Mini Course on Bohmian Mechanics
Lecture 2

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Spin

The Pauli equation

Consider N spin-$\frac{1}{2}$ particles.

$\psi(t) : \mathbb{R}^{3N} \to (\mathbb{C}^2)^{\otimes N}$,
$\psi = \psi_{s_1 \ldots s_N}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, t)$

The appropriate version of the Schrödinger eq is the Pauli eq

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{k=1}^{N} \frac{1}{2m_k} \Big(-i\hbar \nabla_{\boldsymbol{q}_k} - \boldsymbol{A}(\boldsymbol{q}_k) \Big)^2 \psi - \sum_{k=1}^{N} \frac{\hbar}{2m_k} \boldsymbol{\sigma}_k \cdot \boldsymbol{B}(\boldsymbol{q}_k) \psi + V\psi$$

with $\boldsymbol{A}$ the vector potential, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ the magnetic field, V the electric potential, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices, and $\boldsymbol{\sigma}_k$ acting on s_k.

Bohmian mechanics with spin

You might have expected that Bohm needs little spinning balls. But it is much easier.

Equation of motion: [Bell 1966]

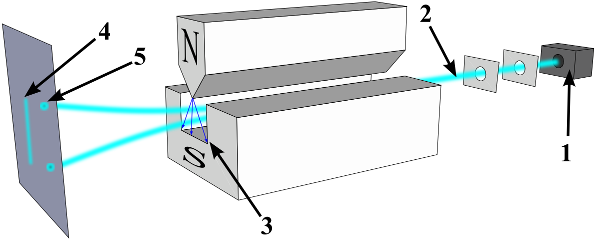
$$\frac{d\boldsymbol{Q}_k(t)}{dt} = \frac{\boldsymbol{j}_k}{\rho}(Q(t),t) = \frac{\hbar}{m_k}\mathrm{Im}\frac{\psi^*\nabla_k\psi}{\psi^*\psi}(Q(t),t)$$

where $\phi^*\psi = \sum_{s=1}^{2^N} \phi_s^*\psi_s$ inner product in spin-space, $s = (s_1...s_N)$.

So the electron is still a point, and not spinning. Spin is merely in the wave fct (the wave fct is spinor-valued).

The Stern-Gerlach experiment

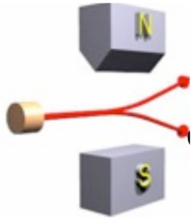
Then what does a “spin measurement” do in BM?



Picture credit: `http://en.wikipedia.org/wiki/Stern-Gerlach_experiment`

Otto Stern, Walther Gerlach 1922

The Stern-Gerlach experiment in Bohmian mechanics



Stern–Gerlach experiment

Wave packet $\psi = \binom{\psi_\uparrow}{\psi_\downarrow}$ splits into two packets, one purely $\uparrow$, the other purely $\downarrow$. Then detect the position of the particle: If it is in the spatial support of the $\uparrow$ packet, say that the outcome is "up."

So, the "measurement" is not literally a measurement (i.e., not a determination of a pre-existing value). The outcome is a random value generated in the experiment. That is common with "quantum measurements" in Bohmian mechanics (or GRW or MW), except for position measurements.

Prediction

Since $Q \sim |\psi|^2$, $\mathbb{P}(\text{up}) = \|\psi_\uparrow\|^2$ and $\mathbb{P}(\text{down}) = \|\psi_\downarrow\|^2$
Empirically correct. Same in direction $\boldsymbol{a} \in \mathbb{R}^3$. In particular, BM is compatible with non-commuting operators $\boldsymbol{a} \cdot \boldsymbol{\sigma}$.

Notabene

- Spin is not in the primitive ontology. There is no hidden variable for spin; no "actual value" of z-spin, in contrast to actual positions.
- BM would break down without ontological positions, but not without ontological spin.
- In particular, there is no need to have beables for all observables.

On fundamental theories of the universe

What BM illustrates about fundamental laws

- BM does not include (as yet) QED, QCD, quantum gravity.
- But one can imagine a hypothetical universe governed by non-relativistic BM.
- Then the defining equations are fundamental laws for that universe.
- BM provides a paradigm of what such laws can look like.

Fundamental theories of the universe . . .

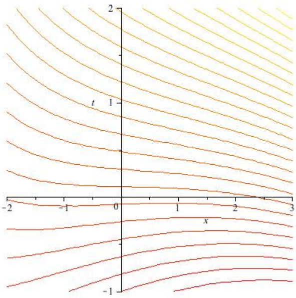
- cannot be formulated in terms of observers making measurements
- have to provide equations valid for all particles all the time ("democracy of electrons"); stochastic terms are OK, "collapse upon measurement" not OK
- should have a primitive ontology describing space-time and where the matter is in it
- are free to postulate the fundamental ontology and the fundamental laws of nature
- are free to make postulates about initial conditions (e.g., being random)
- need not provide actual values for all observables (not even spin)
- need not be formulated only in terms of observable quantities
- have to imply the quantum formalism (empirical rules) as theorems (consequences) of the fundamental laws
- need not "explain" the Schrodinger eq
- need not match classical intuition
- but have to explain the 2-slit experiment etc.

Brief outlook on lecture 3

- Relativistic extension of BM
- Extension of BM to QFT

BM in relativistic space-time

- BM for 1 particle possesses a natural relativistic extension based on the Dirac eq.
- BM for N particle also possesses a natural relativistic extension, but only if the existence of a preferred foliation $\mathcal{F}$ (slicing of space-time into spacelike hypersurfaces) is granted.
- Theorem: Inhabitants can't measure which surfaces belong to $\mathcal{F}$.
- General relativity naturally provides such surfaces: constant timelike distance from the big bang.
- Collapse models can be defined without a preferred foliation.



BM and QFT

- QFT has all of the issues of QM (measurement problem, nonlocality) but further ones in addition (localization, negative energies, divergences).
- BM: Field ontology or particle ontology? (Or something else?)
- Field ontology: The math is not yet fully developed. Similar problems if an infinitude of particles arose.
- Particle ontology: Law of Q is stochastic, involves particle creation and annihilation.
- Both: Models exist for simple QFTs.
- Many open problems, equally for orthodox or collapse versions.

A recent controversy: Sharoglazova's 2025 "challenge" to Bohmian mechanics

Further reading: Upcoming paper by Christian Beck, Sheldon Goldstein, Dustin Lazarovici, R.T., and Nino Zanghì

Article

Energy–speed relationship of quantum particles challenges Bohmian mechanics

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Violetta Sharoglazova[1], Marius Puplauskis[1], Charlie Mattschas[1], Chris Toebes[1] & Jan Klaers[1]✉

Classical mechanics characterizes the kinetic energy of a particle, the energy it holds due to its motion, as consistently positive. By contrast, quantum mechanics describes the motion of particles using wave functions, in which regions of negative local kinetic energy can emerge[1]. This phenomenon occurs when the amplitude of the wave function experiences notable decay, typically associated with quantum tunnelling. Here, we investigate the quantum mechanical motion of particles in a system of two

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Cited by 34 (according to Google Scholar) in less than a year,

e.g., by J. Landgrebe and B. Smith [arXiv 2606.00805], who wrote

[10] Bohm's idea of modeling particles as infinitesimal points in his guide equation is not only incompatible with the Standard Model, which sees particles as having an extension, but also unable to model any magnitude other than position; it is also incompatible with relativity and QFT, and has recently been falsified experimentally [42].

Sharoglazova's question and answer

Question

What is the speed of a quantum particle in a classically forbidden region (potential barrier) where it seems to have negative kinetic energy? (Say, $V(x) = V_0 \, 1_{x \geq 0}$ with $V_0 > 0$ and $E_{\mathrm{kin}} = \hbar^2 k^2/2m - V_0$.)

Relevant eigenstates of H (evanescent, $\kappa = \sqrt{2mV_0/\hbar^2 - k^2}$):

$$\varphi_k(x) = \begin{cases} e^{ikx} + be^{-ikx} & \text{for } x < 0 \\ ce^{-\kappa x} & \text{for } x \geq 0 \end{cases}$$

Her answer

$v = \sqrt{2|E_{\mathrm{kin}}|/m} = \hbar\kappa/m$,
which disagrees with the Bohmian velocity.

Her approach

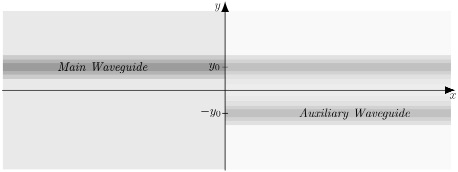
She and her co-authors developed a method intended for measuring v.

(They are fine with trajectories but think Bohm had the wrong formula.)

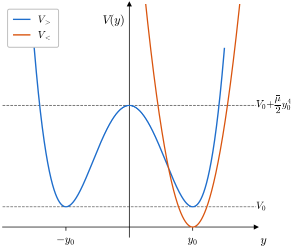
Sharoglazova's experiment

Non-rel. quantum particle in 2D

$$i\hbar \frac{\partial \Psi(x,y,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(x,y) \right] \Psi(x,y,t)$$



brightness $= V(x,y)$



$$V(x,y) = \begin{cases} V_<(y) = 2\mu y_0^2 \big(y - y_0\big)^2 & \text{for } x < 0 \\ V_>(y) = V_0 + \frac{1}{2}\bar\mu (y^2 - y_0^2)^2 & \text{for } x \geq 0 \end{cases}$$

Observed: $\int_{t_1}^{t_2} dt\, |\Psi(x,y,t)|^2$

Sharoglazova's claim should be puzzling from the start

First, it seems like an empirical difference between BM and OQM

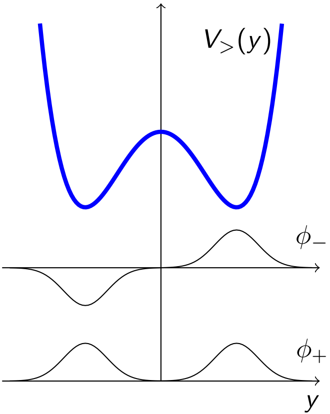
But we know already that there is no such empirical difference.

Even more direct: v_x is inferred from $|\Psi(x,y,t)|^2$

But we know that BM and OQM have the same $|\Psi(x,y,t)|^2$.

How can that be? There must be a mistake somewhere.

Preparation: the double well



"Deep well regime" ($y_0^3\sqrt{\bar\mu m} \gg \hbar$):

The two lowest eigenvalues $E_+ < E_-$ are very close to each other, eigenfunctions

$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_m \pm \phi_a)$

(m for main wave guide, a for auxiliary)

$$\psi_t(y) = \frac{1}{\sqrt{2}}(e^{-iE_+t/\hbar}\phi_+(y) + e^{-iE_-t/\hbar}\phi_-(y))$$
$$= e^{-i\bar{E}t/\hbar}(\cos(J_0t)\phi_m(y) + i\sin(J_0t)\phi_a(y))$$

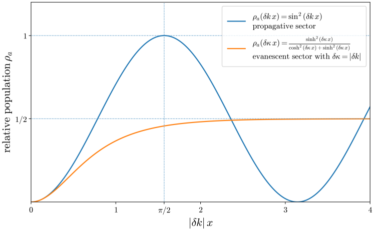
Rabi oscillations with frequency $J_0 = (E_- - E_+)/2\hbar$, i.e., tunneling time $T_{\mathrm{tun}} = 2\pi/J_0$

2D case

eigenstates: for $x > 0$,

$$\varphi_k(x,y) = \frac{1}{\sqrt{2}}\Big(\gamma_+ e^{-\kappa_+ x}\phi_+(y) + \gamma_- e^{-\kappa_- x}\phi_-(y)\Big)$$

$$\approx \gamma e^{-\bar{\kappa}x}\Big(\cosh(\delta\kappa\, x)\,\phi_m(y) + \sinh(\delta\kappa\, x)\,\phi_a(y)\Big)$$



relative population in aux guide

$$\rho_a(x) = \frac{\sinh^2(\delta\kappa\, x)}{\sinh^2(\delta\kappa\, x) + \cosh^2(\delta\kappa\, x)},$$

so equilibration length $X_{\mathrm{eq}} \sim 1/\delta\kappa$

The tunneling time argument

Sharoglazova's argument

- We know the time T_{tun} it takes to tunnel in y-direction from the main waveguide to the aux waveguide.
- We measure the distance $X_{\text{eq}} > 0$ (order of magnitude) until a substantial fraction of the particles have tunneled to the aux waveguide.
- Conclude that particle move distance X_{eq} in x-direction in time T_{tun}, so x-speed $v_x = X_{\text{eq}}/T_{\text{tun}} = \hbar\bar{\kappa}/m$ (order of magnitude).

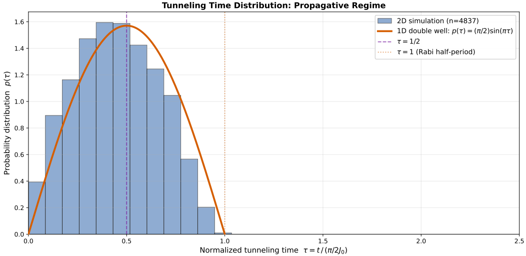
This value is much larger than the avg Bohmian speed of the particles entering $x > 0$. (In the paper, Sharoglazova et al. say the Bohmian speed is 0, which is not correct for a wave packet, but the correct value is still much smaller than $\hbar\bar{\kappa}/m$.)

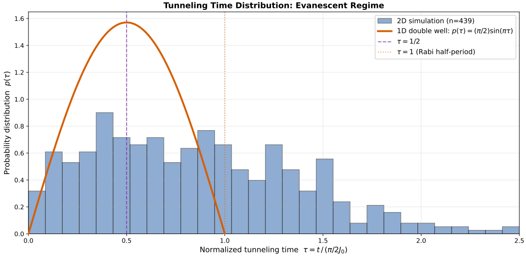
Sharoglazova's mistake

The tunneling time cannot be assumed to be unaffected by the x-dependence of $\Psi(x, y)$ (i.e., entanglement).

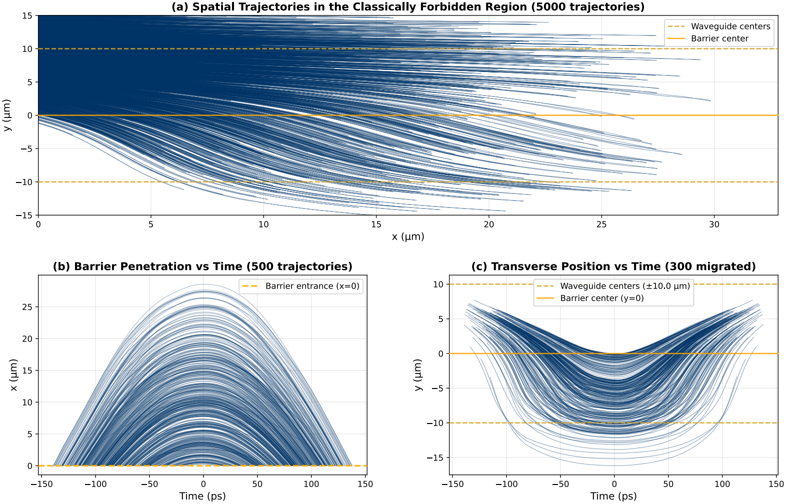
How Sharoglazova's assumption is violated in BM

The assumption was: T_{tun} is fixed by the double well potential, unaffected by entanglement.





Bohmian trajectories for Sharoglazova's experiment



Avg Bohmian x-speed of a particle entering $x > 0$: $\bar{v}_{\mathrm{B}} = \sqrt{\frac{\pi}{8}} \frac{\hbar k_0 \sigma_k}{m \kappa_0}$

Sharoglazova's 2nd argument

was presented in the paper as a "consistency check" on the speed estimate, but could be taken as an independent estimate. The argument can be carried out in 1D (x only).

Dwell time argument (first half)

- A widely accepted expression for the average dwell time of the particle in a 1D region (a, b) is given by the Büttiker formula

$$\tau_d(k) = \frac{m}{\hbar k} \int_a^b dx \, |\varphi_k(x)|^2 \, .$$

- In the present case, $(a, b) = (0, \infty)$ and, for $x > 0$,

$$\varphi_k(x) = \frac{2ik}{ik - \kappa} e^{-\kappa x} \, .$$

- Thus, the average dwell time in the barrier is

$$\tau_d(k) = \frac{2m}{\hbar \kappa} \frac{k}{k^2 + \kappa^2} .$$

Sharoglazova's 2nd argument (cont'd)

Dwell time argument (2nd half)

- The the typical penetration depth is $\lambda = 1/2\kappa$.
- Thus, $v_x \sim \frac{2\lambda}{\tau_d} = \frac{\hbar\kappa}{m}\left(\frac{1+(k/\kappa)^2}{2k/\kappa}\right)$.

For $\hbar^2 k^2/2m$ neither close to 0 nor close to V_0, $(\cdots) = \mathcal{O}(1)$, so this estimate roughly agrees with the first.

Mistake

Büttiker formula = dwell time averaged over all particles in the ensemble, not only those entering the barrier $x > 0$. Thus, the mean dwell time conditionally on entering the barrier is τ_d divided by the penetration probability P. For a wave packet, $P \sim \lambda/\sigma_x = \sigma_k/\kappa_0$, which leads for small σ_k to

$$v_x \sim \sqrt{\frac{2}{\pi}}\frac{\hbar k_0 \sigma_k}{m\kappa_0}, \quad \text{in agreement with } \bar{v}_{\mathrm{B}}.$$

Thus, the corrected argument supports Bohmian mechanics.

Büttiker formula in Bohmian mechanics

- Given a trajectory $X(t)$, the dwell time in region A is $T_A = \int_{-\infty}^{+\infty} dt\, 1_A(X(t))$.
- For any ensemble of trajectories, the mean dwell time is $\tau_A = \mathbb{E}\, T_A = \int dt\, \mathbb{E} 1_A(X(t)) = \int dt\, \mathbb{P}(X(t) \in A) = \int dt \int_A dx\, \rho(x,t)$.
- In Bohmian mechanics, $\rho(x,t) = |\psi(x,t)|^2$.
- Fourier transformation is unitary $\Rightarrow$ $\int_{-\infty}^{+\infty} dt\, |f(t)|^2 = \int_{-\infty}^{+\infty} d\omega\, |\hat{f}(\omega)|^2$.
- Leads to $\tau_A = \int_0^\infty dk\, |\hat{\psi}_{\mathrm{in}}(k)|^2 \, \frac{m}{\hbar k} \int_A dx\, |\varphi_k(x)|^2$.
- As $|\hat{\psi}_{\mathrm{in}}(k)|^2 \to \delta(k - k_0)$, we obtain the Büttiker formula.
- Upshot: in the monochromatic limit, the Büttiker formula is exactly true in Bohmian mechanics.

Thank you for your attention