Mini Course on Bohmian Mechanics
Lecture 1

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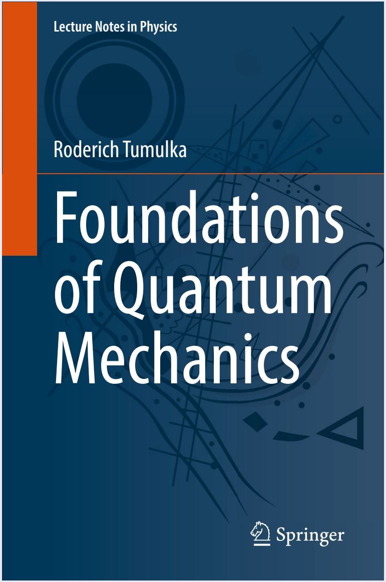
The slides of this talk will be available at my webpage
http://www.math.uni-tuebingen.de/de/forschung/maphy/personen/roderichtumulka/tumulka-talks

Emphasis on

what every researcher in the foundations of quantum mechanics needs to know about Bohmian mechanics

Further reading



(2022)

Definition of the theory

Defining equations

In the non-relativistic version, Bohmian mechanics asserts that electrons, quarks etc. have precise positions $\boldsymbol{Q}_k(t) \in \mathbb{R}^3$ at every time t, subject to the equation of motion:

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k} \,\mathrm{Im} \frac{\nabla_{\boldsymbol{q}_k} \psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, t)}{\psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, t)}\Big|_{\boldsymbol{q}_j = \boldsymbol{Q}_j(t) \forall j},$$

where the wave functoin $\psi : \mathbb{R}^{3N} \times \mathbb{R} \to \mathbb{C}$ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t}(\boldsymbol{q}_1 \ldots \boldsymbol{q}_N, t) = -\sum_{k=1}^{N} \tfrac{\hbar^2}{2m_k} \nabla^2_{\boldsymbol{q}_k} \psi + V(\boldsymbol{q}_1 \ldots \boldsymbol{q}_N) \psi \,.$$

Bohm's equation of motion

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k} \mathrm{Im} \frac{\nabla_{\boldsymbol{q}_k} \psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, t)}{\psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, t)} \Bigg|_{\boldsymbol{q}_j = \boldsymbol{Q}_j(t) \forall j} \qquad (1)$$

can be rewritten as

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\text{current}}{\text{density}} = \frac{\boldsymbol{j}_k(\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_N)}{\rho(\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_N)}$$

with prob. current $\boldsymbol{j}_k = \frac{\hbar}{m_k} \mathrm{Im}[\psi^* \nabla_k \psi]$ and prob. density $\rho = \psi^* \psi$.

Historical curiosity

Bohm (1952) wrote the eq. of motion (1) as a 2nd-order eq. for $d^2\boldsymbol{Q}_k/dt^2$ (by taking d/dt of (1)) and demanded (1) as a constraint condition on the velocity—a convoluted way of defining the same trajectories.

One more axiom of Bohmian mechanics

We write $Q(t) := (\boldsymbol{Q}_1(t), \ldots, \boldsymbol{Q}_N(t)) =:$ configuration at time t

Axiom

At the initial time $t = 0$ of the universe, $Q(0)$ is random with probability density $\rho(Q(0) = q) = |\psi(q, t = 0)|^2$. In short, $Q(0) \sim |\psi_0|^2$.

In particular, assume $\|\psi_0\|^2 = 1$.

Basic properties

Equivariance theorem

If $Q(t_0) \sim |\psi_{t_0}|^2$ for one t_0, then $Q(t) \sim |\psi_t|^2$ for all t.

Sketch of proof: Prob. ρ gets transported by motion with velocity $\boldsymbol{v}_k$ according to

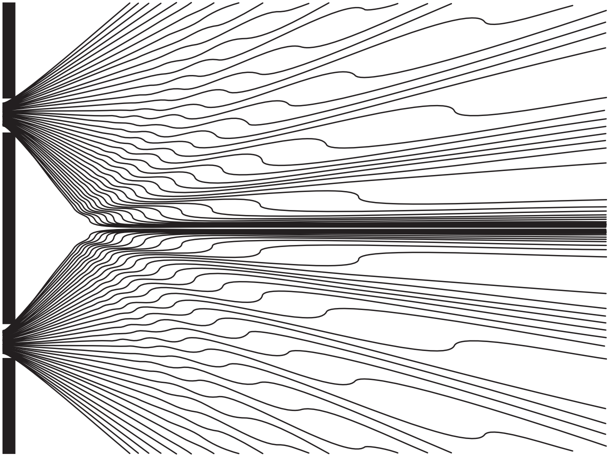
$$\frac{\partial \rho}{\partial t} = -\sum_{k=1}^{N} \nabla_k \cdot (\rho \boldsymbol{v}_k) \,.$$

The Schrödinger eq. implies that

$$\frac{\partial |\psi|^2}{\partial t} = -\sum_{k=1}^{N} \nabla_k \cdot \boldsymbol{j}_k \,.$$

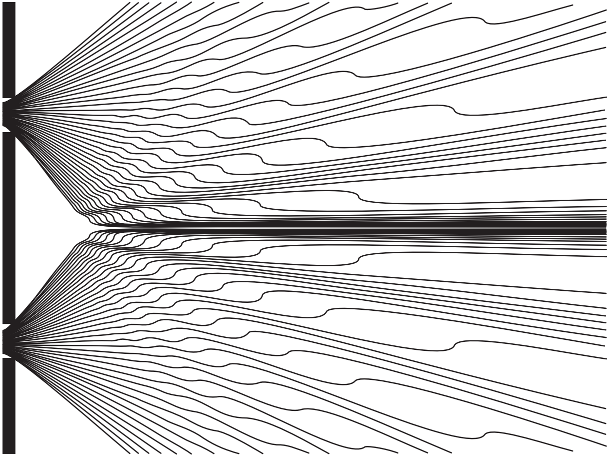
Since $\boldsymbol{v}_k = \boldsymbol{j}_k/|\psi|^2$, if $\rho = |\psi|^2$ then $\partial\rho/\partial t = \partial|\psi|^2/\partial t$.

Example: the double-slit experiment



Drawn by G. Bauer after Philippidis et al.

Shown: A double-slit and 80 possible paths of Bohm's particle. The wave passes through both slits, the particle through only one.



Most paths arrive where $|\psi|^2$ is large—that's how the interference pattern arises. If one slit gets closed, the wave passes through only one slit, which leads to different trajectories and less interference.
Bohmian mechanics takes wave–particle dualism literally: there is a wave, and there is a particle. The path of the particle depends on the wave.

Subsystems

Simple fact

A disentangled subsystem behaves as if alone in the universe. That is: If $q = (x, y)$ with $x \in \mathbb{R}^M$, $y \in \mathbb{R}^{3N-M}$ and $\psi(x, y) = \psi_1(x)\,\psi_2(y)$, then obviously

$$|\psi(x,y)|^2 = |\psi_1(x)|^2\,|\psi_2(y)|^2 \text{ and } v_x = \frac{\hbar}{m}\mathrm{Im}\frac{\nabla\psi_1}{\psi_1} \text{ indep. of } y.$$

Another simple fact

dQ/dt depends only on $\psi_t(Q(t))$ and $\nabla\psi_t(Q(t))$; distant wave packets have no influence on dQ/dt; this will play a role later.

Identical particles

It is sometimes claimed that in QM, particles can only be identical because they don't have trajectories. Not true:

Identical particles in BM

Identical particles work just as well. Just assume that, as usual, ψ is permutation symmetric for bosons and anti-symmetric for fermions.
In fact, BM provides *reasons* for why identical particles should have either symmetric or anti-symmetric wave function. (Ask me about it.)
[Dürr et al. quant-ph/0506173]

Galilean invariant

Collapse of the wave function

Collapse of the wave function in Bohmian mechanics

The wave function Ψ of the universe does not collapse (but evolves according to the Schrödinger equation).
The wave function ψ of a system is the *conditional wave function*

$$\psi(x) = \mathcal{N}\ \Psi(x, Y)$$

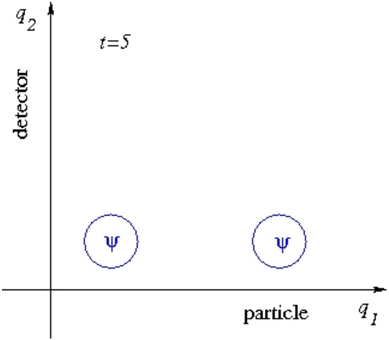
with $\mathcal{N}$ = normalizing constant, x = configuration variable of the system, Y = actual (Bohmian configuration) of the environment.

If x-system and y-system are disentangled, $\Psi(x, y) = \phi(x)\chi(y)$, and don't interact, then the conditional wave function ψ obeys its own Schrödinger eq., but in general it doesn't.

In BM, ψ collapses.

Here is why:

Evolution of Ψ in configuration space of particle + detector:



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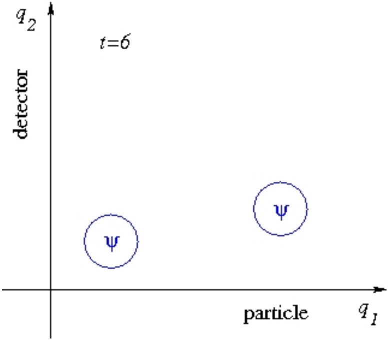

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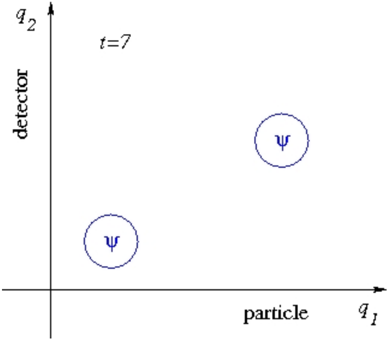

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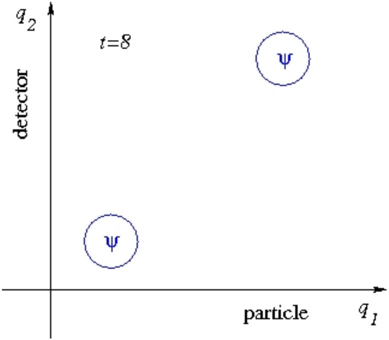


Collapse of ψ in Bohmian mechanics

- Since $Q = (X, Y) \sim |\Psi|^2$, Q lies in one of the packets; say, in the upper.
- Conditional on the configuration Y of the detector, $\psi(x)$ is a cross-section of the upper packet. That is, ψ has collapsed.
- Moreover, decoherence occurs: The two packets of Ψ do not overlap in configuration space and will not overlap any more in the future (for the next 10^{100} years). (As usual with macroscopically different packets.)
- As a consequence, $Q = (X, Y)$ will be guided only by the packet of Ψ containing Q (for the next 10^{100} years).
- Thus, ψ will follow the upper packet for the next 10^{100} years.

Measurement process more generally

Consider an ideal quantum measurement of the observable $A = \sum_\alpha \alpha P_\alpha$ with eigenvalues α and P_α the projection to the corresponding eigenspace. It begins at t_0 and ends at t_1. At t_0, the wave fct of object and apparatus is

$$\Psi(t_0) = \psi(t_0) \otimes \phi$$

with $\psi(t_0)$ = wave fct of the object, ϕ = ready state of the apparatus. (We are using that the apparatus is itself made out of quantum particles: "democracy of electrons.")
By the Schrödinger eq., Ψ evolves to

$$\Psi(t_1) = e^{-iH(t_1-t_0)}\Psi(t_0)\,.$$

Measurement process, continued

We have that $\Psi(t_0) = \psi(t_0) \otimes \phi$ and $\Psi(t_1) = e^{-iH(t_1-t_0)}\Psi(t_0)$.

Suppose first that the object is in an eigenstate ψ_α of A. Then

$$\Psi_\alpha := \Psi(t_1) = e^{-iH(t_1-t_0)}[\psi_\alpha \otimes \phi]$$

should be a state in which the apparatus displays the value α (e.g., by the position of a needle).

Suppose next that $\psi(t_0) = \sum_\alpha c_\alpha \psi_\alpha$ is an arbitrary superposition. Then

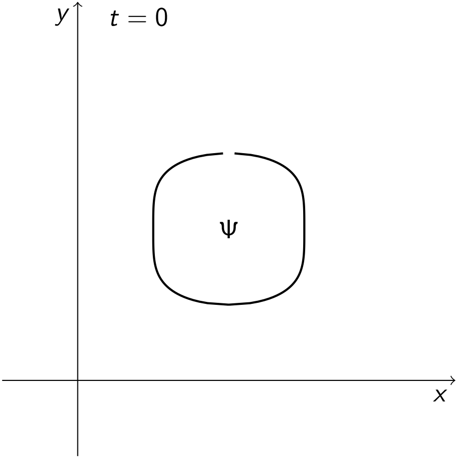
$$\Psi(t_0) = \sum_\alpha c_\alpha \left[\psi_\alpha \otimes \phi\right]$$

and, by linearity of the Schrödinger eq.,

$$\Psi(t_1) = \sum_\alpha c_\alpha \Psi_\alpha \,,$$

i.e., a superposition of wave functions of apparatuses displaying different outcomes.

Evolution of Ψ in configuration space of system x + apparatus y:



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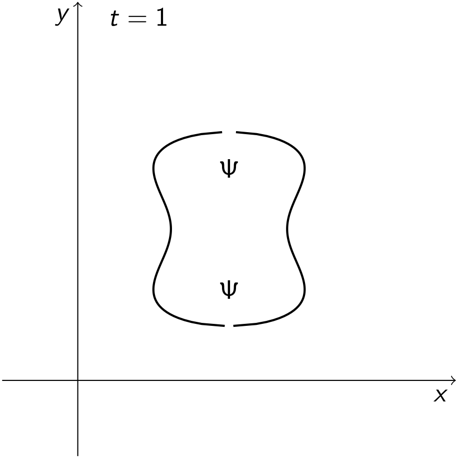

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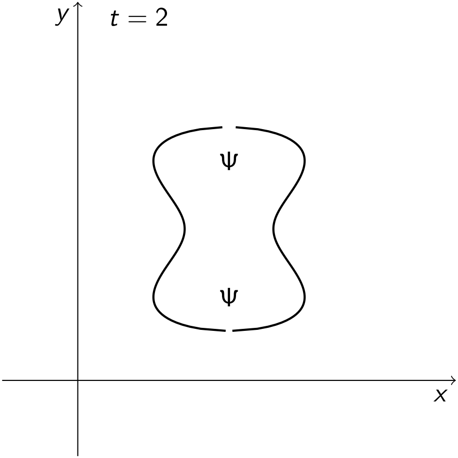

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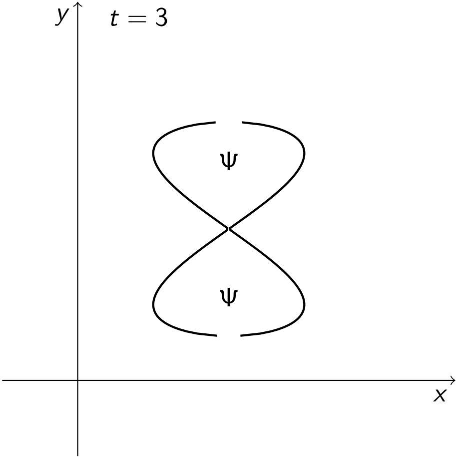

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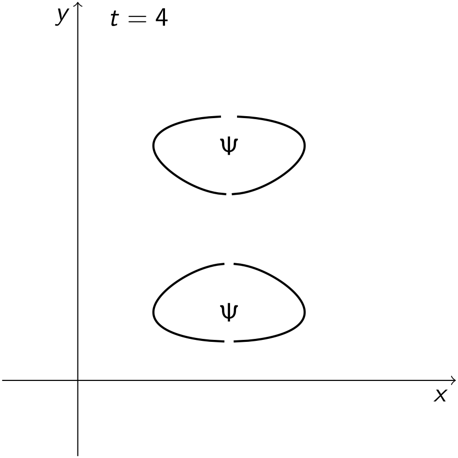

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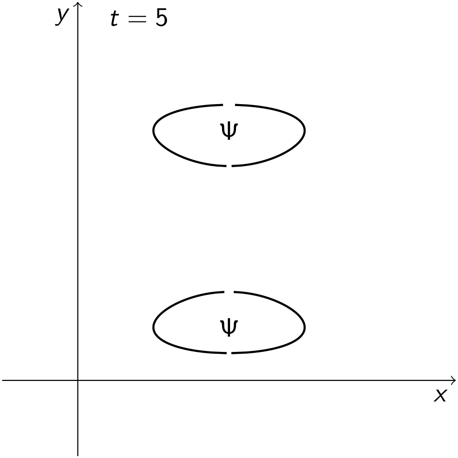

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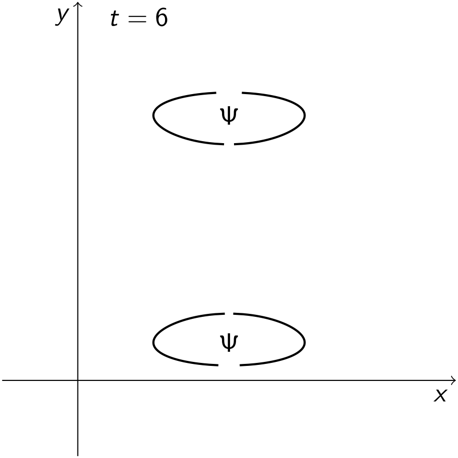


Measurement outcomes in BM

- Y provides the actual position of the needle, and thus the actual outcome $Z = f(Y)$.
- $\mathbb{P}(Z = \alpha) = \|\Psi_\alpha\|^2 = |c_\alpha|^2$, in agreement with the rules of QM.
- If $\Psi_\alpha = \psi_\alpha \otimes \phi_\alpha$ for all α (i.e., if the measurement process doesn't change the state of the object), then the cond. wf is $\psi = \psi_\alpha|_{\alpha=Z}$ (collapse to eigenfunction), in agreement with the rules of QM.
- Moreover, by decoherence, also in Ψ the lower packet can henceforth be ignored.

As a consequence

Observers inhabiting a Bohmian universe (made out of Bohmian particles) observe random-looking outcomes of their experiments whose statistics agree with the rules of quantum mechanics for making predictions.

In short, Bohmian mechanics is empirically adequate.

Can BM be empirically tested against OQM?

No. That follows from the kind of analysis of experiments just given.

More direct argument

- BM and OQM share the same Schrödinger eq.
- Consider this Schrödinger eq for ψ of object + apparatus.
- Consider late time t at which the experiment is over.
- $|\psi_t|^2$ determines prob.s of pointer positions in both BM and OQM.
- Thus, these probabilities are equal.

In OQM, one often does not use the method of solving the Schrödinger eq for object + apparatus (but instead quantization rules or analogies). But at the end of the day, most orthodox physicists would agree that a prediction obtained by that method is correct.

And yet, people proposed tests:

Das, Maudlin, Cavendish; Sharoglazova et al.; Drezet, Amblard; ...

Does that make BM irrelevant?

Feynman (1967): For those people who insist that the only thing that is important is that the theory agrees with experiment, I would like to imagine a discussion between a Mayan astronomer and his student. The Mayans were able to calculate with great precision predictions, for example, for eclipses and for the position of the moon in the sky, the position of Venus, etc. It was all done by arithmetic. They counted a certain number and subtracted some numbers, and so on. There was no discussion of what the moon was. There was no discussion even of the idea that it went around. They just calculated the time when there would be an eclipse, or when the moon would rise at the full, and so on. Suppose that a young man went to the astronomer and said, 'I have an idea. Maybe those things are going around, and they are balls of something like rocks out there, and we could calculate how they move in a completely different way from just calculating what time they appear in the sky.' 'Yes,' says the astronomer, 'and how accurately can you predict the eclipses?' He says, 'I haven't developed the thing very far yet.' Then says the astronomer, 'Well, we can calculate eclipses more accurately than you can with your model, so you must not pay any attention to your idea because obviously the mathematical scheme is better.'

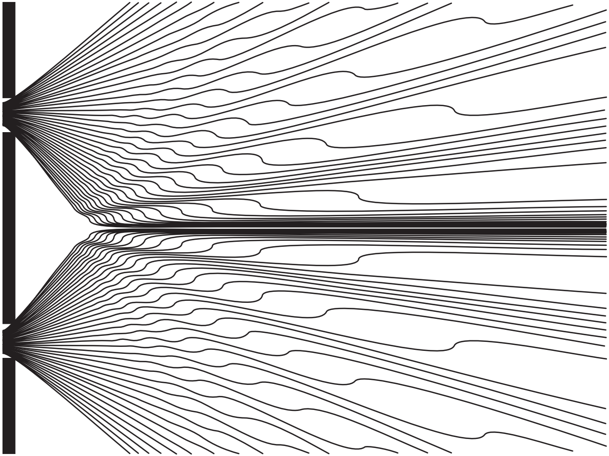
Empirical tests

Collapse theories such as GRW and CSL

make empirical predictions slightly different from BM/OQM. It's worth the effort to do experimental tests! We want to know.
But that does not mean that a theory such as BM that can't be empirically tested against OQM is irrelevant. We might live in a Bohmian universe.

Further properties

But how can BM be compatible with Heisenberg's uncertainty relation?



No-superluminal-signaling theorem in BM

Alice's lab A and Bob's lab B at spacelike separation during the time interval $[t_1, t_2]$ in some Lorentz frame.
$H = H_A \otimes I_B + I_A \otimes H_B$ (no interaction terms).
Time evolution from t_1 to t_2 factorizes, ($\mathcal{T}e$ = time-ordered exp)

$$U = \mathcal{T}e^{-i\int_{t_1}^{t_2} H(t)\,dt} = \mathcal{T}e^{-i\int_{t_1}^{t_2} H_A(t)\,dt} \otimes \mathcal{T}e^{-i\int_{t_1}^{t_2} H_B(t)\,dt} = U_A \otimes U_B\,.$$

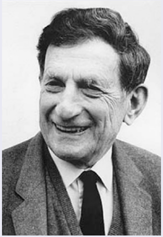
Message that Alice wants to send is contained in external fields in $H_A(t)$.
Bob carries out any experiment on his lab at any time t_2; distribution of Bob's outcome Z is independent of H_A: Indeed, for $P_B(z)$ = projection to config. space region with Bob's instrument pointing to z,

$$\begin{aligned}\mathbb{P}(Z=z) &= \langle U\Psi|[I_A \otimes P_B(z)]|U\Psi\rangle \\ &= \langle\Psi|[(U_A^\dagger I_A U_A)\otimes(U_B^\dagger P_B(z) U_B)]|\Psi\rangle \\ &= \langle\Psi|[I_A \otimes (U_B^\dagger P_B(z) U_B)]|\Psi\rangle\,.\end{aligned}$$

Klaus Tausk (1967), Bell (1976), P. Eberhard (1978), G.C. Ghirardi (1980)
(and recently denied by Maudlin, Das, Cavendish; more later).

History

- 1924: Einstein toyed with the idea that photons may have trajectories obeying an equation of motion similar to that of Bohmian mechanics. John Slater joined him.
- 1926: Louis de Broglie discovered Bohmian mechanics, called it "pilot-wave theory."
- 1945: Nathan Rosen (the R of EPR) independently discovered Bohmian mechanics.
- 1952: David Bohm independently discovered Bohmian mechanics. He was the first to realize that the theory is empirically adequate.



David Bohm (1917–1992)

Primitive ontology

- ontology = what exists, according to a theory (Bell: "beables")
- primitive ontology = things in a theory representing matter in space and time (Bell: "local beables")
- E.g. in BM, ontology = (Q, ψ); primitive ontology = Q
- When we say "the cat sat on the mat" in BM, we imply that there actually is a certain amount of matter in a certain location. We don't just mean "the wave function lies in a certain subspace," which now seems like a contortion. Language can be taken literally, and the everyday picture of reality is largely correct.
- Primitive ontology can be easily added also in GRW/CSL or MWI:
  - flash ontology in GRW [Bell 1990, Tumulka quant-ph/0406094, Allori et al. 1206.0019]
  - matter density ontology in GRW/CSL [Ghirardi et al. 1111.1425]
  - matter density ontology in MWI [Schrödinger 1927, Allori et al. 0903.2211]

Thank you for your attention