The Spatial Quantum Zeno "Paradox"

Tumulka's
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1. THE "PARADOX" in simple terms

2. WHY "ZENO"?

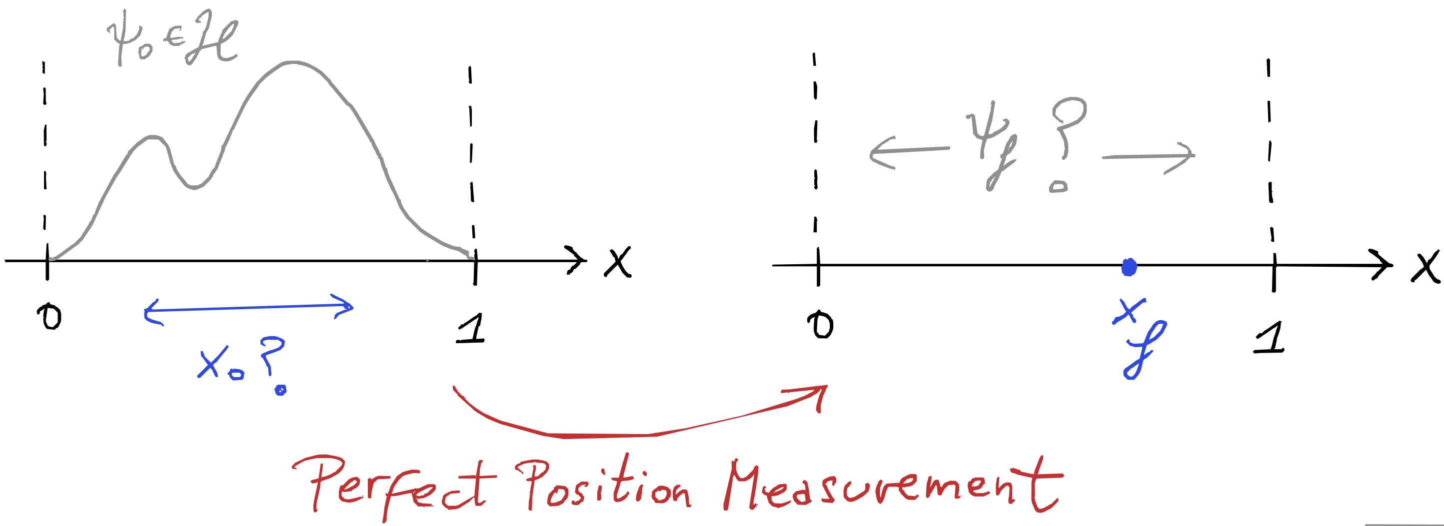
3. THE GENERAL STATEMENT

4. SOLUTION to the PARADOX:

A NEW TYPE of QUANTUM STATE?

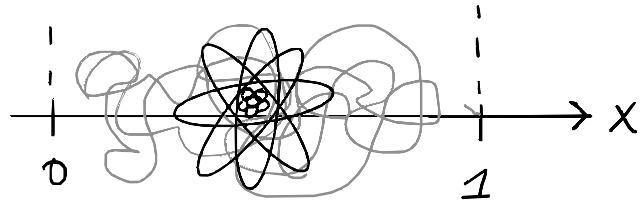
1. The Paradox in Simple Terms

"A particle precisely found in physical space is nowhere to be found in Hilbert space"

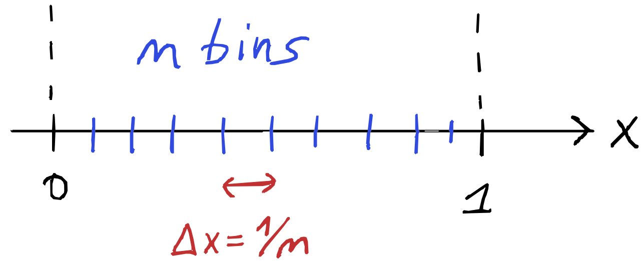


RECIPE to CHECK the PARADOX

Step 1: Confine a Quantum particle to a BOX

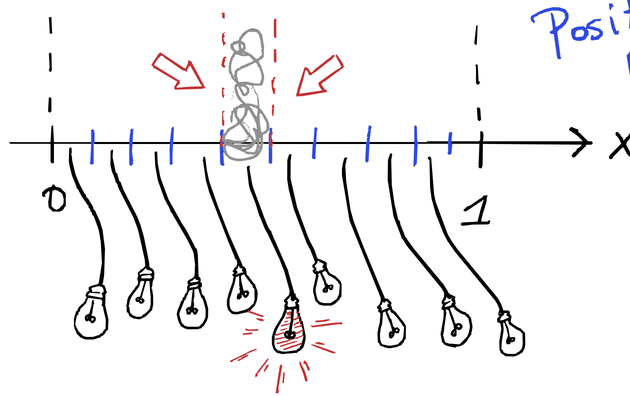


Step 2: Divide the box in m BINS



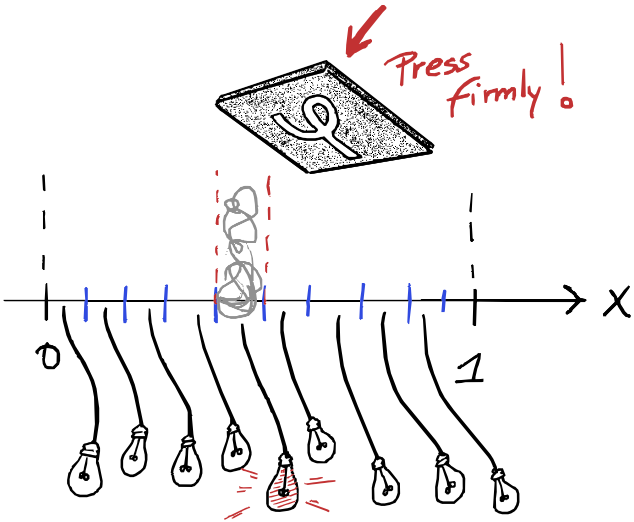
Step 3: Find the bin WHERE it is located

(causes "collapse" of wavefunction)



Position Measmt. of prec. Δx

Step 4: Find if particle "projects" to $\phi \in \mathcal{H}$ i.e., make "yes-no" projective measurement of $|\phi\rangle\langle\phi|$



- The result:

$$\lim_{\Delta x \to 0} \mathbb{P}\left(\begin{matrix}\text{particle} \\ \text{"projects" to } \phi\end{matrix}\right) = 0 \quad \forall \phi \in \mathcal{H}.$$

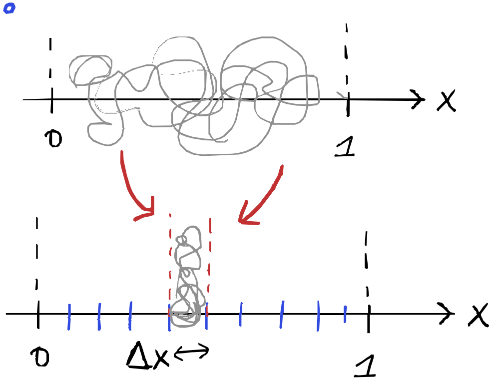
PRECISE LOCATION KNOWN $\Rightarrow$ FITS NO MOULD φ!

The Paradox

- $\nexists\, \psi \in \mathcal{H}$ & $\nexists$ density matrix $\hat{\rho}$ on $\mathcal{H}$

s.th. $\mathbb{P}\left(\begin{array}{c}\text{"yes-no" projective} \\ \text{measurement of } |\phi\rangle\langle\phi| \\ \text{yields "yes"}\end{array}\right) = 0 \quad \forall \phi \in \mathcal{H}$

- So:



After perfect position measurement, a particle CANNOT be described by a $\psi \in \mathcal{H}$ or dens. mat $\hat{\rho}$ on $\mathcal{H}$!

Particle "ESCAPES" Hilbert space!?

CLAIM: $\nexists\ \psi \in \mathcal{H}$ & $\nexists\ \hat{\rho}$ dens mat on $\mathcal{H}$

s.th. $\mathbb{P}\left(\begin{array}{c}\text{"yes-no" projective} \\ \text{measurement of } |\phi\rangle\langle\phi| \\ \text{yields "yes"}\end{array}\right) = 0 \quad \forall\, \phi \in \mathcal{H}.$

Check: • For unit $\psi \in \mathcal{H}$ choose $\phi = \psi$. Then,

$$\mathbb{P}(|\phi\rangle\langle\phi| \text{ yes}) = |\langle\phi|\psi\rangle|^2 = \|\psi\|^2 = 1 \neq 0.$$

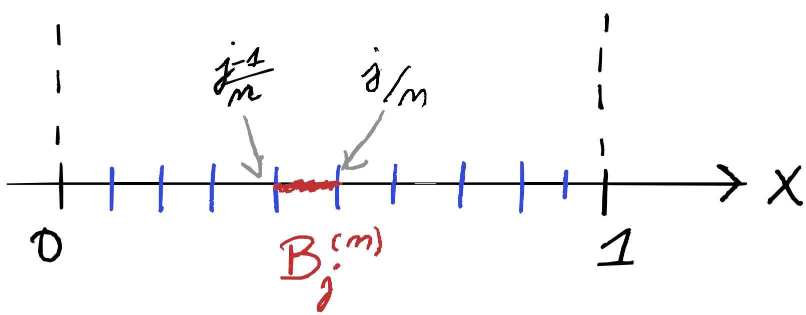
• For density mat $\hat{\rho}$, diagonalize $\hat{\rho} = \sum_{j=1}^{\infty} p_j |\psi_j\rangle\langle\psi_j|$

$$\Rightarrow \left\{\begin{array}{c}\text{choose} \\ \phi = \psi_j\end{array}\right\} \Rightarrow \mathbb{P}\left(\begin{array}{c}|\phi\rangle\langle\phi| \\ \text{yes}\end{array}\right) = \operatorname{tr}(|\phi\rangle\langle\phi|\hat{\rho}) = p_j \neq 0.$$

The Recipe in Math Terms

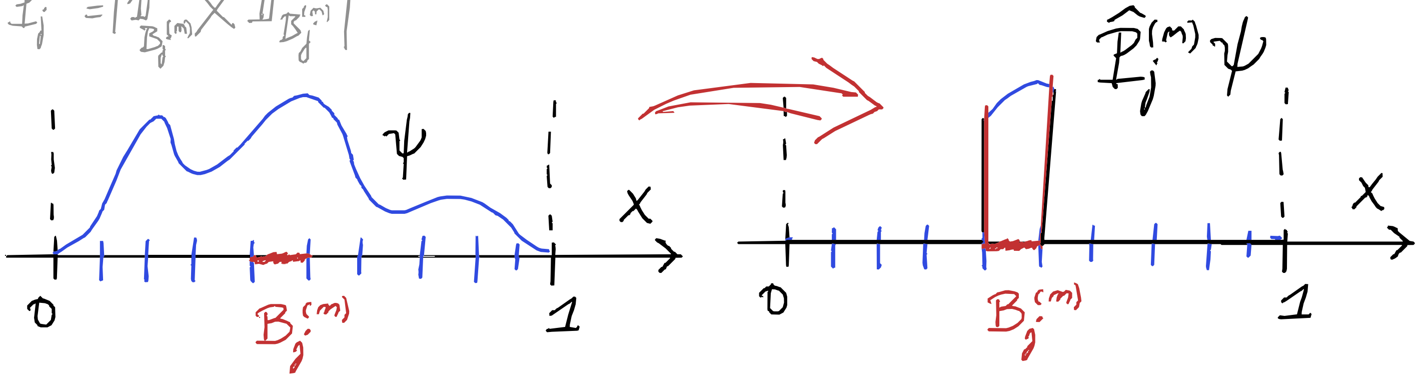
Step 1: Confine a Quantum particle to a box $[0,1]$ $\rightsquigarrow$ Prepare a quantum particle with state $\psi \in L^2([0,1], dx)$

Step 2: Divide the box in m bins $\rightsquigarrow$ $B_j^{(m)} := \left[\frac{j-1}{m}, \frac{j}{m}\right) \quad j \in \{1, \dots, m\}$



Step 3: Find the bin where it is located $\leadsto$ **Discretized position PVM:**

$$\mathbb{P}_j^{(m)} = |\mathbb{1}_{B_j^{(m)}}\rangle\langle \mathbb{1}_{B_j^{(m)}}|$$



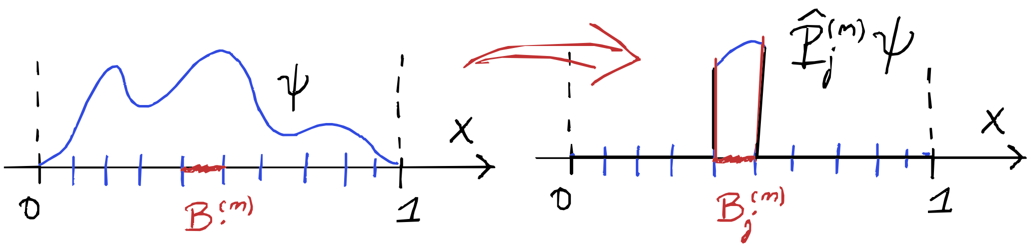
$$\left(\hat{\mathbb{P}}_j^{(m)}\psi\right)(x) := \begin{cases} \psi(x) & \text{if } x \in B_j^{(m)} \\ 0 & \text{if } x \notin B_j^{(m)} \end{cases}$$

Imprecise Position Observable Op.

$$\hat{X}^{(m)} := \sum_{j=1}^{m} \frac{j}{m}\, \hat{\mathbb{P}}_j^{(m)}$$

Step 3: Find the bin where it is located $\Big\} \rightsquigarrow$ Projective measurement of $\hat{X}^{(m)}$

$$\psi \xrightarrow[\text{if outcome is: } X = j/m]{\text{COLLAPSE}} \frac{\hat{P}_j^{(m)}\psi}{\|\hat{P}_j^{(m)}\psi\|} =: \psi'_{j,m}$$



Step 4: $|\phi\rangle\langle\phi|$ Yes-no measurement

$$\mathbb{P}\left(\begin{matrix}|\phi\rangle\langle\phi| \\ \text{yes}\end{matrix} \;\middle|\; X = j/m\right) = |\langle\phi|\psi'_{j,m}\rangle|^2$$

- Prob. irrespective of posit. result: 1. [redacted]

$$\mathbb{P}\left(|\phi\rangle\langle\phi| \text{ yes}\right) = \sum_{j=1}^{m} \mathbb{P}\left(|\phi\rangle\langle\phi| \text{ yes } \& \ X = j/m\right) =$$

$$= \sum_{j=1}^{m} \mathbb{P}\left(|\phi\rangle\langle\phi| \text{ yes} \,\middle|\, X = j/m\right) \cdot \mathbb{P}\left(X = j/m\right) =$$

$$= \sum_{j=1}^{m} \frac{|\langle\phi|\hat{\mathcal{P}}_j^{(m)}\psi\rangle|^2}{\|\hat{\mathcal{P}}_j^{(m)}\psi\|^2} \cdot \|\hat{\mathcal{P}}_j^{(m)}\psi\|^2 =$$

$$= \sum_{j=1}^{m} |\langle\phi|\hat{\mathcal{P}}_j^{(m)}\psi\rangle|^2$$

- Claim:

$$\lim_{m\to\infty} \sum_{j=1}^{m} |\langle\phi|\hat{\mathcal{P}}_j^{(m)}\psi\rangle|^2 = 0 \qquad \forall \psi, \phi \in \mathcal{H}$$

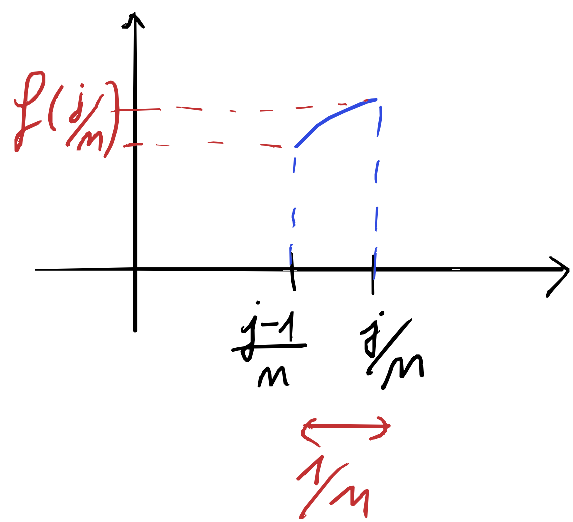
Physicist-way to "prove" it

- For regular enough $\psi, \phi : [0,1] \to \mathbb{C}$ & $m \gg 1$,

$$\langle \phi, \hat{P}_j^{(m)} \psi \rangle = \int_{\frac{j-1}{m}}^{j/m} \overline{\phi(x)}\, \psi(x)\, dx \approx \frac{1}{m}\, \overline{\phi(j/m)}\, \psi(j/m).$$

- Then:

$$\sum_{j=1}^{m} \left| \langle \phi, \hat{P}_j^{(m)} \psi \rangle \right|^2 \approx$$



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- Then:

$$\sum_{j=1}^{m} |\langle \phi, \hat{P}_j^{(m)} \psi \rangle|^2 \approx \sum_{j=1}^{m} \frac{1}{m^2} |\phi(j/m)|^2 \cdot |\psi(j/m)|^2$$

$m \gg 1$??

- Note Riemann sum

$$\frac{1}{m} \int_0^1 |\phi(x)|^2 |\psi(x)|^2 dx$$

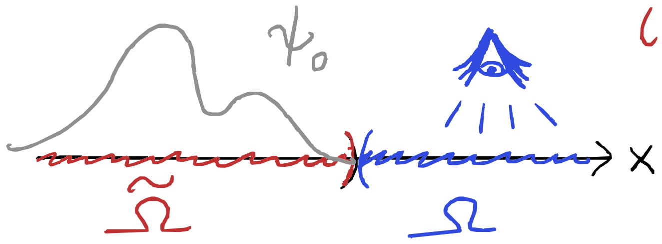
- Take $\lim_{m \to \infty}$ $\rightsquigarrow 0$ //.

True prove ++ involved!
(Lebesg. diff. thm.)

2. WHY "ZENO"?

THE ORIGINAL QUANTUM ZENO PARADOX

(i) Let qtm particle in $\tilde{\Omega}$ at $t=0$, whose free time evol. would make $\mathbb{P}$ to find it in Ω eventually $\neq 0$.



(ii) Every Δt check if already arrived to Ω.

This is ARRIVAL TIME MEASMT. of precision Δt

- ISSUE?

$$\lim_{\Delta t \to 0} \mathbb{P}\left(\begin{array}{l}\text{Find particle}\\ \text{in } \Omega \text{ at time } t\end{array}\right) = 0 \quad \forall t.$$

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"Motion (or time) is an illusion (Zeno)". Proof by contradict. (Zeno): At any single instant, an arrow in flight is at rest, occupying a specific space.* Since time is composed of such motionless instants, there can be no motion.

- The arrow is neither moving to where it is (bec. it's already there) nor where it is not (bec. no time elapses for it to move there).

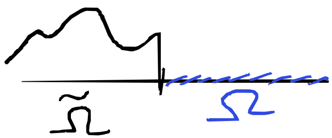
(- Assumes time is composed of "nows" (moments) and NOTHING ELSE.)

ALTVE PHRASING:

- An arrow in apparent motion at each moment travels no distance (bec. it occupies an equal space for the whole instant). But the entire period of "motion" contains only instants, all of which contain an arrow at rest, and so, the arrow cannot be moving!

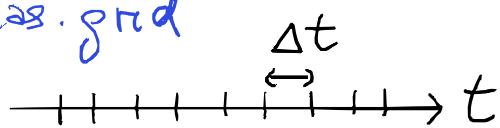
ANALOGY: Given $\psi_0 \in L^2(\mathbb{R})$,

- Prepare



by project $|\mathbb{1}_{\tilde{\Omega}}\rangle\langle\mathbb{1}_{\tilde{\Omega}}|$

- Meas. grid

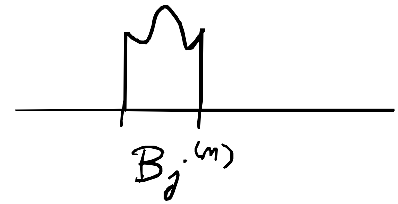


- Check $|\phi\rangle\langle\phi|$ for any $\phi \in L^2(\Omega)$ (at each t in grid)

- Then

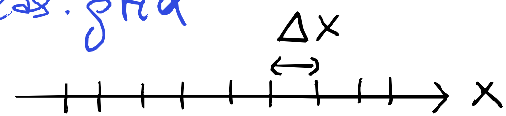
$$\lim_{\Delta t \to 0} \mathbb{P}\left(\begin{array}{c} |\phi\rangle\langle\phi| \text{ "yes"} \\ \text{at some } t \end{array} \right) = 0$$

- Prepare



by project $|\mathbb{1}_{B_j^{(m)}}\rangle\langle\mathbb{1}_{B_j^{(m)}}|$

- Meas. grid



- Check $|\phi\rangle\langle\phi|$ for any $\phi \in L^2([0,1])$

- Then

$$\lim_{\Delta x \to 0} \mathbb{P}\left(\begin{array}{c} |\phi\rangle\langle\phi| \text{ "yes"} \\ \text{"at some } \phi\text{"} \end{array} \right) = 0$$

- Thus, we called it

the SPATIAL Quantum

Zeno Effect / "Paradox"!

"A particle precisely found in physical space

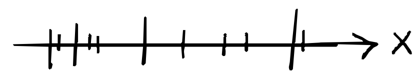
is nowhere to be found in Hilbert space"

3. The General Statement

Result & conseqs. still true if:

(i) Replace initial ψ by a density matrix $\hat{\rho}$

(ii) Take grids with uneven spacing



(iii) Replace $[0,1]$ by box $Q = [0,1]^d$ of dim $d \in \mathbb{N}$

(iv) Replace Q by whole $\mathbb{R}^d$ configurat. space

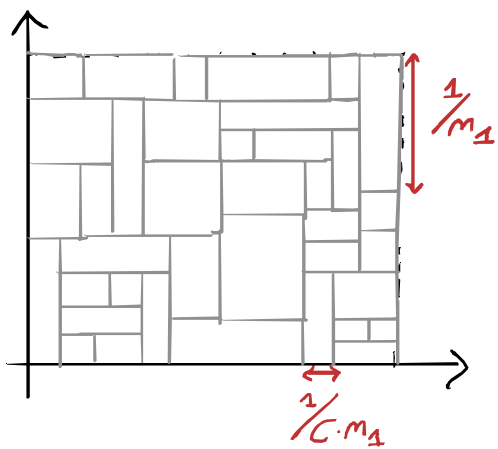
• **DEF.: GRID SCHEME for $[0,1)^d$**

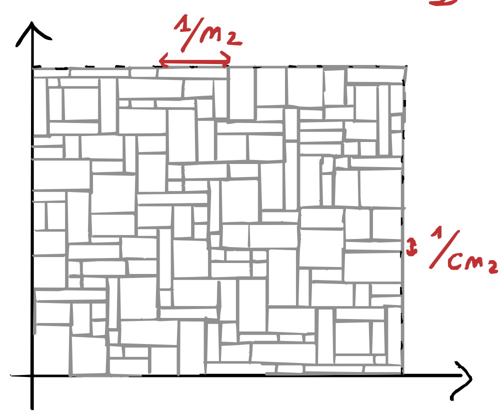
for each $m \in \mathbb{N}$ a PARTITION of $[0,1)^d$ into

$\left\{ N_m \text{ RECTANGULAR BINS of EDGE LENGTHS in ORDER } 1/m \right\} \Rightarrow \left(N_m \xrightarrow[m \to \infty]{} \infty \right)$

Example $d=2$

$m_1 < m_2$





• Similarly, def. GRID SCHEME for $\mathbb{R}^d$.

- JOINT DISCRETIZED POSITION PVM

$$\hat{\mathbb{P}}_j^{(m)} := |\mathbb{1}_{B_j^{(m)}}\rangle\langle\mathbb{1}_{B_j^{(m)}}| \qquad \Big\{\ B_j^{(m)} \in \text{m-th GRID of SCHEME}$$

- Experiment:

  e.g. N particles in 3D $(d=3N)$

  Step 1: Prepare dim d syst. in mixed state $\hat{\rho}$

  Step 2: Joint position meas. (meas. of config.) of precision $\Delta x \sim 1/m$

  Step 3: "yes-no" $|\phi\rangle\langle\phi|$ meas $\phi \in \mathcal{H}$

Then: $\mathbb{P}\left(\begin{matrix}|\phi\rangle\langle\phi| \text{ yes if} \\ m, \hat{\rho}\end{matrix}\right) = \mathrm{tr}\left[|\phi\rangle\langle\phi| \sum_{j=1}^{N_m} \hat{\mathbb{P}}_j^{(m)} \hat{\rho}\, \hat{\mathbb{P}}_j^{(m)}\right].$

- **THEOREM:** For any $d \in \mathbb{N}$, any $\hat{\rho}$, any ϕ and any grid scheme

$$\lim_{n\to\infty} \mathbb{P}\left(\begin{array}{l} |\phi\rangle\langle\phi| \text{ yields "yes"} \\ \text{if prep. } \hat{\rho} \text{ \& localized } \Delta x \sim 1/n \end{array} \right) = 0$$

statement true if replace $[0,1)^d \rightsquigarrow \mathbb{R}^d$.

- **"PARADOX":** No $\Psi \in L^2(\mathbb{R}^d)$ or mixed $\hat{\rho}$ on $L^2(\mathbb{R}^d)$ satisfies $\mathbb{P}(|\phi\rangle\langle\phi| \text{ "yes"}) = 0 \ \forall \phi \in L^2(\mathbb{R}^d)$.

$\Rightarrow$ PERFECT CONFIGURATION MEASUREMENT CANNOT be described by HILBERT SP. $L^2(\mathbb{R}^d)$.

- We said "POSITION MEASUREMENT" but exact same problem for ANY observable with purely ABSOLUTELY CONTINOUS SPECTRUM!

  e.g. MOMENTUM measurements, ENERGY measurements for free Schrödinger or Dirac particles etc.

4. Solution to "Paradox": A New Type of Quantum State?

- Physicist answer: Position post-meas. state is

DIRAC DELTA $|x_0\rangle$ in "posit-rep" $\langle x|x_0\rangle = \delta(x-x_0)$

After all $\hat{x} =$ "$\int_{x_0 \in \mathbb{R}} x_0 \, |x_0\rangle\langle x_0| \, dx_0$"

(So $|x_0\rangle$ is "EIGENSTATE")

$\Rightarrow$ But $\delta \notin \mathcal{H}$ so, result is trivial!

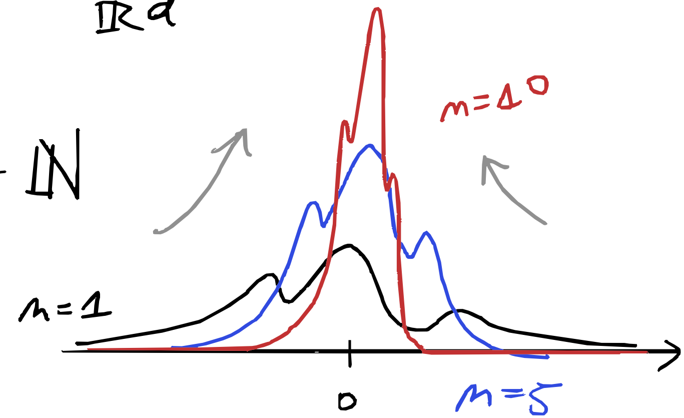
- Flaw: Mathcally too NAÏVE! True answer: "$\sqrt{\delta}$"!

WHAT is a $\delta(\cdot - x_0)$?

LEMMA: Given $g \in L^1(\mathbb{R}^d)$: $\int_{\mathbb{R}^d} |g|(x)\,dx = 1$,

def.:

$$g_m(x) := m\,g(mx), \quad m \in \mathbb{N}$$



Then,

$$\lim_{m\to\infty} \langle g_m, f \rangle = f(0) \qquad \forall f \in \mathcal{C}_b(\mathbb{R}^d)$$

So: $$\lim_{m\to\infty} \langle g_m(\cdot - x_0), f \rangle = f(x_0) \qquad \forall x_0 \in \mathbb{R}^d$$

Hence, $$\text{“}\lim_{m\to\infty} g_m(\cdot - x_0)\text{”} = \delta(\cdot - x_0)$$

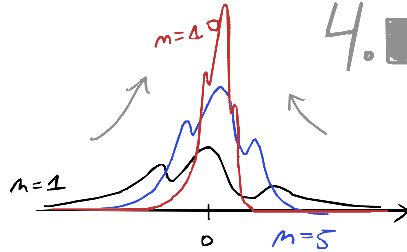
ISSUE: Even if $g_m \in L^2(\mathbb{R}^d)$,

$$\left\{ \begin{aligned} \|g_m\|^2_{L^2(\mathbb{R}^d)} &= \int_{x\in\mathbb{R}^d} |g_m|^2(x)\,dx = m^2 \int_{x\in\mathbb{R}^d} |g|^2(mx)\,dx = \\ &\underset{\left\{\begin{array}{l} y = mx \\ dx = \frac{1}{m}dy \end{array}\right\}}{\overset{\uparrow}{=}} m \int_{y\in\mathbb{R}^d} |g|^2(y)\,dy = m\cdot\|g\|^2_{L^2} \xrightarrow[m\to\infty]{} \infty \end{aligned} \right.$$

$$\Longrightarrow \quad \|\text{``}\delta\text{''}\|^2_{L^2} = \infty\,!$$

$\Rightarrow$ Not valid as a quantum state even heuristically!

INSTEAD: Define



$$\tilde{g}_m(x) := \sqrt{|g_m|(x)}$$

$$\|\tilde{g}_m\|^2_{L^2(\mathbb{R}^d)} = \int_{x\in\mathbb{R}^d} |g_m|(x)\,dx = \frac{1}{m}\int_{x\in\mathbb{R}^d} |g|(mx)\,dx$$

$$= \int_{y\in\mathbb{R}^d} |g|(y)\,dx = 1 \qquad \forall m\in\mathbb{N}$$

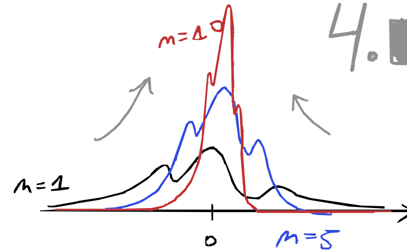
- Since "$\lim_{m\to\infty}\sqrt{|g_m|}$" $= \sqrt{\delta}$

"$\|\sqrt{\delta}\|_{L^2(\mathbb{R}^d)}$" $= \langle\sqrt{\delta},\sqrt{\delta}\rangle = 1$

Acceptable "norm"

INSTEAD: Define

$$\tilde{g}_m(x) := \sqrt{|g_m|(x)}$$



LEMMA: For each $V \in \mathcal{C}_b(\mathbb{R}^n)$

$$\text{“}\langle \sqrt{\delta}, V(\hat{x}) \sqrt{\delta} \rangle\text{”} := \lim_{m\to\infty} \langle \tilde{g}_m, V(\hat{x})\, \tilde{g}_m \rangle = V(0)$$

$\Rightarrow$ Well-defined expect. values for posit. observabls.

- Also, $\forall \phi \in L^2(\mathbb{R}^n)$

$$\text{“}\langle \sqrt{\delta}, |\phi\rangle\langle\phi| \sqrt{\delta} \rangle\text{”} = \lim_{m\to\infty} \langle \tilde{g}_m, |\phi\rangle\langle\phi| \tilde{g}_m \rangle = 0$$

⚠ $\nexists\, \psi \in L^2(\mathbb{R}^n)$ or $\hat{\rho}$ with this property!

- Thus, $\sqrt{\delta}$ is NOT in $\mathcal{H}$! Rings Bell! 4.

- But well-defined "norm" & "expectation values"!

⇝ Define it as (SINGULAR) ALGEBRAIC STATE ?

- RECALL: Given a C^*-algebra $\mathcal{A}$ (e.g. $\mathcal{L}(\mathcal{H})$)

  an (algebraic) STATE is a form $\omega : \mathcal{A} \longrightarrow \mathbb{C}$:

  (i) LINEAR (ii) $\omega(Id) = 1$ (iii) $\omega(B^*B) \geq 0$

- It is a generalization of

$$\omega(B) = \langle \psi, B\psi \rangle \quad \text{or} \quad \omega(B) = tr[\hat{\rho} B]$$

(Expected values of $\psi, \hat{\rho}$ fully characterize them)

THM (similar proof as BURGARTH's 2023) : Def. domain 4.

$$\mathcal{D}_{\sqrt{\delta}} := \left\{ B \in \mathcal{L}(L^2(\mathbb{R}^d)) \;\middle|\; \exists \lim_{m\to\infty} \langle \tilde{g}_m, B \tilde{g}_m \rangle \right\}$$

& for $B \in \mathcal{D}_{\sqrt{\delta}}$,

$$\sqrt{\delta}(B) := \text{“}\langle \sqrt{\delta}, B \sqrt{\delta} \rangle\text{”} := \lim_{m\to\infty} \langle \tilde{g}_m, B \tilde{g}_m \rangle$$

• Then, $\mathcal{D}_{\sqrt{\delta}}$ is: (i) $*$-closed Banach sp. (NOT algebra)

(ii) containing $C^0_\infty(\hat{x})$, $C^0_\infty(\hat{p})$, $\mathcal{K}(\mathcal{H})$, Weyl CCR algebra

• $(\sqrt{\delta}, \mathcal{D}_{\sqrt{\delta}})$ can be extended (possibly non uniquely) to a PURE SINGULAR STATE on $\mathcal{L}(L^2(\mathbb{R}^d))$.

- Similarly for each $x_0 \in \mathbb{R}^d$

Quantum state $\sqrt{\delta(\cdot - x_0)} =: \sqrt{\delta_{x_0}}$

definable as algebraic state:

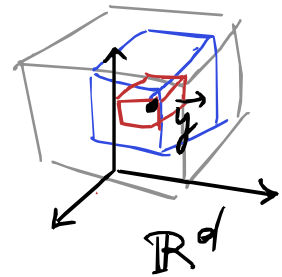
$$\langle \sqrt{\delta_{x_0}}, B \sqrt{\delta_{x_0}} \rangle := \lim_{n \to \infty} \langle \tilde{g}_n(\cdot - x_0), B\, \tilde{g}_n(\cdot - x_0) \rangle .$$

PARTIAL ANSWER TO PARADOX

4.

DEF.: Given $\hat{\mathbb{P}}_y^{(m)}$ projectors

def. for $\psi \in L^2(\mathbb{R}^d)$



grid scheme and $\mathbb{R}^d$

$$\psi_y^m := \frac{\hat{\mathbb{P}}_y^{(m)}\psi}{\|\hat{\mathbb{P}}_y^{(m)}\psi\|}$$

collapsed ψ after posit. measmt. locates and y with $\Delta x \sim 1/m$

PROP.: For all $B \in \mathrm{Weyl}\,GR,\ \mathcal{C}^\circ_\Delta(\hat{x}),\ \mathcal{C}^\circ_\infty(\hat{p}),\ \mathcal{K}(L^2(\mathbb{R}^d))$

$$\lim_{m\to\infty} \langle \psi_y^m, B\psi_y^m \rangle = \langle \sqrt{\delta_y}, B\sqrt{\delta_y} \rangle .$$

PARTIAL ANSWER TO PARADOX

COR.: Quantum state of particle system perfectly localized in config. space at $x_0 \in \mathbb{R}^d$ is the "densely defined" algebraic state $\left(\sqrt{\delta_{x_0}}, Q\sqrt{\delta_{x_0}}\right)$.

PARTIAL ANSWER TO PARADOX

4. [redacted]

COR.: Quantum state of particle system perfectly localized in config. space at $x_0 \in \mathbb{R}^d$ is the "densely defined" algebraic state

$$\left(\sqrt{\delta_{x_0}},\ \mathcal{D}\sqrt{\delta_{x_0}}\right).$$

WHY is this PARTIAL?

- In paradox we considered

NOT $\psi^m_{x_0}$ but unconditional

mixed state:

$$\sum_{x_0 \in \mathrm{Grid}} \frac{1}{m} \, |\psi^m_{x_0}\rangle\langle\psi^m_{x_0}|$$

⇒ Need to rigorize also mixture of $\sqrt{\delta}$'s

$$\text{"} \sum_{x_0 \in \mathbb{R}^d} \frac{1}{|\mathbb{R}^d|} \sqrt{\delta_{x_0}} \text{"} \quad !$$

To be continued...

Thanks for your Attention