Introduction to Relativity

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Seminar on Foundations of QM, 22 April 2026

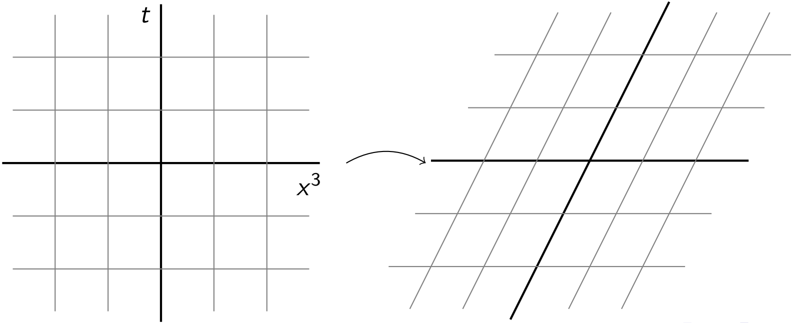
Special relativity summarized in one sentence

Space-time is an affine Minkowski space, that is, an affine space with metric of signature $+---$.

Galilean relativity

- space-time point $(t, \boldsymbol{x})$
- Galilean boost

$$\begin{aligned} t' &= t \\ x_1' &= x_1 + v_1 t \\ x_2' &= x_2 + v_2 t \\ x_3' &= x_3 + v_3 t \end{aligned}$$



Galilean relativity

- Both Newtonian and Bohmian mechanics are invariant under Galilean transformations (= boosts, rotations, composites).
- That is, transformation of a solution yields another solution (with a possibly different wave function in Bohmian mechanics).
- It is in principle impossible to determine empirically how fast any object (such as the Earth) is moving.
- Two classic views:

Newton

Space is at rest. Therefore, there is a fact about the actual velocity of an object, but this velocity cannot be measured empirically.

Leibniz

There are no facts in nature about the absolute velocity of an object, only about relative velocities. There is no fact about the identity of space points at different times.

Leibniz' view

Space-time as a 4-dimensional real affine space A equipped with an equivalence class of affine-linear functions $t : A \to \mathbb{R}$ ("time"), where equivalence means that two functions differ by addition of a real constant (so there is no fact about which time is time 0), and the structure of a Euclidean space on each level set of t ("time slice") such that every translation of A is an isometry on each time slice.

Einstein's special relativity

In contrast to Newtonian and Bohmian mechanics, Maxwell's equations, the fundamental equations governing the electromagnetic field according to classical electrodynamics, are not invariant under Galilean transformations. However, they are invariant under a similar family of linear transformations $\mathbb{R}^4 \to \mathbb{R}^4$, the Lorentz transformations.

Einstein's principle of relativity (1905)

The true symmetry of space-time is given by the Lorentz transformations. All laws of nature are invariant under Lorentz transformations.

Minkowski space

Definition

Minkowski space is a 4d real vector space M equipped with a symmetric bilinear form $(\cdot,\cdot) : M \times M \to \mathbb{R}$ (called *metric*) which in a suitable basis b_0, b_1, b_2, b_3 is given by

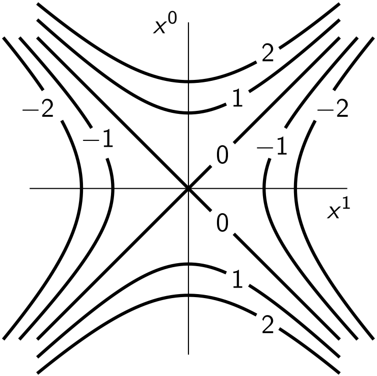
$$(x, y) = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 = \sum_{\mu,\nu=0}^{3} x^\mu y^\nu \eta_{\mu\nu}$$

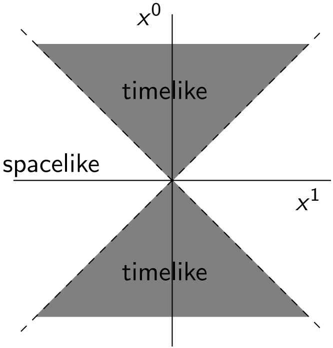
$$\text{with } \eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} .$$

Such a basis is called a Lorentz frame.

indefinite; $x^0 = ct$ with $c =$ speed of light; we write x^μ for x; Einstein sum convention

$$x^\mu y^\nu \eta_{\mu\nu} \text{ instead of } \sum_{\mu,\nu=0}^{3} x^\mu y^\nu \eta_{\mu\nu} .$$


x^0
x^1
2
1
0
0
1
2
-2
-1
-1
-2


x^0
x^1
timelike
spacelike
timelike

Lorentz transformations

Definition

A Lorentz transformation is a linear mapping $\Lambda : M \to M$ that preserves the metric,

$$(\Lambda x, \Lambda y) = (x, y) \quad \forall x, y \in M\,.$$

Alternatively, it is the coordinate expression of such a mapping relative to a Lorentz frame, i.e., a matrix $\Lambda^\mu{}_\nu$ such that

$$\Lambda^\mu{}_\lambda \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\lambda\rho} \text{ or } \Lambda^t \eta \Lambda = \eta\,.$$

These matrices (also called "pseudo-orthogonal") form a group called the Lorentz group or $O(1, 3)$.

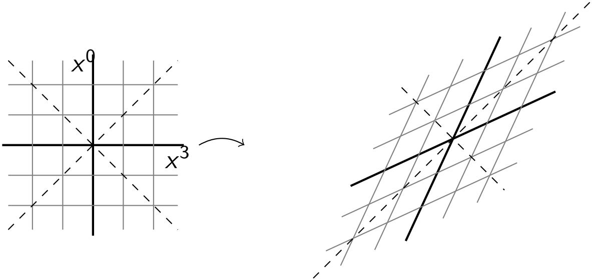
Lorentz transformations

Example

The Lorentz boost in the x^3 direction with rapidity ξ is the matrix

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh\xi & & & \sinh\xi \\ & 1 & & \\ & & 1 & \\ \sinh\xi & & & \cosh\xi \end{pmatrix}.$$

When composing Lorentz boosts in x^3 direction, rapidities add.



Consequences

- Analog of Newton's view: space is at rest, but we cannot find out which frame that is.
- Analog of Leibniz's view: there is no absolute rest frame.
- However, different Lorentz frames disagree about which space-time points are simultaneous.
- This suggests that there is no fact about which space-time points really are simultaneous, or about the temporal order of spacelike separated points.
- Einstein was a Leibnizian: Space and time as such do not exist; only space-time exists, and it is an affine Minkowski space.

Affine spaces

What is an affine space A?

- One definition: An affine space A is a translate $a+U$ of a subspace U in a vector space V.
- A different type of definition (relevant here): An affine space is a set A together with an action of (the additive group of) a vector space U that is simply transitive.
- A group action $\text{act} : G \to S_A = \{f : A \to A \text{ bij.}\}$ is called *(simply) transitive* if for any $x, y \in A$ there is a (unique) $g \in G$ with $\text{act}_g(x) = y$.
- One can think of U as the set of translations of A, or of A as "a copy of U without marking the origin."
- Note that a subspace $U \subset V$ acts simply transitively on any translate $a + U$.
- We write $u = b - a$ if $\text{act}_u(a) = b$.

Affine Minkowski space

How to define Euclidean space E_n

One definition: E_n is a metric space (with distance function $d_{E_n} : E_n \times E_n \to [0,\infty)$) isometric to $\mathbb{R}^n$ with $d(\boldsymbol{x},\boldsymbol{y}) = (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$.
Another one: E_n is an affine space whose vector space U_n is equipped with a positive definite symmetric bilinear form.

Definition of affine Minkowski space A

A is an affine space whose vector space U is a Minkowski space (i.e., is equipped with a symmetric bilinear form of signature $+ - - -$).

Remark: Another, equivalent definition: A is a Lorentzian 4-manifold isometric to Minkowski space.

pseudo-Cartesian coordinates $A \to \mathbb{R}^4$ ("inertial frame," "~~observer~~")

Choose origin $\mathbf{0} \in A$, Lorentz frame in U.
Coordinates($\boldsymbol{a}$) := coefficients($\boldsymbol{a} - \mathbf{0}$).

Poincaré group

The Poincaré transformations are those of the form

$$x^\mu \mapsto a^\mu + \Lambda^\mu{}_\nu x^\nu$$

with Λ a Lorentz transformation.

- They are also called "inhomogeneous Lorentz transformations."
- They form a group, the Poincaré group.
- They correspond to changes of pseudo-Cartesian coordinates in affine Minkowski space.
- They are the isomorphisms of affine Minkowski space A (i.e., mappings $\varphi : A \to A$ such that there is a pseudo-orthogonal $\Lambda : U \to U$ with $\mathrm{act}_{\Lambda(u)} \circ \varphi = \varphi \circ \mathrm{act}_u$). (Equivalently, the isometries of the Lorentzian manifold A to itself.)

Dual space

- covector = linear form $V \to \mathbb{R}$ (say, $\dim V < \infty$)
- $V' = \{\text{covectors}\}$ dual space
- dual basis is defined by $\hat{b}_i(b_j) = \delta_{ij}$
- inner product (positive definite) defines isomorphism (linear bijection) $V \to V'$, $v \mapsto v\cdot$
- same coefficients relative to $\{\hat{b}_i\}$ if $\{b_j\}$ is orthonormal
- Minkowski metric still defines isomorphism $V \to V'$, $v \mapsto (v, \cdot)$
- but coefficients change:

$$x_\mu = \eta_{\mu\nu} x^\nu \,,$$

$$(x, y) = x^\mu y_\mu = x_\mu y^\mu$$

- Gradient is a covector: $\partial f / \partial x^\mu = \partial_\mu f$

Arc length

$[s_1, s_2] \to M : s \mapsto X(s)$ timelike curve, arc length (proper length, invariant length)

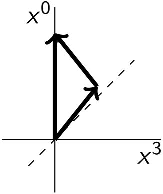
$$c\tau := \int_{s_1}^{s_2} ds \sqrt{\frac{dX^\mu}{ds}\frac{dX_\mu}{ds}}\,.$$

Example

If $x - y$ is future-timelike, then the straight line segment from y to x has invariant length $c\tau = \sqrt{(x^\mu - y^\mu)(x_\mu - y_\mu)}$.

Rule

Invariant length$/c$ is time along the curve as measured by ideal clocks.



Twin example
time dilation
arc length parameterization $X(\tau)$

Classical electrodynamics: paradigm of relativistic theory

The world consists of space-time, an electromagnetic field, and N particles. Space-time is an affine Minkowski space M, and each particle $i \in \{1, \dots, N\}$ has a path $\tau \mapsto X_i(\tau)$ ("world line"). Relativistic Newtonian equation of motion with Lorentz force

$$m_i \frac{d^2 X_{i\mu}}{d\tau^2} = q_i F_{\mu\nu}(X_i(\tau)) \frac{dX_i^\nu}{d\tau} \,,$$

$m_i > 0$ and $q_i \in \mathbb{R}$ ("charge") constants, $F : M \to M \otimes M$ is called the electromagnetic field. It is anti-symmetric, $F_{\nu\mu} = -F_{\mu\nu}$, and governed by Maxwell's equations

$$\begin{aligned} \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= 0 \\ \partial^\mu F_{\mu\nu} &= 4\pi J_\nu \,, \end{aligned}$$

where J is called the charge current density and given by

$$J^\nu(x) = \sum_{i=1}^{N} q_i \int d\tau \, \delta^4(x - X_i(\tau)) \,.$$

Classical electrodynamics

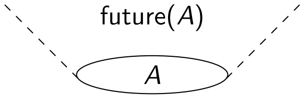
Problem: inconsistent

For a charge current that is concentrated on curves, every solution $F_{\mu\nu}$ of the Maxwell equations will diverge on these curves, in fact at the rate

$$F_{\mu\nu}(x) \sim \frac{1}{r^2}$$

The equation of motion demands that we evaluate $F_{\mu\nu}$ exactly at $x = X_i(\tau)$, where it is not defined.

We ignore this problem and pretend the theory was well defined.
Each eq is Lorentz invariant.



The theory is local:
Events at spacelike separation cannot influence each other.
Unlike quantum mechanics, by Bell's theorem.

Length contraction

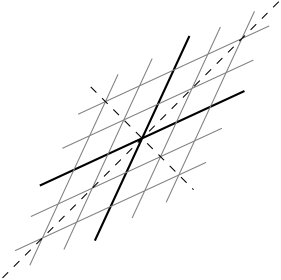
Suppose a "measuring rod at rest" is a time-independent solution of classical electrodynamics. Suppose

- the "left end" has $x^3 = 0$ for all x^0,
- the "right end" $x^3 = L$ for all x^0.

Then in a different frame, by Lorentz transformation,

- the left end is at $x'^3 = (\tanh\xi)x'^0 = vt'$,
- the right end at $x'^3 = (\tanh\xi)x'^0 + (\cosh\xi)^{-1}L = vt' + \sqrt{1-v^2/c^2}\,L$.

Thus, the coordinate difference is $\sqrt{1-v^2/c^2}\,L < L$.



Consequence

A rod that is uniformly moving and has settled into the stable state is shorter!

4-volume

Theorem. *Lorentz transformations* Λ *have* $\det(\Lambda) = \pm 1$.

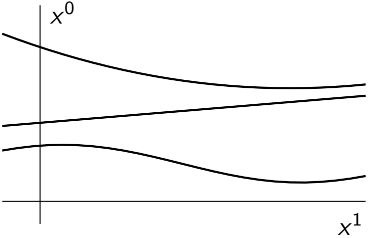
Proof. Since $\Lambda^t \eta \Lambda = \eta$, $\quad \det(\Lambda^t)\det(\eta)\det(\Lambda) = \det(\eta)$. ☐

Corollary. Any two pseudo-Cartesian coordinate systems map the Lebesgue volume measure on $\mathbb{R}^4$ to the same measure on affine Minkowski space M (called 4-volume).

Cauchy surface

Definition

A Cauchy surface is a subset Σ of M which is intersected by every inextensible timelike-or-lightlike curve exactly once.



More or less the same as "spacelike hypersurface."

Cauchy surfaces often play the role of "instants of time," for example for specifying initial data ("Cauchy data"), e.g. of $F_{\mu\nu}$.

Volume on a Cauchy surface

Parameterize a Cauchy surface Σ by a mapping $\Phi : \mathbb{R}^3 \supseteq B \to \Sigma$. Natural measure of surface area, denoted by $V(d^3x)$ in the following, where d^3x is a 3d surface element:

$$\int_\Sigma V(d^3x)\, f(x)\,.$$

Proposition

In any bijective parameterization $\Phi : \mathbb{R}^3 \to \Sigma$ (so $x = \Phi(\boldsymbol{y})$),

$$\int_\Sigma V(d^3x)\, f(x) = \int_{\mathbb{R}^3} d^3\boldsymbol{y}\, \sqrt{-\det({}^3g(\boldsymbol{y}))}\, f(\Phi(\boldsymbol{y}))\,,$$

where ${}^3g$ is the Riemann metric on Σ,

$${}^3g_{ij} = \frac{\partial \Phi^\mu}{\partial y^i} \frac{\partial \Phi^\nu}{\partial y^j} \eta_{\mu\nu}$$

for $i, j \in \{1, 2, 3\}$.

A brief look at general relativity

Suppose the metric is now position-(and time-)dependent: Let u, v be two (tangent) vectors based at the space-time point x, and suppose

$$(u, v) = u^\mu v^\nu g_{\mu\nu}(x)\,.$$

That leads to general relativity.

For comparison, on a surface Σ in Euclidean space, the arc length of a curve $X : [s_1, s_2] \to \Sigma$ is

$$\int_{s_1}^{s_2} ds \sqrt{\frac{dX^i}{ds}\frac{dX^j}{ds} g_{ij}(X(s))}\,,$$

and $g_{ij}(x)$ is called the Riemann metric.

$g_{\mu\nu}(x)$ can be diagonalized to diag$(+1, -1, -1, -1)$, but not necessarily by tangent vectors of coordinate lines. (Space-time is a Lorentzian manifold.)
The curvature of $g_{\mu\nu}(x)$ is in general non-zero (while in affine Minkowski space it is 0). (Space-time is curved.)

Thank you for your attention