Bohmian mechanics: its meaning and its consequences

Roderich Tumulka

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Summer School Fundamental Problems in Physics
Trieste, 19 June 2025

Further reading:
R. Tumulka, *Foundations of Quantum Mechanics* (Springer 2022)

Definition of Bohmian mechanics

Bohmian mechanics

"[Bohmian mechanics] exercises the mind in a very salutary way." J.S. Bell (1984)

The universe consists of

- a 3d Euclidean space $\mathbb{R}^3$,
- N material points ("particles") moving in $\mathbb{R}^3$ with time $t \in \mathbb{R}$,
- such that the position $\boldsymbol{Q}_k(t) \in \mathbb{R}^3$ of particle $k \in \{1...N\}$ at time t changes according to the following equation of motion:

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k} \frac{\mathrm{Im}[\Psi^\dagger \nabla_{\boldsymbol{q}_k} \Psi]}{\Psi^\dagger \Psi}\bigg|_{\boldsymbol{q}_j = \boldsymbol{Q}_j(t) \forall j}, \qquad (1)$$

where the "wave function of the universe" $\Psi : (\mathbb{R}^3)^N \times \mathbb{R} \to \mathbb{C}^\ell$ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t}(\boldsymbol{q}_1 ... \boldsymbol{q}_N, t) = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla^2_{\boldsymbol{q}_k} \Psi + V(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)\, \Psi \,.$$

Bohm's equation of motion (1)

has the form

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\text{current}}{\text{density}} = \frac{\boldsymbol{j}_k(\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_N)}{\rho(\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_N)}$$

with prob. current $\boldsymbol{j}_k$ and prob. density $\rho = |\Psi|^2$. We write $Q(t) := (\boldsymbol{Q}_1(t), \ldots, \boldsymbol{Q}_N(t)) =:$ configuration at time t

Equivalance theorem

If $Q(t_0) \sim |\psi_{t_0}|^2$ for one t_0, then $Q(t) \sim |\psi_t|^2$ for all t.

Here, $X \sim \mu$ means "the variable X is random with distribution μ."

Historical curiosity

Bohm (1952) wrote the eq. of motion (1) as a 2nd-order eq. for $d^2\boldsymbol{Q}_k/dt^2$ (by taking d/dt of (1)) and demanded (1) as a constraint condition on the velocity—a convoluted way of defining the same trajectories.

One more axiom of Bohmian mechanics

Axiom

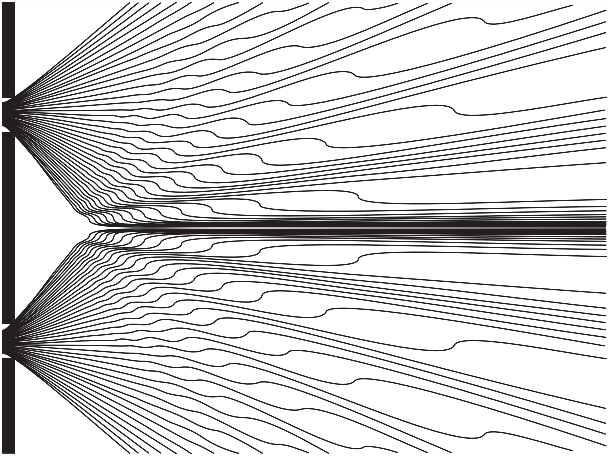
At the initial time $t = 0$ of the universe, $Q(0) \sim |\psi_0|^2$.

In particular, assume $\psi_0 \in L^2(\mathbb{R}^{3N}, \mathbb{C})$ with $\|\psi_0\|^2 = 1$.

Weaker version of this axiom suffices:

$Q(0)$ is typical relative to $|\psi_0|^2$; that is, it looks as if it were random with distribution $|\psi_0|^2$.

Example: the double-slit experiment



Drawn by G. Bauer after Philippidis et al.

Shown: A double-slit and 80 possible paths of Bohm's particle. The wave passes through both slits, the particle through only one.

Bohmian mechanics takes wave–particle dualism literally: there is a wave, and there is a particle. The path of the particle depends on the wave.

No-signaling theorem in BM (a variation of [Ghirardi 1980])

Informally

It is not possible to send messages to spacelike separated regions.

More precisely

Suppose system x (think of particles in Alice's lab) is entangled with system y (particles in Bob's lab), but there is no interaction term in the Hamiltonian. Suppose system z (Bob's message) is initially disentangled from xy,

$$\Psi_{t_0}(x, y, z) = \phi(x, y)\chi(z)\,,$$

$$H = H_x \otimes I_y \otimes I_z + I_x \otimes H_{yz}\,.$$

Then for any $t \geq t_0$, the marginal distribution of $X(t)$ is independent of χ and independent of H_{yz}, including external fields.

Key to understanding: In a hypothetical universe governed by BM, also a measurement apparatus/instrument needle/observer consists of Bohmian particles. Any macroscopic record can be encoded in particle positions (think of the configuration of ink on paper).

Collapse of the wave function in Bohmian mechanics

The wave function Ψ of the universe does not collapse (but evolves according to the Schrödinger equation).
The wave function ψ of a system is the *conditional wave function*

$$\psi(x) = \mathcal{N}\ \Psi(x, Y)$$

with $\mathcal{N}$ = normalizing constant, x = configuration variable of the system, Y = actual (Bohmian configuration) of the environment.

If x-system and y-system are disentangled, $\Psi(x, y) = \phi(x)\chi(y)$, and don't interact, then the conditional wave function ψ $(= \phi)$ obeys its own Schrödinger eq., but in general it doesn't.

In BM, ψ collapses.

The measurement problem

Measurement process more generally

Consider an ideal quantum measurement of the observable $A = \sum_\alpha \alpha P_\alpha$ with eigenvalues α and P_α the projection to the corresponding eigenspace. It begins at t_0 and ends at t_1. At t_0, the wave fct of object and apparatus is

$$\Psi(t_0) = \psi(t_0) \otimes \phi$$

with $\psi(t_0) =$ wave fct of the object, $\phi =$ ready state of the apparatus. By the Schrödinger eq., Ψ evolves to

$$\Psi(t_1) = e^{-iH(t_1-t_0)}\Psi(t_0)\,.$$

Measurement process, continued

We have that $\Psi(t_0) = \psi(t_0) \otimes \phi$ and $\Psi(t_1) = e^{-iH(t_1 - t_0)}\Psi(t_0)$.

Suppose first that the object is in an eigenstate ψ_α of A. Then

$$\Psi_\alpha := \Psi(t_1) = e^{-iH(t_1 - t_0)}[\psi_\alpha \otimes \phi]$$

should be a state in which the apparatus displays the value α (e.g., by the position of a needle).

Suppose next that $\psi(t_0) = \sum_\alpha c_\alpha \psi_\alpha$ is an arbitrary superposition. Then

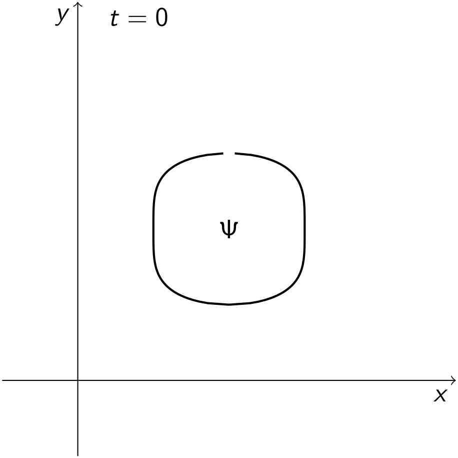
$$\Psi(t_0) = \sum_\alpha c_\alpha \left[\psi_\alpha \otimes \phi\right]$$

and, by linearity of the Schrödinger eq.,

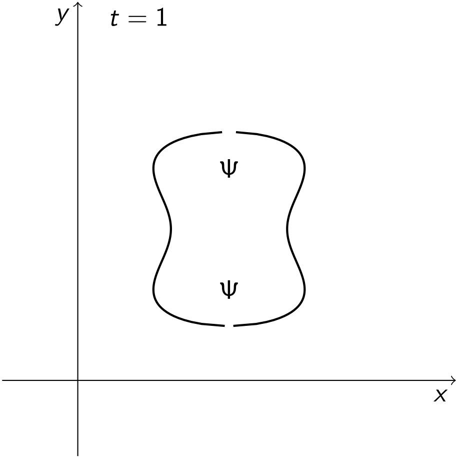
$$\Psi(t_1) = \sum_\alpha c_\alpha \Psi_\alpha \,,$$

i.e., a superposition of wave functions of apparatuses displaying different outcomes.

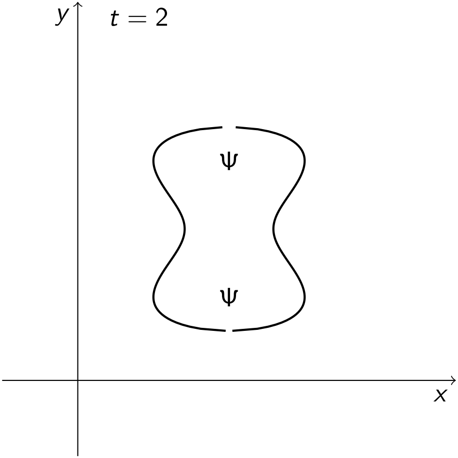
Evolution of Ψ in configuration space of system x + apparatus y:



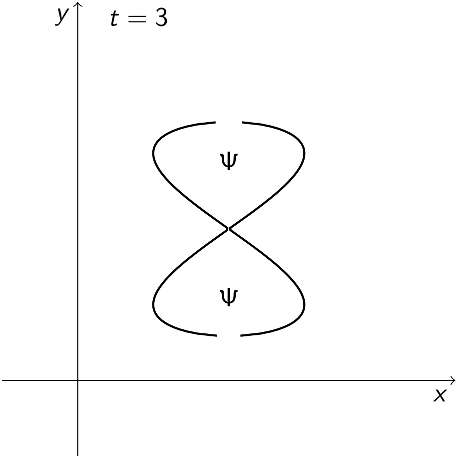
Evolution of Ψ in configuration space of system x + apparatus y:



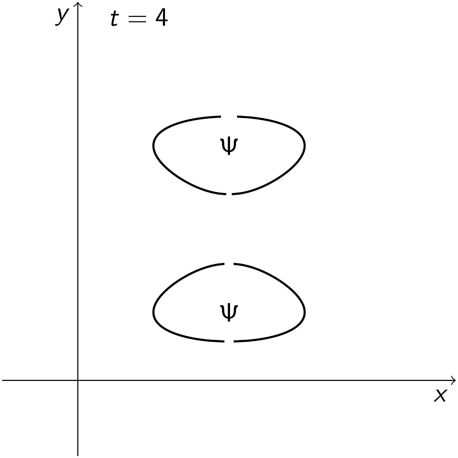
Evolution of Ψ in configuration space of system x + apparatus y:



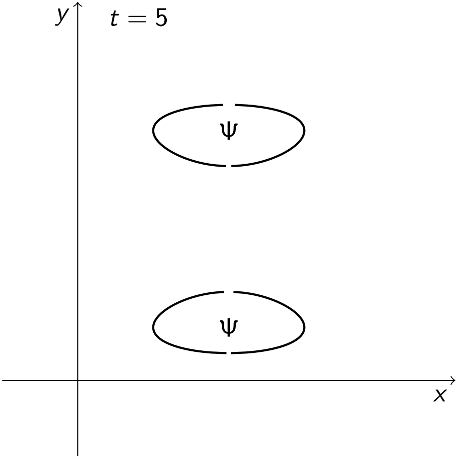
Evolution of Ψ in configuration space of system x + apparatus y:



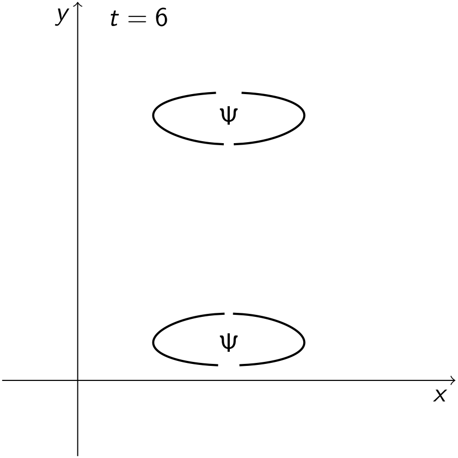
Evolution of Ψ in configuration space of system x + apparatus y:



Evolution of Ψ in configuration space of system x + apparatus y:



Evolution of Ψ in configuration space of system x + apparatus y:



Measurement outcomes in BM

- Y provides the actual position of the needle, and thus the actual outcome $Z = f(Y)$.
- $\mathbb{P}(Z = \alpha) = \|\Psi_\alpha\|^2 = |c_\alpha|^2$, in agreement with the rules of QM.
- If $\Psi_\alpha = \psi_\alpha \otimes \phi_\alpha$ for all α (i.e., if the measurement process doesn't change the state of the object), then the cond. wf is $\psi = \psi_\alpha|_{\alpha=Z}$ (collapse to eigenfunction), in agreement with the rules of QM.
- Moreover, by decoherence (meaning here that the two packets will not overlap for 10^{100} years), also in Ψ the lower packet can henceforth be ignored.

As a consequence

Observers inhabiting a Bohmian universe (made out of Bohmian particles) observe random-looking outcomes of their experiments whose statistics agree with the rules of quantum mechanics for making predictions.

Why OQM has a measurement problem

- The apparatus consists of electrons and quarks, so it should be possible to treat it like a quantum system with a wave fct ϕ on $\mathbb{R}^{3N}$, $N > 10^{23}$.
- If we do, then $\Psi(t_0) = \psi \otimes \phi$ evolves according to the Schrödinger eq. to $\Psi(t_1) = \sum_\alpha c_\alpha \Psi_\alpha$, where Ψ_α corresponds to a needle pointing to α. A superposition of different outcomes.
- $\Psi(t_1)$ doesn't say what the actual outcome is.
- We might have expected a state $\Psi(t_1)$ with a unique needle position.
- We might have expected a random state because the outcome should be random.

Let's pin down the problem

3 assumptions

1. In each run of the experiment, there is a unique outcome.
2. The wave function is a complete description of a system's physical state in reality. (There are no further variables.)
3. The time evolution of the wave function of an isolated system, not entangled with the outside, is always given by the Schrödinger eq.

Together, they lead to a contradiction: By 3, $\Psi(t_1)$ is generically a superposition of Ψ_α corresponding to different outcomes. Thus, $\Psi(t_1)$ doesn't select an outcome. If there were further variables (such as Bohm's Q), they could select an outcome, but by 2 there aren't. Thus, there is no unique outcome, in contradiction to 1.

Consequence

We need to drop one of the 3 assumptions.

Bohmian mechanics drops 2, collapse theories 3, many-worlds 1.

The question that the discussion circles about

What is actually there in reality?

POVMs

Positive-operator-valued measure (POVM)

An operator R is *positive* $\Leftrightarrow \langle\psi|R|\psi\rangle \geq 0$ for all ψ.
Equivalently, $R = R^\dagger$ and spectrum$(R) \subseteq [0,\infty)$.

Definition

A POVM on a discrete set $\mathscr{Z}$ is a family $(E_z)_{z\in\mathscr{Z}}$ of positive operators such that $\sum_{z\in\mathscr{Z}} E_z = I$.
(A POVM in the continuum associates with subsets $S \subseteq \mathscr{Z}$ a positive operator $E(S)$ such that $E(\mathscr{Z}) = I$ and
$E(S_1 \cup S_2 \cup \ldots) = E(S_1) + E(S_2) + \ldots$ if $S_i \cap S_j = \emptyset$ for $i \neq j$.)

Main theorem about POVMs in BM: a generalization of Born's rule

For any experiment with outcome Z on a system with Hilbert space $\mathscr{H}$, there is a POVM E on the set $\mathscr{Z}$ of possible outcomes such that for every $\psi \in \mathscr{H}$ with $\|\psi\| = 1$,

$$\mathbb{P}_\psi(Z = z) = \langle\psi|E_z|\psi\rangle\,.$$

Example: For an ideal quantum measurement of the observable $A = \sum_\alpha \alpha P_\alpha$, $\mathscr{Z} =$ spectrum(A) and $E_z = P_\alpha$.

Proof of the main theorem about POVMs from BM

Let the experiment be over at time t_1, and read off the result from the apparatus display, $Z = f(Q(t_1))$. Let $P(\cdot)$ be the position POVM on $\mathbb{R}^{3N}$, i.e.,

$$P(S)\psi(q) = 1_S(q)\,\psi(q)$$

for any $S \subseteq \mathbb{R}^{3N}$. Let $U = \exp(-iH(t_1 - t_0)/\hbar)$. Then

$$\mathbb{P}(Z = z) = \int\limits_{f^{-1}(z)} dq\,|\Psi_{t_1}(q)|^2 = \langle\psi|E_z|\psi\rangle_{\mathscr{H}}$$

with

$$E_z = \langle\phi|U^\dagger P(f^{-1}(z))U|\phi\rangle_{\text{app}}\,.$$

$\square$

No-hidden-variable theorem

"Hidden variable" can mean

- Any further variable assumed to exist in addition to ψ (such as Q in BM, which however is not hidden at all!)
- The assumption that every observable has an actual value already before a quantum measurement. (Not the case in BM.)

Let us look at the latter view and suppose that with every self-adjoint operator A there is associated a physical quantity v_A, the actual value of the observable A, and that a quantum measurement of A simply reveals the value v_A. Can it be this way?

No-hidden-variable theorem

Suppose $3 \leq \dim \mathscr{H} < \infty$. Let $\mathscr{A}$ be the set of all self-adjoint operators on $\mathscr{H}$, fix $\psi \in \mathscr{H}$ with $\|\psi\| = 1$. The Born distribution for $A \in \mathscr{A}$ is

$$\mathrm{Prob}(\alpha) = \|P_\alpha \psi\|^2 = \langle \psi | P_\alpha \psi \rangle$$

for $A = \sum_\alpha \alpha P_\alpha$. For pairwise-commuting A, B, C with $A = \sum_{\alpha\beta\gamma} \alpha P_{\alpha\beta\gamma}$, $B = \sum_{\alpha\beta\gamma} \beta P_{\alpha\beta\gamma}$, $C = \sum_{\alpha\beta\gamma} \gamma P_{\alpha\beta\gamma}$, the joint Born distribution is

$$\mathrm{Prob}(\alpha, \beta, \gamma) = \|P_{\alpha\beta\gamma}\psi\|^2 . \tag{2}$$

NHV theorem
[Gleason 1957, Kochen and Specker 1967]

Consider a joint distribution of random variables v_A for all $A \in \mathscr{A}$. Suppose that a quantum measurement of any $A \in \mathscr{A}$ yields v_A. Suppose further that whenever $A, B \in \mathscr{A}$ commute, then a quantum measurement of A doesn't change the value of v_B (nor that of v_A). Then the joint distribution of v_A, v_B, v_{A+B} disagrees with the joint Born rule (2).

Upshot

It's not convincingly possible that there is an actual value v_A for every observable A.

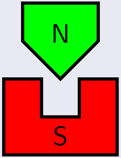
But BM is deterministic...

...so the outcome Z of an experiment is a function $F(X_0, Y_0, \psi, \phi)$ of the initial data at t_0, $\Psi(t_0) = \psi \otimes \phi$ and $Q(t_0) = (X_0, Y_0)$.

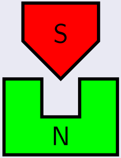
Why isn't Z a v_A?

Because it depends on Y_0 and ϕ, not just on A.

Example: Two experiments that are quantum measurements of σ_z



One is a Stern-Gerlach experiment in the z direction.



The other uses a magnet with inverted polarity and calls the outcome "down" if the particle is found in the upper packet.

On the same X_0 and ψ, the two experiments sometimes give different results. ("contextuality")

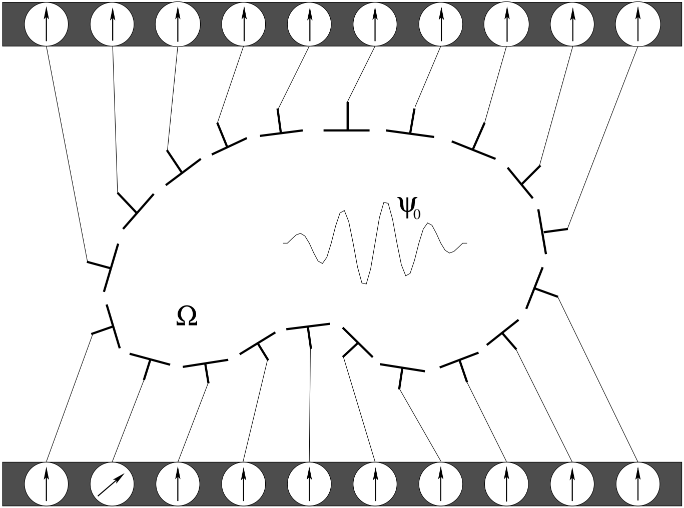
"The result of an experiment depends on the experiment."

[Dürr, Goldstein, Zanghì 2004]

...and not just on A. Different experiments belonging to the same observable may yield different results but the same probability distribution of results.

Time of detection

Problem of detection time and place



$T \in [0, \infty), \boldsymbol{X} \in \partial\Omega, Z = (T, \boldsymbol{X})$ redrawn after [Daumer et al. quant-ph/9512016]

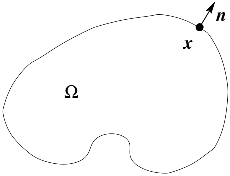
Problem of detection time

- in orthodox quantum mechanics (OQM), there is no time operator
- Pauli 1933: it's impossible to have an operator for detection time
- quantum Zeno effect [Turing 1950s]: seems impossible ("A watched pot never boils," Misra and Sudarshan 1977)
- Allcock's paradox [1969]
- several suggestions:
  - Aharonov and Bohm 1961: compute classical arrival time from $\boldsymbol{x}(0), \boldsymbol{p}(0)$, then quantize to obtain an operator
  - Kijowski [1974]
  - Maccone: $t \mapsto |\psi(\boldsymbol{x}, t)|^2$ for fixed $\boldsymbol{x} \in \partial\Omega$
  - "absorbing boundary rule" [Werner 1987, Tumulka 2016]
  - Das and Dürr 2018: equate with Bohmian arrival time

Absorbing boundary rule

- Solve the 1-particle Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$ with "absorbing boundary condition" (ABC)

$$\boldsymbol{n}(\boldsymbol{x})\cdot\nabla\psi(\boldsymbol{x}) = i\kappa\,\psi(\boldsymbol{x})$$



at every $\boldsymbol{x}\in\partial\Omega$, where $\boldsymbol{n}(\boldsymbol{x})=$ outward unit normal vector to $\partial\Omega$ at $\boldsymbol{x}$, and $\kappa>0$ a constant.

- ABC implies that the probability current $\boldsymbol{j}^\psi = \frac{\hbar}{m}\mathrm{Im}[\psi^*\nabla\psi]$ points outward at $\partial\Omega$:

$$\boldsymbol{n}\cdot\boldsymbol{j} = \frac{\hbar}{m}\mathrm{Im}[\psi^*\boldsymbol{n}\cdot\nabla\psi] = \frac{\hbar}{m}\mathrm{Im}[\psi^* i\kappa\psi] = \frac{\hbar}{m}\kappa|\psi|^2 \geq 0\,.$$

- $\mathbb{P}_{\psi_0}\Big(T\in dt, \boldsymbol{X}\in d^2\boldsymbol{x}\Big) = \boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{j}^{\psi_t}(\boldsymbol{x})\,dt\,d^2\boldsymbol{x}$ assuming $\|\psi_0\|=1$.
- If the experiments get interrupted at time t before detection, the collapsed wave function is $\psi_t/\|\psi_t\|$.

Properties

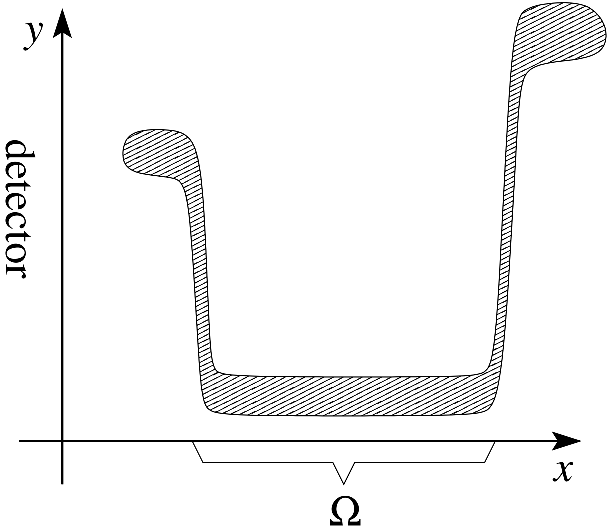
- $\|\psi_t\|^2 = \mathbb{P}_{\psi_0}(T > t)$ "survival probability," decreasing in t
- The time evolution of ψ, $W_t = \exp(-iHt/\hbar)$, is not unitary (Hamiltonian not self-adjoint) due to loss at $\partial\Omega$
- distribution is given by a POVM
- $E_\kappa\big(dt \times d^2\boldsymbol{x}\big) = \frac{\hbar\kappa}{m} W_t^\dagger \, |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \, W_t \, dt \, d^2\boldsymbol{x}$,
  $E_\kappa(T = \infty) = \lim_{t\to\infty} W_t^\dagger W_t$
- In Bohmian mechanics, the particle with $|\psi_0|^2$-distributed initial condition $\boldsymbol{X}(0)$ moves according to the equation of motion

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{j}^{\psi_t}(\boldsymbol{X}(t))}{|\psi_t(\boldsymbol{X}(t))|^2}$$

  until it hits $\partial\Omega$ at time T and place $\boldsymbol{X} = \boldsymbol{X}(T)$, and gets absorbed. $\mathbb{P}_{\psi_0}\big(\boldsymbol{X}(t) \in d^3\boldsymbol{x}\big) = |\psi_t(\boldsymbol{x})|^2 \, d^3\boldsymbol{x}$.
- energy-time uncertainty relation $\Delta E \, \Delta T \geq \hbar/2$ with E referring to $-\frac{\hbar^2}{2m}\nabla^2$ on $L^2(\mathbb{R}^3)$

Heuristic derivation

configuration space:



Current research:

- Derive ABR from a microscopic quantum-mechanical model of a detector, given as a system of $N \gtrsim 10^{23}$ particles
  [joint work with R. Kaimal]
- No signaling theorem: assuming the existence of an apparatus represented by the ABR, it is not possible to send faster-than-light signals [joint work with C. Peters and S. Tahvildar-Zadeh],
  more precisely [contra claims of W. Cavendish]:

No-signaling theorem for ABR (work in progress)

Consider N non-interacting Dirac particles, a (timelike) surface $S \subset \mathbb{R}^4$ of ABR-detectors, and a (spacelike) Cauchy surface Σ. Suppose Alice can see the detection events on $S \cap \text{past}(\Sigma)$ and Bob can influence the shape/location of Σ and detector parameters (such as $\kappa(x)$) in future(Σ). Then the distribution of Alice's observations is independent of Bob's choices.

Time of arrival

In BM in the absence of detectors, there is a fact about when and where the particle's trajectory first intersects $\partial\Omega$: the arrival time T_{WOD} and arrival place $\boldsymbol{X}_{WOD}$ (WOD = without detectors).

- Distribution

$$\mathbb{P}(\boldsymbol{X}_{WOD} \in d^2\boldsymbol{x}, T_{WOD} \in dt) = \begin{cases} \boldsymbol{j}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) \text{ at the 1st crossing} \\ 0 \text{ at 2nd or later crossing} \end{cases}$$

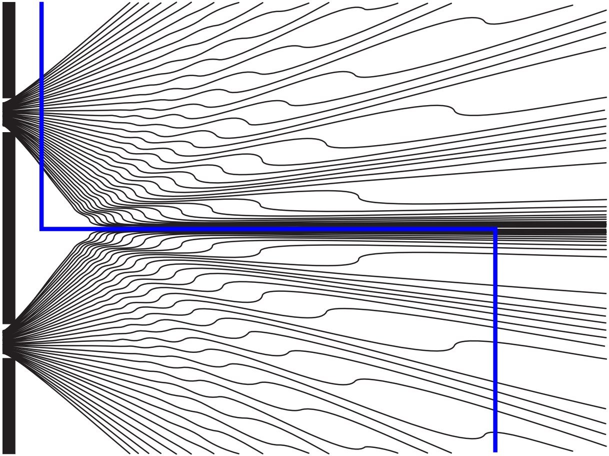
- Das and Dürr [1802.07141] hypothesized that

$$\mathbb{P}(\boldsymbol{X}_D \in d^2\boldsymbol{x}, T_D \in dt) = \mathbb{P}(X_{WOD} \in d^2\boldsymbol{x}, T_{WOD} \in dt)\,,$$

  in short $\mathbb{P}_D = \mathbb{P}_{WOD}$. I disagree.

- If the hypothesis were true, that would be nice for Bohmians: It would allow BM to make a testable prediction that OQM can't make. If confirmed experimentally, maybe all physicists would become Bohmians.

Example illustrating that trajectories in the presence of detectors will be different from those in their absence:



Thus, $\boldsymbol{X}_{WID} \neq \boldsymbol{X}_{WOD}$ (WID = with detector) and in general $T_{WID} \neq T_{WOD}$. What can you expect of $\boldsymbol{X}_D$ then?

$\mathbb{P}_D$ vs $\mathbb{P}_{WID}$ vs $\mathbb{P}_{WOD}$

- In the far-field regime (scattering regime) $t \to \infty, |\boldsymbol{x}| \to \infty$, $\mathbb{P}_D \to \mathbb{P}_{WOD}$.

  [Conjectured by Daumer et al. quant-ph/9512016, supported by current research with R. Kaimal, C. Beck, and D. Lazarovici.]
- Das and Dürr computed $\mathbb{P}_{WOD}$ for a setup with a spin-$\frac{1}{2}$ particle in $\Omega = \mathbb{R}^2 \times [0, L]$ and $\psi_0(\boldsymbol{x}) = \varphi(\boldsymbol{x}) \otimes |\boldsymbol{n}\rangle$ with $|\boldsymbol{n}\rangle \in \mathbb{C}^2$ and found striking dependence of $\mathbb{P}_{WOD}$ on $|\boldsymbol{n}\rangle$.

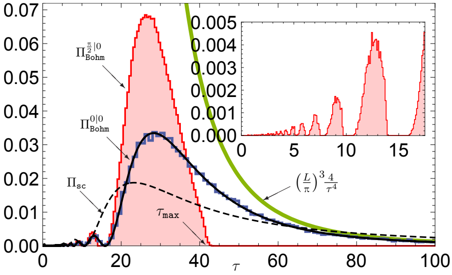
Theorem [Goldstein, Tumulka, Zanghì 2309.11835, 2405.04607]

In the example of Das and Dürr, $\mathbb{P}_{WOD}$ is not given by a POVM (and thus is $\neq \mathbb{P}_D$), not even approximately. (The spin dependence is crucial.)

- Detlef Dürr sadly passed away in 2021.
  Das, Maudlin and Cavendish insist that $\mathbb{P}_D = \mathbb{P}_{WOD}$.

$\mathbb{P}_D = \mathbb{P}_{WOD} \;\Rightarrow\;$ superluminal signaling

- Alice and Bob share 100 EPR pairs in $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = |\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle$.
- If Alice wants to send "1," she measures σ_z on each of her particles, so Bob's particles collapse to either $|\uparrow\rangle$ or $|\downarrow\rangle$.
- If Alice wants to send "0," she measures σ_x, so Bob's particles collapse to either $|\rightarrow\rangle$ or $|\leftarrow\rangle$.
- In a small, local Ω, Bob measures T_D on each of his particles. For $|\boldsymbol{n}\rangle = |\uparrow\rangle$ or $|\boldsymbol{n}\rangle = |\downarrow\rangle$, the statistics is the blue curve; for $|\boldsymbol{n}\rangle = |\rightarrow\rangle$ or $|\boldsymbol{n}\rangle = |\leftarrow\rangle$, the red curve. [from Das and Dürr 1802.07141]



- If a hypothesis H implies superluminal signaling, you should become skeptical, as no one has observed superluminal signaling yet.
- Moreover, in that case you know for sure that H is false in BM, as a no-signaling theorem holds in BM.

A moral

The words "measurement" and "observation" suggest that the apparatus plays a merely passive role. But this is often not the case, and the apparatus must be included in the consideration.

Thank you for your attention