Philosophical Questions in Statistical Mechanics, Quantum Mechanics, and Mathematics

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What kind of philosophical questions we will look at

- Does "philosophy" mean "vague talk"?
  Not here!
- Are philosophical questions irrelevant to the scientist?

> "How can we define precisely what knowledge is? Or causation?"
> (Maybe we don't need such definitions, just as we don't need a precise definition of what a spoon is.)

  Not the ones we will look at!
- Do scientists ever disagree with each other?
  You may be surprised! The questions we consider are quite controversial among scientists and philosophers.

What the issues are about

- in statistical mechanics:
  - the meaning of thermal equilibrium and entropy
  - the meaning and justification of using ensembles
- in quantum mechanics:
  - the interpretation
- in mathematics:
  - Do undecidable statements have truth values?

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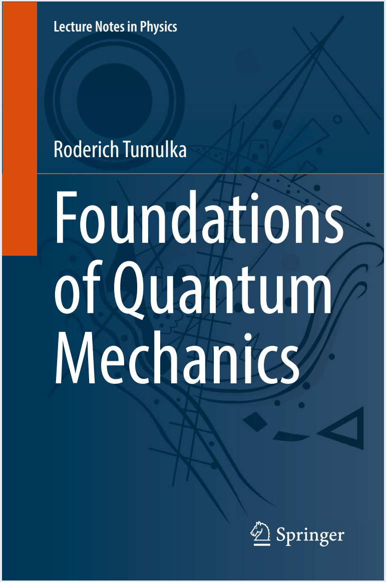


Statistical mechanics

What is thermal equilibrium? Two views

(in both quantum and classical mechanics)

Individualist view

A system is in thermal equilibrium if it is in an appropriate pure state (given by a wave function or point in phase space).

Ensemblist view

A system is in thermal equilibrium if it is in an appropriate statistical state (given by a density matrix or probability measure on phase space).

http://arxiv.org/abs/1003.2129
http://arxiv.org/abs/1903.11870

Individualist equilibrium in classical mechanics

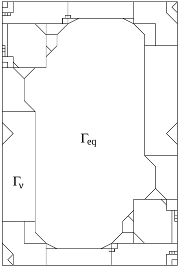
- State: point $X = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$ in phase space
- energy shell
  $\Gamma = \{X : E \leq H(X) \leq E + \delta E\}$
- depending on a choice of macro-variables, partition Γ into macro-states Γ_ν corresponding to different (small ranges of) values of the macro-variables,

$$\Gamma = \bigcup_\nu \Gamma_\nu$$

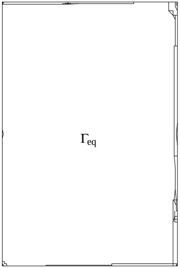
- one cell Γ_{eq} has the overwhelming majority of volume,

$$\frac{\mathrm{vol}\,\Gamma_{\mathrm{eq}}}{\mathrm{vol}\,\Gamma} \approx 1.$$

- Def: A system is in equilibrium $\Leftrightarrow$ its phase point lies in the set Γ_{eq}.



or rather:



Ensemblist equilibrium

Classical mechanics

Def: A system is in thermal equilibrium $\Leftrightarrow$ its state X is random with probability density

$$\rho = \rho_{can} = \frac{1}{Z} e^{-\beta H}$$

(canonical ensemble) or

$$\rho = \rho_{mc}$$

(micro-canonical ensemble).

Quantum mechanics

Def: A system is in thermal equilibrium $\Leftrightarrow$ its quantum state is a mixture with density matrix

$$\rho = \rho_{can} = \frac{1}{Z} e^{-\beta H}$$

(canonical ensemble) or

$$\rho = \rho_{mc}$$

(micro-canonical ensemble).

Ensemblist equilibrium vs. individualist equilibrium

Individualists argue that a classical system has an X but not a ρ.

The ensemblist formulation has

- the problem that an individual system can't be in equilibrium and
- the virtue of being precise—as it doesn't invoke the decomposition $\cup_\nu \Gamma_\nu$ or $\oplus_\nu \mathscr{H}_\nu$.

An ensemblist could regard an individual system as being in equilibrium by regarding the mixed state ρ as representing an observer's knowledge. But which observer? Is thermal equilibrium subjective?

Some ensemblists argue that the differences between observers are of no practical importance when N is large.

Individualists argue that the arbitrariness of the Γ_ν is of no practical importance when N is large.

Entropy

Entropy of ensemble

Classical: Gibbs entropy $S_{\mathrm{G}} = -k \int_{\mathbb{R}^{6N}} dq\, dp\, \rho(q,p) \log \rho(q,p)$

Quantum: von Neumann entropy $S_{\mathrm{vN}} = -k\ \mathrm{tr}(\rho \log \rho)$

Problematical for an individualist because $S = 0$ for a pure state.

Entropy of individual system

Classical: partition $\Gamma = \bigcup_\nu \Gamma_\nu$ of phase space
Boltzmann entropy $S_{\mathrm{B}}(X) = k \log \mathrm{vol}(\Gamma_\nu)$ if $X \in \Gamma_\nu$

Quantum: orthogonal decomposition $\mathscr{H} = \bigoplus_\nu \mathscr{H}_\nu$ of Hilbert space
quantum Boltzmann entropy $S_{\mathrm{E}}(\psi) = k \log \dim \mathscr{H}_\nu$ if $\psi \in \mathscr{H}_\nu$

Individualists are unhappy with S_{G} and S_{vN}

The individualist considers an individual closed system in a pure state.

Entropy of ensemble

Classical: Gibbs entropy $S_{\mathrm{G}} = -k \int_{\mathbb{R}^{6N}} dq\, dp\, \rho(q,p) \log \rho(q,p)$

Quantum: von Neumann entropy $S_{\mathrm{vN}} = -k\ \mathrm{tr}(\rho \log \rho)$

- What is the entropy of an individual system? Does ρ then encode the knowledge of an observer? (Which observer?) Is entropy then subjective, a measure of somebody's knowledge?
  [Yes, said Leo Szilard 1929].
- Even for an ensemble (say, $n = 10^6$ systems with wave functions ψ_i, so $\rho = n^{-1} \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|$), the individualist finds S_{vN} problematical: Can't we create arbitrary ensembles? Ones in which the width of the ensemble has nothing to do with the thermodynamic entropy of the ψ_i?

Quantum mechanics

Everbody agrees on

rules for empirical predictions: unitary evolution, Born's rule, collapse rule

What we may want more

We would like to know: how does nature do it, what is the explanation of the observed outcome statistics? What actually happens?

The Copenhagen interpretation

- uses “classical pictures,” but doesn’t take them seriously
- insists that there are no particle positions
- refuses to provide clear claims about what is real instead
- demands that we focus on observables

"Positivists" vs "realists"

Here, "positivism" is the view that

- a statement is unscientific or even meaningless if it can't be tested experimentally
- an object is not real if it can't be observed
- a variable is not well defined if it can't be measured.

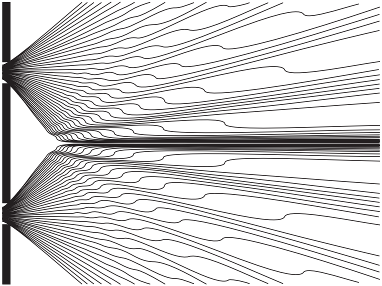
Feynman didn't like that:

"Does this mean that my observations become real only when I observe an observer observing something as it happens? This is a horrible viewpoint. Do you seriously entertain the thought that without observer there is no reality?" (1959)

Richard Feynman (1918–1988)

Realist proposal: Bohm's trajectories

The natural trajectories are the flow lines of the probability 4-current $(\rho, j) = (|\psi|^2, \mathrm{Im}\, \psi^* \nabla \psi)$. Bohm's (1952) proposal is to take them seriously, to hypothesize that electrons are literally point particles.



Drawn by G. Bauer after Philippidis et al.

(Most contemporaries hated that. They had spent years practicing Copenhagen philosophy, and now difficult philosophy might be replaced by a simple equation.) The configuration Q_t is $|\psi_t|^2$-distributed at all t.

It works.

Inhabitants of a universe governed by Bohmian mechanics would make observations in agreement with the rules of QM.

Consequence

No empirical test between Bohmian mechanics and Copenhagen QM is possible.

So what is a theory like Bohm's good for?

- For understanding
  (cf. Copernicus vs Ptolemy)
- For precise reasoning
  (cf. mathematicians' definitions of the integral)

Another realist proposal: Spontaneous collapse

Key idea:

The Schrödinger equation is only an approximation, valid for systems with few particles ($N < 10^4$) but not for macroscopic systems ($N > 10^{23}$). The true evolution law for the wave function is non-linear and stochastic (i.e., inherently random) and avoids superpositions (such as Schrödinger's cat) of macroscopically different contributions.

Put differently, regard the collapse of ψ as a physical process governed by mathematical laws.

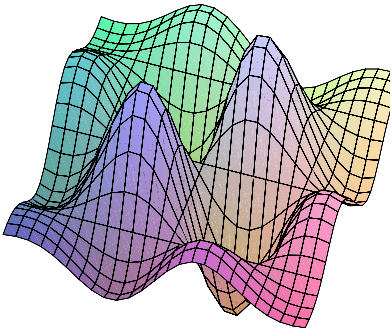
GianCarlo Ghirardi (1935–2018)

Explicit equations by Ghirardi, Rimini, and Weber [1986]

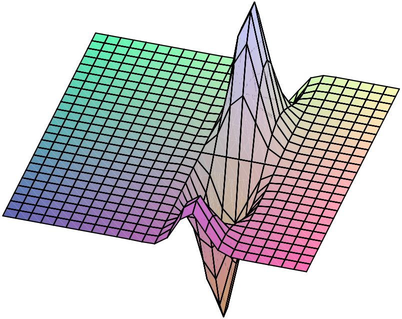
The predictions of the GRW theory deviate very very slightly from the quantum formalism. At present, no experimental test is possible.

GRW's spontaneous collapse

before the "spontaneous collapse":



and after:



Mathematics

Example: the continuum hypothesis

Continuum hypothesis (CH) [Georg Cantor 1878]

For every infinite subset S of $\mathbb{R}$, there exists either a bijection $\varphi : S \to \mathbb{N}$ or a bijection $\varphi : S \to \mathbb{R}$.

Theorem [Kurt Gödel 1938]

The negation of CH ($\neg$CH) cannot be deduced from the standard axioms of set theory (Zermelo-Fraenkel axioms $=:$ ZF).

Theorem [Paul Cohen 1963]

CH cannot be deduced from ZF.

Thus, CH is undecidable from ZF.

In particular, one could use ZF $\cup$ CH as axioms of set theory or else use ZF $\cup$ $\neg$CH as axioms, without ever running into a contradiction.

Truth value

On the other hand, we understand what CH is saying; intuitively, it is a meaningful statement and thus should have a truth value (i.e., be either true or false).

"Realist" or "Platonist" position

CH has a truth value, even if we don't know it and perhaps will never know it. ZF is simply an incomplete/insufficient set of axioms.

"Positivist" position

We can choose whether to take CH to be true or false, there is no fact "out there" about whether it is "really" true. It is a matter of definition. The only scientific meaning one could give to the concept of "true" for a mathematical statement is that it can be proved, so neither CH nor $\neg$CH is true.

Mathematicians are divided about this issue.

"Reasonless truths"

Tim Maudlin (philosopher and Platonist) argued that we should expect that many, many true mathematical statements can't be proved from standard axioms (such as *Principia Mathematica* $=:$ PM [Bertrand Russell and Alfred Whitehead 1913]) with the following example [2010]:

- Define that $x, y \in \mathbb{R}$ "match" ($x \sim y$) if and only if in their decimal expansion after the decimal point, they have equal first digits, or equal digits 2 and 3, or equal digits 4–6, or equal digits 7–10, or etc.
  $3.1|41|592|6535|89\ldots$
  $3.8|35|219|6535|30\ldots$
- Independent random numbers (uniformly from $[0,1]$) match with probability $1 - \frac{9}{10} \cdot \frac{99}{100} \cdot \frac{999}{1000} \cdots \approx 0.11$.
- Consider statements S of the type $54^{1/18} \sim \pi^{78}$ or $\sin(29 + \sqrt{3}) \sim \sqrt{\tanh(1/12)}$.
- Consider lots of such examples; $\approx 89\%$ of them should be false.
- If S is true, it can be proven. But if S is false, then one would expect there is no deeper reason for that, it just so happens. Therefore, one would expect that $\neg S$ can't be deduced from PM (or any set of intuitively plausible axioms), even though we will usually not have proof that it can't (in contrast to Gödel's 1931 example of a statement that is undecidable from PM).

Similar issues

Axiom of choice (AC)

For every non-empty family S of non-empty sets, there is a "choice function" f on S, i.e., such that $f(A) \in A$ for every $A \in S$.

AC has also been proven to be undecidable from ZF. Most Platonists believe that AC is true.

Non-standard analysis [Abraham Robinson 1969]

If the expression "power set" is not required to mean power set, then there exist inequivalent models of the axioms of $\mathbb{R}$. Some contain elements greater than every natural number.

Is this relevant for practical applications?

Not very much. The axiom of choice is regularly used in functional analysis. But mainly, the issue is relevant to understanding vs misunderstanding math.

Thank you for your attention