Mini Course on Bohmian Mechanics
Lecture 3

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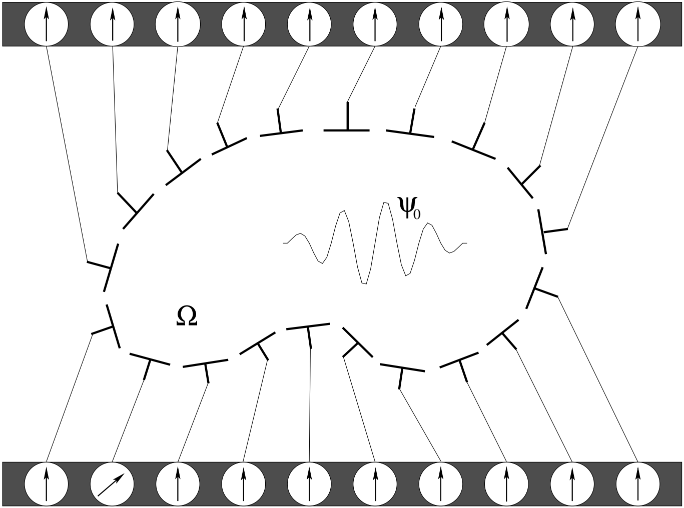
Time of detection and time of arrival

Problem of detection time and place



$T \in [0, \infty), \boldsymbol{X} \in \partial\Omega, Z = (T, \boldsymbol{X})$ redrawn after [Daumer et al. quant-ph/9512016]

Problem of detection time

- in orthodox quantum mechanics (OQM), there is no time operator
- Pauli 1933: it's impossible to have an operator for detection time
- quantum Zeno effect [Turing 1950s]: seems impossible ("A watched pot never boils," Misra and Sudarshan 1977)
- Allcock's paradox [1969]
- several suggestions:
    - Aharonov and Bohm [1961]: compute classical arrival time from $\boldsymbol{x}(0), \boldsymbol{p}(0)$, then quantize to obtain an operator
    - Kijowski [1974]
    - Maccone [2018]: $t \mapsto |\psi(\boldsymbol{x}, t)|^2$ for fixed $\boldsymbol{x} \in \partial\Omega$
    - "absorbing boundary rule" [Werner 1987, Tumulka 2016]
    - Das and Dürr [2018]: equate with Bohmian arrival time

Hard vs soft detectors

Soft detectors take a while before noticing a particle flying through the detector volume.
Hard detectors register a particle immediately when it arrives at the detector surface $\partial\Omega$.

Model for soft detectors [Allcock 1969]: negative imaginary potential

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - iW(\boldsymbol{x})\psi(\boldsymbol{x})$$

($W(\boldsymbol{x}) \geq 0$ detector strength) implies

$$\partial_t\rho = -\nabla\cdot\boldsymbol{j} - \frac{2}{\hbar}W(\boldsymbol{x})\rho(\boldsymbol{x})\,.$$

That is, particle gets removed from the experiment at rate $\frac{2}{\hbar}W(\boldsymbol{x})$, corresponding to

$$\mathbb{P}\big(\boldsymbol{X}_D \in d^3\boldsymbol{x},\, T_D \in dt\big) = \tfrac{2}{\hbar}W(\boldsymbol{x})|\psi(\boldsymbol{x},t)|^2\, d^3\boldsymbol{x}\, dt\,.$$

Arrival times in BM

In BM in the absence of detectors, there is a fact about when and where the particle's trajectory first intersects $\partial\Omega$: the arrival time T_{WOD} and arrival place $\boldsymbol{X}_{WOD}$ (WOD = without detectors).

- Distribution

$$\mathbb{P}(\boldsymbol{X}_{WOD} \in d^2\boldsymbol{x}, T_{WOD} \in dt) = \begin{cases} \boldsymbol{j}(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) \text{ at the 1st crossing} \\ 0 \text{ at 2nd or later crossing} \end{cases}$$

Das and Dürr [1802.07141]

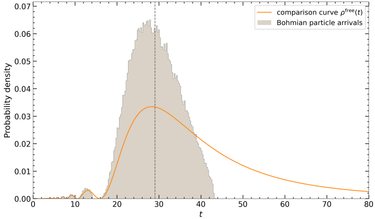
- hypothesized that

$$\mathbb{P}(\boldsymbol{X}_D \in d^2\boldsymbol{x}, T_D \in dt) = \mathbb{P}(X_{WOD} \in d^2\boldsymbol{x}, T_{WOD} \in dt)\,,$$

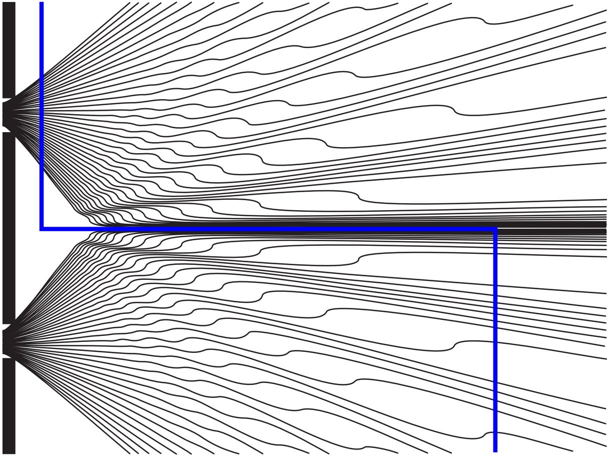
  in short $\mathbb{P}_D = \mathbb{P}_{WOD}$. I disagree.
- If the hypothesis were true, that would be nice for Bohmians: It would allow BM to make a testable prediction that OQM can't make. If confirmed experimentally, maybe all physicists would become Bohmians.

Das and Dürr computed $\mathbb{P}_{WOD}$ for a setup with

- a spin-$\frac{1}{2}$ particle in $\mathbb{R}^2 \times [0, \infty)$
- reflecting wall at $x_3 = 0$
- waveguide potential $\frac{m}{2}\omega^2(x_1^2 + x_2^2)$
- $\psi_0(\boldsymbol{x}) = \varphi(\boldsymbol{x}) \otimes |\boldsymbol{n}\rangle$ with $|\boldsymbol{n}\rangle \in \mathbb{C}^2$
- arrival at $x_3 = L$
- and found striking dependence of $\mathbb{P}_{WOD}$ on $|\boldsymbol{n}\rangle$.



Example illustrating that trajectories in the presence of detectors will be different from those in their absence:



Thus, $\boldsymbol{X}_{WID} \neq \boldsymbol{X}_{WOD}$ (WID = with detector) and in general $T_{WID} \neq T_{WOD}$. What can you expect of $\boldsymbol{X}_D$ then?

T_{WID} vs T_D

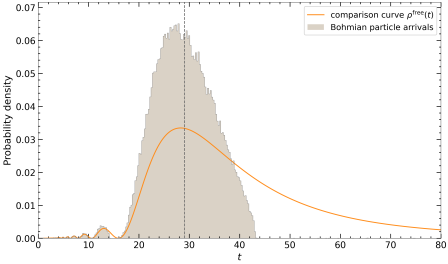
They may differ because

- detectors might not immediately register the particle,
- they might have limited resolution or other inaccuracies,
- or probability might flow back from post-detection to pre-detection configurations.

But one would hope that for ideal detectors, $(\boldsymbol{X}_D, T_D) = (\boldsymbol{X}_{WID} = T_{WID})$, while even ideal detectors could not make $(\boldsymbol{X}_D, T_D)$ agree with $(\boldsymbol{X}_{WOD}, T_{WOD})$.

If $\mathbb{P}_D = \mathbb{P}_{WOD}$ then superluminal signaling is possible

- Alice and Bob share 100 EPR pairs in $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = |\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle$.
- If Alice wants to send "1," she measures σ_z on each of her particles, so Bob's particles collapse to either $|\uparrow\rangle$ or $|\downarrow\rangle$.
- If Alice wants to send "0," she measures σ_x, so Bob's particles collapse to either $|\rightarrow\rangle$ or $|\leftarrow\rangle$.
- In a small, local Ω, Bob measures T_D on each of his particles. For $|\boldsymbol{n}\rangle = |\uparrow\rangle$ or $|\boldsymbol{n}\rangle = |\downarrow\rangle$, the statistics is the brown histogram; for $|\boldsymbol{n}\rangle = |\rightarrow\rangle$ or $|\boldsymbol{n}\rangle = |\leftarrow\rangle$, the orange curve.



- If a hypothesis H implies superluminal signaling, you should become skeptical, as no one has observed superluminal signaling yet.
- Moreover, in that case you know for sure that H is false in BM, as a no-signaling theorem holds in BM.

A moral

The words “measurement” and “observation” suggest that the apparatus plays a merely passive role. But this is often not the case, and the apparatus must be included in the consideration.

Positive-operator-valued measure (POVM)

Definition

A POVM on a discrete set $\mathscr{Z}$ is a family $(E_z)_{z\in\mathscr{Z}}$ of positive operators such that $\sum_{z\in\mathscr{Z}} E_z = I$.
(A POVM in the continuum associates with subsets $S \subseteq \mathscr{Z}$ a positive operator $E(S)$ such that $E(\mathscr{Z}) = I$ and
$E(S_1 \cup S_2 \cup \ldots) = E(S_1) + E(S_2) + \ldots$ if $S_i \cap S_j = \emptyset$ for $i \neq j$.)

Main theorem about POVMs in BM: a generalization of Born's rule

For any experiment with outcome Z on a system with Hilbert space $\mathscr{H}$, there is a POVM E on the set $\mathscr{Z}$ of possible outcomes such that for every $\psi \in \mathscr{H}$ with $\|\psi\| = 1$,

$$\mathbb{P}_\psi(Z = z) = \langle\psi|E_z|\psi\rangle\,.$$

Example: For an ideal quantum measurement of the observable $A = \sum_\alpha \alpha P_\alpha$, $\mathscr{Z} = \text{spectrum}(A)$ and $E_z = P_\alpha$.

Proof of the main theorem about POVMs from BM

Let the experiment be over at time t_1, and read off the result from the apparatus display, $Z = f(Q(t_1))$. Let $P(\cdot)$ be the position POVM on $\mathbb{R}^{3N}$, i.e.,

$$P(S)\psi(q) = 1_S(q)\,\psi(q)$$

for any $S \subseteq \mathbb{R}^{3N}$. Let $U = \exp(-iH(t_1 - t_0)/\hbar)$. Then

$$\mathbb{P}(Z = z) = \int\limits_{f^{-1}(z)} dq\,|\Psi_{t_1}(q)|^2 = \langle\psi|E_z|\psi\rangle_{\mathscr{H}}$$

with

$$E_z = \langle\phi|U^\dagger P(f^{-1}(z))U|\phi\rangle_{\mathrm{app}}\,.$$

$\square$

$\mathbb{P}_D$ vs $\mathbb{P}_{WOD}$

Theorem [Goldstein, Tumulka, Zanghì 2309.11835, 2405.04607]

In the example of Das and Dürr, $\mathbb{P}_{WOD}$ is not given by a POVM (and thus is $\neq \mathbb{P}_D$), not even approximately. (The spin dependence is crucial.)

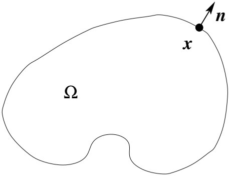
Detlef Dürr sadly passed away in 2021.
Das, Maudlin and Cavendish insist that $\mathbb{P}_D = \mathbb{P}_{WOD}$.

Absorbing boundary rule

- Solve the 1-particle Schrödinger equation $i\hbar\dfrac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$ with absorbing boundary condition (ABC)

  $$\boldsymbol{n}(\boldsymbol{x})\cdot\nabla\psi(\boldsymbol{x}) = i\kappa\,\psi(\boldsymbol{x})$$

  at every $\boldsymbol{x}\in\partial\Omega$, where $\boldsymbol{n}(\boldsymbol{x}) =$ outward unit normal vector to $\partial\Omega$ at $\boldsymbol{x}$, and $\kappa > 0$ a constant.



- ABC implies that the probability current $\boldsymbol{j}^\psi = \frac{\hbar}{m}\mathrm{Im}[\psi^*\nabla\psi]$ points outward at $\partial\Omega$:

  $$\boldsymbol{n}\cdot\boldsymbol{j} = \frac{\hbar}{m}\mathrm{Im}[\psi^*\boldsymbol{n}\cdot\nabla\psi] = \frac{\hbar}{m}\mathrm{Im}[\psi^* i\kappa\psi] = \frac{\hbar}{m}\kappa|\psi|^2 \geq 0\,.$$

- $\mathbb{P}_{\psi_0}\Big(T\in dt, \boldsymbol{X}\in d^2\boldsymbol{x}\Big) = \boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{j}^{\psi_t}(\boldsymbol{x})\,dt\,d^2\boldsymbol{x}$ assuming $\|\psi_0\| = 1$.
- If the experiments get interrupted at time t before detection, the collapsed wave function is $\psi_t/\|\psi_t\|$.

Properties

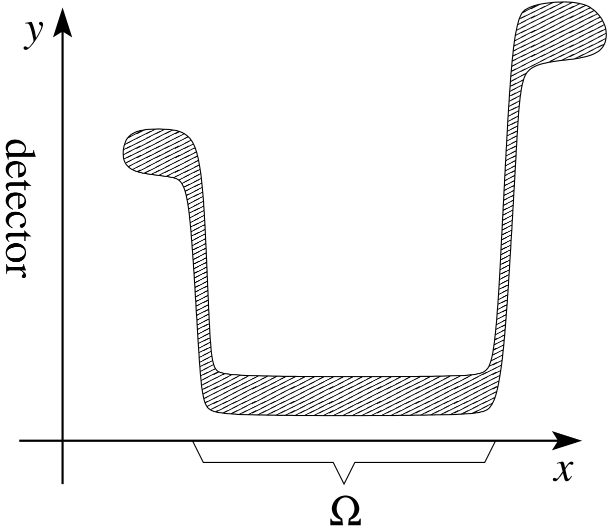
- $\|\psi_t\|^2 = \mathbb{P}_{\psi_0}(T > t)$ "survival probability," decreasing in t
- The time evolution of ψ, $W_t = \exp(-iHt/\hbar)$, is not unitary (Hamiltonian not self-adjoint) due to loss at $\partial\Omega$
- distribution is given by a POVM
- $E_\kappa\big(dt \times d^2\boldsymbol{x}\big) = \frac{\hbar\kappa}{m}\, W_t^\dagger\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, W_t\, dt\, d^2\boldsymbol{x}$,
  $E_\kappa(T = \infty) = \lim_{t\to\infty} W_t^\dagger W_t$
- In Bohmian mechanics, the particle with $|\psi_0|^2$-distributed initial condition $\boldsymbol{X}(0)$ moves according to the equation of motion

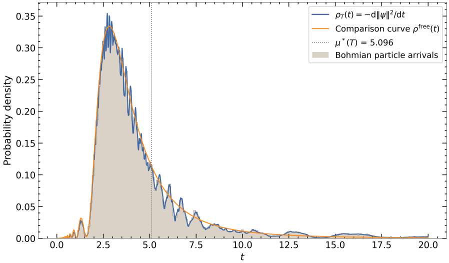
  $$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{j}^{\psi_t}(\boldsymbol{X}(t))}{|\psi_t(\boldsymbol{X}(t))|^2}$$

  until it hits $\partial\Omega$ at time T and place $\boldsymbol{X} = \boldsymbol{X}(T)$, and gets absorbed. $\mathbb{P}_{\psi_0}\big(\boldsymbol{X}(t) \in d^3\boldsymbol{x}\big) = |\psi_t(\boldsymbol{x})|^2\, d^3\boldsymbol{x}$.
- energy-time uncertainty relation $\Delta E\; \Delta T \geq \hbar/2$
  with E referring to $-\frac{\hbar^2}{2m}\nabla^2$ on $L^2(\mathbb{R}^3)$

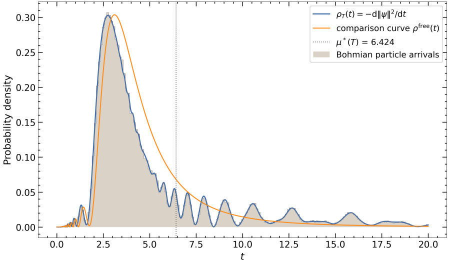
Heuristic derivation

configuration space:





Prediction by ABC for detection time distribution in a wave guide; numerical simulation [Jozani and Tumulka 2603.22044]



Same for imaginary potential in a layer $z_1 < z < L$ [ibid.]

Scattering (far-field) regime

Detectors on $\mathbb{S}_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$, lim $R \to \infty$, folklore knowledge:

$$\mathbb{P}\left(\frac{\boldsymbol{X}_D}{R} \in d^2\boldsymbol{u},\; \frac{T_D}{R} \in d\tau\right) \longrightarrow \underbrace{\frac{m^3}{\hbar^3\tau^4}\left|\widehat{\Psi}_0\left(\frac{m\boldsymbol{u}}{\hbar\tau}\right)\right|^2}_{=:\sigma(\boldsymbol{u},\tau)} d^2\boldsymbol{u}\, d\tau \qquad (1)$$

Remarks:

- PVM: $\sigma =$ joint Born distr. of the observables $|\hat{\boldsymbol{p}}|^{-1}\hat{\boldsymbol{p}}$ and $m|\hat{\boldsymbol{p}}|^{-1}$.
- Usual justification: $\int_{|\boldsymbol{x}|<R} |\Psi_t^{\mathrm{free}}(\boldsymbol{x})|^2\, d^3\boldsymbol{x}$ should $= \mathbb{P}(T_D > t)$, and that changes in $[t, t+dt]$ (to leading order in R) by $-\int_{\mathbb{S}_R} \frac{\boldsymbol{x}}{R} \cdot \boldsymbol{j}(\boldsymbol{x},t)\, d^2\boldsymbol{x} = -dt \int_{\mathbb{S}_R} R^{-3}\sigma(\boldsymbol{x}/R, t/R)\, d^2\boldsymbol{x}$, leading to (1).
- But that quantity need not even be ≥ 0 before lim $R \to \infty$. And doesn't show that detectors *actually* yield σ. Not satisfactory.
- Another argument: In BM, $(\boldsymbol{X}_{WOD}/R, T_{WOD}/R) \sim \sigma$ in the limit [Daumer et al. quant-ph/9512016]. Not satisfactory.
- Another argument: Classically, $\boldsymbol{X}_D/R \to |\boldsymbol{p}|^{-1}\boldsymbol{p}$ and $T_D/R \to m|\boldsymbol{p}|^{-1}$; then quantize. Not satisfactory.
- Absorbing boundary rule would be inappropriate here because hard detectors lead to partial reflection; soft detectors are OK as we can afford large errors in T_D if small compared to R.

Better justification of scattering cross section formula (1)

[Kaimal and Tumulka 2601.01625] From macroscopic models of detectors:

Derivation 1

Put imaginary potential $-i\lambda 1_{|\boldsymbol{x}|>R}$.
$\mathbb{P}\big(\boldsymbol{X}_D \in d^3\boldsymbol{x}, T_D \in dt\big) = \frac{2}{\hbar}\lambda 1_{|\boldsymbol{x}|>R}\,|\Psi(\boldsymbol{x},t)|^2 d^3\boldsymbol{x}\,dt$.
Take limit $R \to \infty$, $\lambda \to 0$, $R\lambda \to \infty$.

Derivation 2

Repeated (stroboscopic) near-projective quantum measurements of (approximately) the observable $1_{|\boldsymbol{x}|>R}$ at times $\mathscr{T}, 2\mathscr{T}, 3\mathscr{T}, \ldots$; take the limit $R \to \infty$, $\mathscr{T} \to \infty$, $\mathscr{T}/R \to 0$.

Byproduct

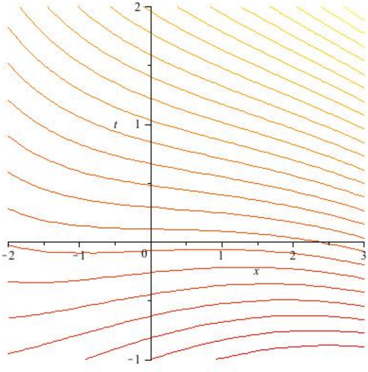
In the limit, $T_D - T_{WID} \longrightarrow \infty$ and $T_{WID} - T_{WOD} \longrightarrow \infty$ while

$$\frac{T_D - T_{WID}}{R} \longrightarrow 0 \quad \text{and} \quad \frac{T_{WID} - T_{WOD}}{R} \longrightarrow 0\,.$$

Bohmian mechanics in relativistic space-time

Bohmian mechanics in relativistic space-time

- If a preferred foliation (= slicing) of space-time into spacelike hypersurfaces ("time foliation" $\mathcal{F}$) is permitted, then there is a simple, convincing analog of Bohmian mechanics, $\mathrm{BM}_{\mathcal{F}}$. [Dürr et al. 1999] Without a time foliation, no version of Bohmian mechanics is known that would make predictions anywhere near quantum mechanics. (And I have no hope that such a version can be found in the future.)



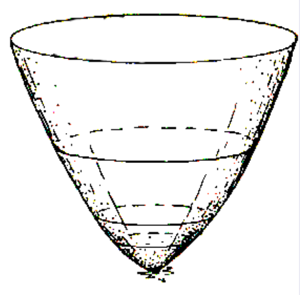
What does it mean for a theory to be relativistic?

Maybe there is no single property of a theory that can be regarded as "being relativistic." Rather, there are several relevant properties:

1. **Lorentz invariance.** Any Lorentz transform of any solution is another solution.
   - Can be made true trivially (e.g., for non-rel. theory) without changing predictions. Thus, necessary but not sufficient for anybody's notion of being relativistic.
2. **Commutation.** Field operators $[\phi(x), \phi(y)] = 0$ for spacelike separated x, y.
   - Easy to satisfy, seems not sufficient for being relativistic.
3. **No signaling faster than light.** Necessary but not sufficient.
4. **Locality à la EPR and Bell.** Violated in nature.
5. **No additional structure.** Don't introduce $\mathcal{F}$, use only $g_{\mu\nu}$ and ψ.
   - It seems possible to define foliations from $g_{\mu\nu}$ and/or ψ.
6. **Microscopic parameter independence.** If regions A, B are spacelike separated, then $\mathbb{P}(PO_A|\Phi_A, \Phi_B, \lambda) = \mathbb{P}(PO_A|\Phi_A, \lambda)$ for external fields Φ and hidden variables λ.
   - True in relativistic GRWf and GRWm, false in $\mathrm{BM}_{\mathcal{F}}$.

Perhaps, the semantic question what we should mean by "relativistic" is irrelevant. The possibility seems worth considering that our universe has a time foliation.

Simplest choice of time foliation $\mathcal{F}$



Drawing: R. Penrose

Let $\mathcal{F}$ be the level sets of the function
T : space-time $\to \mathbb{R}$,
$T(x) =$ timelike-distance$(x,$ big bang$)$.

E.g., T(here-now) $=$ 13.7 billion years

Alternatively, $\mathcal{F}$ might be defined in terms of the quantum state vector ψ, $\mathcal{F} = \mathcal{F}(\psi)$ [Dürr, Goldstein, Norsen, Struyve, Zanghì 2014]

Or, $\mathcal{F}$ might be determined by an evolution law (possibly involving ψ) from an initial time leaf.

Bohmian mechanics for a single Dirac particle

No time foliation needed in this case.

Dirac equation:

$$i\hbar\gamma^\mu\partial_\mu\psi = m\psi \qquad \text{or} \qquad i\hbar\frac{\partial\psi}{\partial t} = -i\hbar\boldsymbol{\alpha}\cdot\nabla\psi + m\beta\psi$$

Equation of motion:

$$\frac{dX^\mu}{ds} \propto \overline{\psi}(X^\nu(s))\,\gamma^\mu\,\psi(X^\nu(s))$$

or, equivalently,

$$\frac{d\boldsymbol{X}}{dt} = \frac{\psi^*\boldsymbol{\alpha}\,\psi}{\psi^*\psi}(\boldsymbol{X},t) = \frac{\boldsymbol{j}}{\rho}(\boldsymbol{X},t)$$

world lines $=$ integral curves of current 4-vector field $j^\mu = \overline{\psi}\gamma^\mu\psi$
world lines are timelike or lightlike at every point

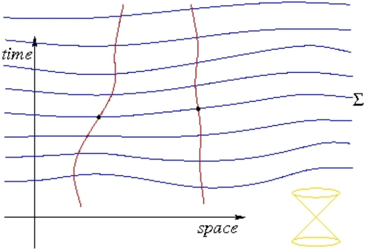
$|\psi|^2$ is conserved in every Lorentz frame.

BM$_{\mathcal{F}}$ [Dürr et al. 1999]

Consider N particles. Suppose that, for every $\Sigma \in \mathcal{F}$, we have ψ_Σ on Σ^N. $Q(\Sigma) = (Q_1 \cap \Sigma, \ldots, Q_N \cap \Sigma)$ = configuration on Σ

Equation of motion:

$$\frac{dQ_k^\mu}{ds} = \mathit{expression}\Big[\psi(Q(\Sigma))\Big]$$



Example for N Dirac particles

$\psi_\Sigma : \Sigma^N \to (\mathbb{C}^4)^{\otimes N}$. Equation of motion:

$$\frac{dQ_i^{\mu_i}(s)}{ds} \propto \overline{\psi}(Q(\Sigma))\big[\gamma^{\mu_1} \otimes \cdots \otimes \gamma^{\mu_N}\big]\psi(Q(\Sigma)) \prod_{k \neq i} n_{\mu_k}(Q_k \cap \Sigma)$$

with $n_\mu(x)$ = unit normal vector to Σ at $x \in \Sigma$.

Key facts about $\mathrm{BM}_{\mathcal{F}}$

Known in the case of N non-interacting Dirac particles, expected to be true also, say, one day, in full QED:

Equivariance

Suppose initial configuration is $|\psi|^2$-distributed. Then the configuration of crossing points $Q(\Sigma) = (Q_1 \cap \Sigma, \ldots, Q_N \cap \Sigma)$ is $|\psi_\Sigma|^2$-distributed (in the appropriate sense) on every $\Sigma \in \mathcal{F}$.

Predictions

[Lienert and Tumulka 1706.07074, Lill and Tumulka 2104.13861]

The detected configuration is $|\psi_\Sigma|^2$-distributed on *every* spacelike Σ.
No superluminal signaling.

As a consequence,

$\mathcal{F}$ is invisible, i.e., experimental results reveal no information about $\mathcal{F}$.
(A limitation to knowledge)

Thank you for your attention