior–boundary condition (IBC) basic form

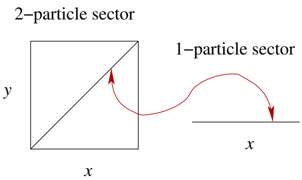
For q on the boundary of $\mathcal{Q}$,

$$\psi(q) = \alpha\,\psi(q')\,,$$

where q' is an interior point of $\mathcal{Q}$ and $\alpha = \alpha(q) \in \mathbb{R}$ independent of ψ.

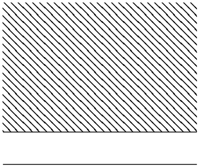
Here, "boundary" $=$ diagonal;
boundary config: where $\boldsymbol{x}_i = \boldsymbol{y}_j$;
interior config q': one y-particle removed

Ex: $q = (\boldsymbol{x}, \boldsymbol{x})$, $q' = \boldsymbol{x}$



Toy example of a configuration space with boundary

- Consider configuration space $\mathcal{Q} = \mathcal{Q}^{(1)} \cup \mathcal{Q}^{(2)}$ with $\mathcal{Q}^{(1)} = \mathbb{R}$ and $\mathcal{Q}^{(2)} = \big\{(x,y) \in \mathbb{R}^2 : y > 0\big\}$.
- Correspondingly, $\mathscr{H} = \mathscr{H}^{(1)} \oplus \mathscr{H}^{(2)}$ with $\mathscr{H}^{(n)} = L^2(\mathcal{Q}^{(n)})$.



- boundary $\partial\mathcal{Q} = \big\{(x,y) \in \mathbb{R}^2 : y = 0\big\}$
- IBC: For every $x \in \mathbb{R}$, $\quad \psi^{(2)}(x,0) = \frac{2mg}{\hbar^2}\psi^{(1)}(x) \qquad (1)$
- Hamiltonian:

$$(H\psi)^{(1)}(x) = -\tfrac{\hbar^2}{2}\partial_x^2\psi^{(1)}(x) - g\,\partial_y\psi^{(2)}(x,0)$$

$$(H\psi)^{(2)}(x,y) = -\tfrac{\hbar^2}{2}\Big(\partial_x^2 + \partial_y^2\Big)\psi^{(2)}(x,y) \quad \text{for } y > 0\,.$$

Theorem (TT 2015 arXiv:1506.00497)

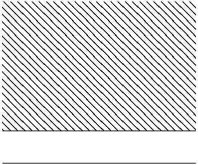
On a domain $\mathscr{D} \subseteq \mathscr{H}$ of functions satisfying the IBC (1), H is rigorously defined and self-adjoint.

Probability balance

IBC: $\psi^{(2)}(x,0) = \frac{2mg}{\hbar^2}\psi^{(1)}(x)$ (1)

$$(H\psi)^{(1)}(x) = -\frac{\hbar^2}{2}\partial_x^2\psi^{(1)}(x) - g\,\partial_y\psi^{(2)}(x,0)$$

$$(H\psi)^{(2)}(x,y) = -\frac{\hbar^2}{2}\left(\partial_x^2 + \partial_y^2\right)\psi^{(2)}(x,y)$$



- Loss of probability in $\mathcal{Q}^{(2)}$ at $(x,0)$: $-j_y^{(2)}(x,0)$
- is compensated by gain in $\mathcal{Q}^{(1)}$:

$$\frac{\partial|\psi^{(1)}|^2}{\partial t}(x) = -\partial_x j^{(1)} - \frac{2}{\hbar}\,\mathrm{Im}\big[\psi^{(1)}(x)^*\,g\,\partial_y\psi^{(2)}(x,0)\big]$$

$$\overset{(1)}{=} -\partial_x j^{(1)} - j_y^{(2)}(x,0)\,.$$

Bohmian trajectories

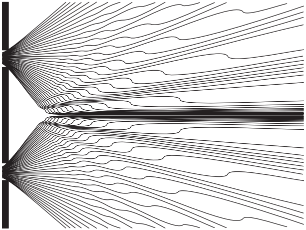
- Bohm's equation of motion:

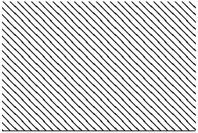
$$\frac{dQ}{dt} = \frac{j^\psi(Q)}{\rho^\psi(Q)} = \frac{\hbar}{m}\mathrm{Im}\frac{\psi^*\nabla\psi}{\psi^*\psi}$$

- in every sector $\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}$.
- When the trajectory hits the boundary at $(x,0)$, it jumps to x.
- When $Q(t) = x \in \mathcal{Q}^{(1)}$, it jumps to $(x,0) \in \mathcal{Q}^{(2)}$ with rate

$$\sigma_t(x \to (x,0)) = \frac{j_y^{(2)}(x,0)^+}{\rho^{(1)}(x)}$$

$$= \frac{\hbar}{m}\frac{\mathrm{Im}^+\left[\psi^{(2)}(x,0)^*\,\partial_y\psi^{(2)}(x,0)\right]}{|\psi^{(1)}(x)|^2}$$





- Markov process
- time reversal invariant
- $Q_t \sim |\psi_t|^2$ at all t

Simplified QFT model ("Lee model")

- There is only 1 x-particle, and it is fixed at the origin. $\mathscr{H} = \mathscr{F}_y^+$
- configuration space $\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n}$
- boundary configurations: any particle at the origin ($r = 0$ in spherical coordinates)
- IBC $\lim_{r \searrow 0} r\psi(y^n, r\boldsymbol{\omega}) = \frac{g\, m_y}{2\pi\hbar^2\sqrt{n+1}}\, \psi(y^n)$ for all $\boldsymbol{\omega} \in \mathbb{S}^2$ (2)
- $$H_{IBC}\psi = H_y^{\mathrm{free}}\psi + \frac{g\sqrt{n+1}}{4\pi} \int_{\mathbb{S}^2} d^2\boldsymbol{\omega} \lim_{r \searrow 0} \frac{\partial}{\partial r}\Big[r\psi(y^n, r\boldsymbol{\omega})\Big] + \frac{g}{\sqrt{n}} \sum_{j=1}^{n} \delta^3(\boldsymbol{y}_j)\, \psi(y^n \setminus \boldsymbol{y}_j) \qquad (3)$$

Theorem (LSTT 2017 arXiv:1703.04476)

On a suitable dense domain $\mathscr{D}_{IBC}$ of ψs in $\mathscr{F}_y^+$ satisfying the IBC (2), H_{IBC} is well defined, self-adjoint, and positive.

Comparison $H_{\mathrm{orig}}, H_{IBC}$

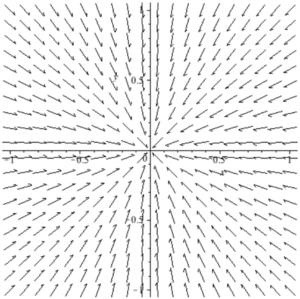
$$H_{\mathrm{orig}}\psi = H_{\mathrm{free}}\psi + g\sqrt{n+1}\,\psi(y^n,\mathbf{0}) + \\ + \frac{g}{\sqrt{n}}\sum_{j=1}^{n}\delta^3(\boldsymbol{y}_j)\,\psi(y^n\setminus\boldsymbol{y}_j)\,,$$

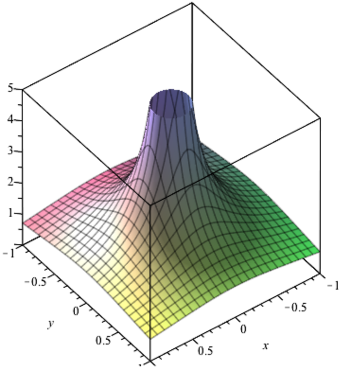
$$H_{IBC}\psi = H_{\mathrm{free}}\psi + \\ + \frac{g\sqrt{n+1}}{4\pi}\int_{\mathbb{S}^2} d^2\boldsymbol{\omega}\;\underbrace{\lim_{r\searrow 0}\frac{\partial}{\partial r}\Big[r\psi(y^n,r\boldsymbol{\omega})\Big]}_{b} \\ + \frac{g}{\sqrt{n}}\sum_{j=1}^{n}\delta^3(\boldsymbol{y}_j)\,\psi(y^n\setminus\boldsymbol{y}_j)$$

$\psi^{(1)}(r\boldsymbol{\omega}) = a\,r^{-1} + b + o(1)$; IBC: $a = \alpha_n\,\psi^{(0)}$
In $\mathbb{R}^d$ instead of $\mathbb{R}^3$, replace $r^{-1} \to r^{-(d-1)/2}$ for $d \geq 3$, $\log r$ for $d = 2$, r^0 for $d = 1$.

Why it works: flux of probability into a point

- probability current $\boldsymbol{j}_{\boldsymbol{y}_j}(y^n) = \dfrac{\hbar}{m_y}\mathrm{Im}\,\psi^*\nabla_{\boldsymbol{y}_j}\psi$
- $\dfrac{\partial\big|\psi(y^n)\big|^2}{\partial t} = -\displaystyle\sum_{j=1}^{n}\nabla_{\boldsymbol{y}_j}\cdot\boldsymbol{j}_{\boldsymbol{y}_j} + (n+1)\underbrace{\lim_{r\searrow 0} r^2\int_{\mathbb{S}^2} d^2\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\boldsymbol{j}_{\boldsymbol{y}_{n+1}}(y^n, r\boldsymbol{\omega})}_{\text{flux into } \mathbf{0} \text{ on } (n+1)\text{-sector}}$
- motion towards $\mathbf{0}$ $\Rightarrow$ $\rho \sim 1/r^2$ as $r \to 0$

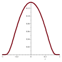


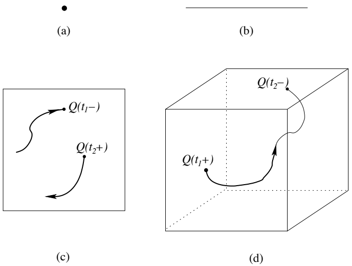


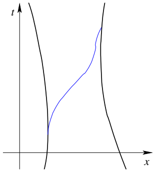
Bohmian picture

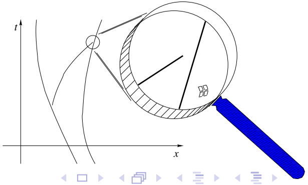
- $t \mapsto Q(t) \in \mathcal{Q}$ piecewise continuous, jumps between $\mathcal{Q}^{(n)}$ and $\mathcal{Q}^{(n+1)}$
- within $\mathcal{Q}^{(n)}$, Bohm's law of motion

$$\frac{dQ}{dt} = \frac{\hbar}{m} \mathrm{Im} \frac{\nabla \psi^{(n)}}{\psi^{(n)}} (Q(t))$$

- with IBC:
- when $Q(t) \in \mathcal{Q}^{(n)}$ reaches $\boldsymbol{y}_j = \boldsymbol{0}$, it jumps to $(y^n \setminus \boldsymbol{y}_j) \in \mathcal{Q}^{(n-1)}$
- emission of new y-particle at $\boldsymbol{0}$ at random time with random direction
- with UV cut-off:
- emission and absorption occurs anywhere in a ball around $\boldsymbol{0}$ (= in the support of φ )







Trajectories starting at $r = 0$

- It turns out that for functions in the domain of H_{IBC}, $\psi^{(1)}(r\omega) = a\, r^{-1} + b + o(1)$ with a and b independent of ω.
- As a consequence, the last velocity of the y-particle before absorption at $\mathbf{0}$ is purely radial.
- Hence, the y-trajectory reaches $\mathbf{0}$ in a definite direction and has an endpoint in spherical coordinates (r, ω) at $(0, \omega_{\text{final}})$.
- Different trajectories in $\mathcal{Q}^{(1)}$ reaching $\mathbf{0}$ at the same time have different ω_{final}.
- Time reverse: for every t and ω_{initial} there is a unique trajectory beginning at $\mathbf{0}$ at time t in the direction ω_{initial} whenever $\lim_{r \to 0} j_r^{(1)}(r\omega_{initial}, t) > 0$.
- Thus, whenever Q_t jumps from $\mathcal{Q}^{(0)}$ to $\mathbf{0} \in \mathcal{Q}^{(1)}$, there starts a trajectory at $\mathbf{0}$ for each direction.
- The process chosses the direction randomly.
- This is key for showing that the IBC process Q_t is well defined.

Comparison to renormalization procedure

- Consider $H_{\text{cutoff}} = H_\varphi$ with $\varphi =$ , limit $\varphi \to \delta^3$.
- Then there exist constants $E_\varphi \to \infty$ and a self-adjoint operator H_∞ such that

$$H_\varphi - E_\varphi \to H_\infty\,.$$

[van Hove 1952, Nelson 1964, see also Dereziński 2003]

Theorem (LSTT 2017 arXiv:1703.04476)

$H_\infty = H_{IBC} + const.$

Previously, it was unknown what the domain of H_∞ looks like and how H_∞ acts on it.

Shown non-rigorously:

The Bohmian process Q_t^φ for H_φ converges in distribution, as $\varphi \to \delta^3$, to the IBC process Q_t^{IBC}.

Moving x-particles

Hilbert space $\mathscr{H} = \mathscr{F}_x^- \otimes \mathscr{F}_y^+$

$$H_{\mathrm{orig}} = H_x^{\mathrm{free}} + H_y^{\mathrm{free}} + g \int_{\mathbb{R}^3} d^3\boldsymbol{x}\; a_x^\dagger(\boldsymbol{x}) \Big(a_y(\boldsymbol{x}) + a_y^\dagger(\boldsymbol{x}) \Big) a_x(\boldsymbol{x})$$

There is a natural formulation of the IBC and H_{IBC} in this setting.

Theorem (Lampart and Schmidt 2017 forthcoming)

Let the space dimension be $d \in 1, 2, 3$. On a suitable dense domain $\mathscr{D}_{IBC}$ of ψs in $\mathscr{H}$ satisfying the IBC, H_{IBC} is well defined, self-adjoint, and positive.

With moving x-particles in 3d, the growth of ψ near the boundary is no longer $a\, r^{-1} + b + o(1)$ but instead

$$\psi^{(1)}(r\omega) = a\, r^{-1} + b \log r + c + o(1)$$

where $b = \tilde{\alpha}\, a$ with certain constant $\tilde{\alpha} \in \mathbb{R}$ independent of ψ.

Symmetries

H_{IBC} and the IBC process Q_t respect the symmetries of H_{orig} including (if appropriate)

- Galilean boosts, rotations, space and time translations, time reversal
- gauge invariance

Thank you for your attention