CHAPTER 2

Topological, metric, and normed spaces

Definition 2.1 (Sequence)**.**
A sequence $(x_n) = (x_1, x_2, x_3, \ldots)$ in a set X is a map:

$$x : \mathbb{N} \to X, \qquad n \mapsto x_n$$

A sequence (x_n) in $\mathbb{R}$ *converges* to $x \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \; \forall n \geq N_\varepsilon : \quad |x_n - x| < \varepsilon$$

(if for every $\varepsilon > 0$ it holds 'eventually' that $|x_n - x| < \varepsilon$) and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or simply} \quad x_n \to x\,.$$

Similarly, a sequence (x_n) in $\mathbb{R}^k$ converges to $x \in \mathbb{R}^k$, if

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \; \forall n \geq N_\varepsilon : \quad \|x_n - x\|_2 < \varepsilon$$

where $\|\cdot\|_2$ denotes the usual Euclidean distance (see Example 2.3). We observe that in both examples the notion of *norm* is the key tool in order to define convergence of sequences.

Definition 2.2 (Norm and normed space)**.**
Let V be a vector space either over $\mathbb{R}$ or $\mathbb{C}$. A *norm* $\|\cdot\|$ on V is a map:

$$\|\cdot\| : V \to [0,\infty), \qquad x \mapsto \|x\|$$

with the properties:

1. $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$
2. $\forall x \in V,\ \lambda \in \mathbb{K} : \qquad \|\lambda x\| = |\lambda| \cdot \|x\|$

3. $\forall x, y \in V: \quad \|x+y\| \leq \|x\| + \|y\|$

The pair $(V, \|\cdot\|)$ is called a *normed space*.

Example 2.3. 1. On $V = \mathbb{R}^n$ or $\mathbb{C}^n$ the following maps are norms:

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \qquad \text{euclidean norm}$$
$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \qquad \text{maximum norm}$$
$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| \qquad \text{1-norm}$$

or, more generally, for $p \in [1, \infty)$, we obtain:

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{\frac{1}{p}} \qquad \text{p-norm}$$

2. Let X a set, $(Y, \|\cdot\|_Y)$ a normed space, and

$$V := \{f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y < \infty\}.$$

Then $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_Y$ is a norm on V.

Definition 2.4 (Convergence in normed spaces)**.**
A sequence (x_n) in a normed space $(V, \|\cdot\|)$ converges to $x \in V$ if

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \; \forall n \geq N_\varepsilon : \quad \|x_n - x\| < \varepsilon\,.$$

A norm, however, can only be defined on a vector space and ideally we would like to forget about such structure. In view of the definition of convergence for normed spaces, a notion of "distance" between points should suffice.

Definition 2.5 (Metric and metric space)**.**
Let X be a set. A *metric* d on X is a map:

$$d : X \times X \to [0, \infty)$$

with the following properties:

1. $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$
2. Symmetry: $\forall x, y \in X: \quad d(x, y) = d(y, x)$
3. Triangle inequality: $\forall x, y, z \in X: \quad d(x, z) \leq d(x, y) + d(y, z)$

The pair (X, d) is called a *metric space*

Example 2.6. 1. Let $(V, \|\cdot\|)$ be a normed space. Then $d : V \times V \to [0, \infty)$, $(x, y) \mapsto d(x, y) := \|x - y\|$ defines a metric on V.

2. Let X be as set. The discrete metric on X is defined by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

3. The euclidean unit sphere $S^2 := \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ with the metric:

$$d(x,y) := \arccos(\langle x, y\rangle)$$

is a metric space.

Definition 2.7 (Convergence in metric spaces)**.**
A sequence (x_n) in a metric space (X,d) converges to $x \in X$ if

$$\forall \varepsilon > 0 \, \exists N_\varepsilon \in \mathbb{N} \, \forall n \geq N_\varepsilon : \quad d(x_n, x) < \varepsilon \, .$$

Definition 2.8 (Open sets in a metric space)**.**
Let (X,d) be a metric space.

1. For $x_0 \in X$ and $r > 0$ the set:

$$B_r(x_0) := \{x \in X \mid d(x, x_0) < r\}$$

is called the *open ball*, with the radius r and the centre x_0.

2. A subset of $U \subset X$ is called a *neighbourhood* of $x_0 \in X$, if U contains an open ball around x_0, i.e.

$$\exists r > 0 : \quad B_r(x_0) \subset U.$$

Then x_0 is called an *interior point* of U.

3. A subset $U \subset X$ is called *open*, if it contains only interior points, i.e.

$$\forall x \in U \exists r > 0 : \quad B_r(x) \subset U.$$

Example 2.9. 1. Let (X,d) be a metric space. Then for any $x \in X$ and $r > 0$ the set $B_r(x)$ is open.

2. Let X be equipped with the discrete metric. Then any subset $U \subseteq X$ is open: $B_{\frac{1}{2}} = \{x\} \quad \forall x \in X$

Proposition 2.10. *Let (X,d) be a metric space. Then:*

1. $\emptyset$ and X are open.

2. If $U, V \subset X$ are open, then also $U \cap V$ is open.

3. *If $U_i \subset X$ is open for all $i \in I$, then also $\bigcup_{i \in I} U_i$ is open.*

The second property implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Definition 2.11 (Closed set)**.**
A subset $A \subset X$ of a metric space is *closed*, if its complement is open, i.e. $A^C = \{x \in X \mid x \notin A\}$ is open.

Example 2.12. 1. Let $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$, $[a, \infty)$ are closed, but $[a, b)$ is neither open nor closed.

2. For any metric space (X, d) the sets $\emptyset$ and X are open and closed.

A metric endows a set not only with a notion of convergence, but also with a geometry (distance, angles, etc.), but actually, open sets turn out to be enough in order to define convergence.

Definition 2.13 (Topology and topological spaces)**.**
Let X be a set. A *topology* $\mathcal{T}$ on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X with the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \quad \Rightarrow \quad U \cap V \in \mathcal{T}$
3. $U_i \in \mathcal{T}$ for all $i \in I \quad \Rightarrow \quad \bigcup_{i \in I} U_i \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called the *open sets* and $(X, \mathcal{T})$ is called a *topological space*. $A \subset X$ is *closed*, if $A^C \in \mathcal{T}$. $U \subset X$ is called a *neighbourhood* of a $x_0 \in X$ (and x_0 is an *interior point* of U), if:

$$\exists O \in \mathcal{T} \quad \text{with} \quad x_0 \in O \subset U.$$

Remark 2.14. The property 2. implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Example 2.15. 1. $\mathcal{T} = \{\emptyset, X\}$ forms the so-called *trivial topology* on X.

2. $\mathcal{T} = \mathcal{P}(X)$ forms the so called *discrete topology* on X.

3. According to Proposition 2.10, the open sets in a metric space form a topology.

Definition 2.16 (Relative topology)**.**
$(X, \mathcal{T})$ a topological space and $Y \subset X$ a subset of X. Then $\mathcal{T}|_Y = \{O \cap Y \mid O \subset \mathcal{T}\}$ is a topology on Y, the *subspace* or *relative topology*. The elements $U \subset \mathcal{T}|_Y$ are called *relative open sets*.

Example 2.17.
1. $X = \mathbb{R}$, $Y = [0, 1]$. Then $Y \in \mathcal{T}|_Y$, i.e. Y is relatively open in itself. Also $[0, \frac{1}{2}) \subset Y$ is relatively open, since $[0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap Y$
2. (X, d) a metric space and $Y \subset X$ a subset. Then $(Y, d|_Y)$ is a metric space.
$$d|_Y : Y \times Y \to [0, \infty], \quad (y_1, y_2) \mapsto d|_Y(y_1, y_2) = d(y_1, y_2)$$
3. $(V, \|\cdot\|)$ normed space and $U \subset V$ a vector subspace. Then $(U, \|\cdot\||_U)$ is a normed space.

Definition 2.18 (Interior, closure, boundary)**.**
Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$.

1. The set $\overset{\circ}{Y} = \bigcup\limits_{\substack{U \in \mathcal{T} \\ U \subset Y}} U$ is called the *interior* of Y.
2. The set $\overline{Y} = \bigcap\limits_{\substack{U \in \mathcal{T} \\ U \subset Y^C}} U^C$ is called the *closure* of Y.
3. The set $\partial Y = \overline{Y} \setminus \overset{\circ}{Y}$ is called the *boundary* of Y.

Proposition 2.19.
1. *$\overset{\circ}{Y} \subset Y \subset \overline{Y}$.*
2. *$\overset{\circ}{Y}$ is the largest open set contained in Y.*
3. *Y is open $\Leftrightarrow Y = \overset{\circ}{Y}$.*
4. *$\overline{Y}$ is the smallest closed set containing Y.*
5. *Y is closed $\Leftrightarrow Y = \overline{Y}$.*
6. *$(\overset{\circ}{Y})^C = \overline{(Y^C)}$ and $\left(Y^C\right)^\circ = \left(\overline{Y}\right)^C$.*[1]

Proposition 2.20. *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then:*

1. *$\overset{\circ}{Y}$ is the set of interior points of Y.*

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[1]This can be proven using de Morgan's laws: For a family of sets $(A_i)_{i \in I}$ it holds

(a) $\left(\bigcap\limits_{i \in I} A_i\right)^C = \bigcup\limits_{i \in I} A_i^C$,

(b) $\left(\bigcup\limits_{i \in I} A_i\right)^C = \bigcap\limits_{i \in I} A_i^C$.

2. $x \in \partial Y \quad \Leftrightarrow \quad$ *for any neighbourhood* U *of* x $U \cap Y \neq \emptyset$ *and* $U \cap Y^C \neq \emptyset$.

3. $\mathring{Y} = Y \backslash \partial Y$ *and* $\overline{Y} = Y \cup \partial Y$

Example 2.21. 1. For $Y = [a, b) \subset \mathbb{R}$ we have $\mathring{Y} = (a, b)$, $\overline{Y} = [a, b]$, $\partial Y = \{a, b\}$.

2. For $\mathbb{Q} \subset \mathbb{R}$ we have $\mathring{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial\mathbb{Q} = \mathbb{R}$.

Definition 2.22 (Convergence in topological spaces)**.**
Let X be a topological space. A sequence (x_n) in X *converges* to $a \in X$ and we write:

$$\lim_{n\to\infty} x_n = a,$$

if for any neighbourhood U of the point a there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Remark 2.23. In general convergence points are not unique: On any set X with the trivial topology any sequence converges to every point in X!

Definition 2.24 (Hausdorff spaces)**.**
A topological space $(X, \mathcal{T})$ is *Hausdorff*, if:

$$\forall x, y \in X, x \neq y, \exists U, V \in \mathcal{T}: \quad x \in U, y \in V, \quad U \cap Y = \emptyset\,.$$

Proposition 2.25. *1. In Hausdorff spaces, sequences have at most one limit.*

2. All metric spaces are Hausdorff.

Definition 2.26 (Cluster point)**.**
A point $a \in X$ is called a *cluster point* of a sequence (x_n) if any neighbourhood U of a contains infinitely many elements of (x_n).

Thus far, we have encountered spaces with different level of abstraction,

$$\textit{normed spaces} \quad \subset \quad \textit{metric spaces} \quad \subset \quad \textit{Hausdorff spaces}\,,$$

in which we call always define a notion of convergence and they indeed coincide. Both in physics and mathematics, the analysis is often carried out in so-called *Banach spaces.*

Definition 2.27 (Cauchy sequence)**.**
A sequence (x_n) in a metric space X is called *Cauchy sequence*, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N: \quad d(x_n, x_m) < \varepsilon$$

Proposition 2.28. *Every convergent sequence in a metric space is also a Cauchy sequence.*

Definition 2.29 (Complete metric space, Banach space)**.**

1. A metric space X is called *complete* if every Cauchy sequence in X converges.
2. A complete normed space (the norm induces the metric) is called a *Banach space*.

Example 2.30. 1. $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ (with e.g. the Euclidean norm) are Banach spaces.

2. $(\mathbb{Q}, |\cdot|)$ is not complete, as there exists a Cauchy sequence $(x_n) \subset \mathbb{Q}$ with $\lim_{n\to\infty} x_n = \sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$.

Exercises

1. Let X be a set and $(Y, \|\cdot\|_Y)$ a normed space. Consider the vector space

$$V \doteq \{f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y\}\,.$$

Prove that $\|f\|_\infty \doteq \sup_{x\in X} \|f(x)\|_Y$ is a norm on V.

2. Let (X, d) be a metric space.

(a) Prove that $B_r(x)$ is open for any $r > 0$ and $x \in X$.

(b) Prove that $\emptyset$ and X are open.

(c) Prove that if $Y, Z \subset X$ are open, then $Y \cap Z$ and $Y \cup Z$ are open.

(d) Consider now the metric

$$d(x, y) \doteq \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Prove that any subset $Y \subset X$ is open.

3. Prove that

(a) all sequences in a Hausdorff space have at most one limit.

(b) all metric spaces are Hausdorff.

4. Prove that every convergent sequence in a metric space is a Cauchy sequence.

5. Find two examples of

(a) metric spaces that are not normed spaces.

(b) normed spaces that are not Banach.

(c) metric spaces that are not complete.

(d) topological spaces that are not Hausdorff.