

CONTINUITY, COMPACTNESS AND CONNECTEDNESS

In the last lecture we saw that the abstraction of the “sewing” of $\mathbb{R}^2$, namely the topology, can always be understood as a declaration of which are the convergent nets. As a matter of “coincidence”, note that the literal meaning of the word “net” indeed suggests that nets are the needlework, the stitching of the space: the paths of the thread that constitutes the seams of the otherwise shapeless set.

4.1 Continuity

Now, given two topological spaces X, Y , we could wonder when a mapping $f : X \rightarrow Y$ preserves that needlework, namely, when does it map a convergent net exactly to a convergent net: “a stitching in X to a stitching in Y ”. Since such a mapping would preserve the contiguity/continuity of points we shall call it continuous, yet, for now we will call it “net-continuous”.

Definition 48. Let X, Y be topological spaces and let $f : X \rightarrow Y$ and $a \in X$. We say that f is *net-continuous at a* when it maps a convergent net with limit a to a convergent net with limit $f(a)$, i.e., when

$$\left((x_j)_{j \in D} \subseteq X : x_j \xrightarrow{j \in D} a \right) \implies \left(f(x_j) \xrightarrow{j \in D} f(a) \right).$$

We call it *sequentially continuous at a* when the same condition holds but only for sequences. $\diamond$

In order to make this definition notationally even nicer, one can introduce the following notation —note that so far we have defined $\lim_{n \rightarrow \infty}$ for $n \in \mathbb{N}$ and $\lim_{j \in D}$ but have not defined anything like $\lim_{x \rightarrow b}$ to mean “limit as the point $x \in X$ approaches a point $b \in X$ ”. As it is intuitive from that, it is a notion of limit that holds regardless of the way we approach b .

Definition 49. Let X, Y be topological spaces and let $f : X \rightarrow Y$, $a \in X$ and $b \in Y$. We denote

$$\lim_{x \rightarrow a}^{net} f(x) = b \quad \left(\text{resp.} \quad \lim_{x \rightarrow a} f(x) = b \right),$$

when f maps nets convergent to a into nets convergent to b (resp. for sequences), i.e.,

$$\left((x_j)_{j \in D} \subseteq X : x_j \xrightarrow{j \in D} a \right) \implies \left(f(x_j) \xrightarrow{j \in D} b \right),$$

(resp. for sequences).^[a] Hence, we can re-state Definition 48 as: f is *net-continuous (resp. sequentially continuous)* at $a \in X$ when $\lim_{x \rightarrow a}^{net} f(x) = f(a)$ $\left(\text{resp.} \quad \lim_{x \rightarrow a} f(x) = f(a) \right)$. $\diamond$

^[a]Note that if $f : X \setminus \{a\} \rightarrow Y$, i.e., f is not defined in a , we can still use this notation, the only difference is that now the nets should be restricted to lie in $X \setminus \{a\}$.

Now, because the usual way to give a topology is using the open sets, the notion of continuity defined in terms of them is called *continuity* (instead of say, “open-continuous”). It says that a function is continuous at a point a when for every neighborhood of its image $f(a)$, all the points in neighborhood of a are mapped inside it:

Definition 50. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \rightarrow Y$ and $a \in X$. We say that f is *continuous at a* when

$$\underbrace{\forall U \in \mathcal{T}_Y : f(a) \in U}_{\text{open nhod of } f(a)} \quad \underbrace{\exists V \in \mathcal{T}_X : a \in V}_{\text{open nhod of } a} \quad \text{such that} \quad f(V) \subseteq U.$$

♦

Although reasonable, this definition does not seem as obvious as say, net-continuity. Luckily, they are exactly the same:

Proposition 17. Given topological spaces X, Y ,

$$f : X \rightarrow Y \text{ is continuous} \iff \text{it is net-continuous.}$$

♦

Hence, we can call “net-continuous” simply “continuous”. In particular, because a net-continuous function is trivially sequentially continuous, this proves that general continuity implies sequential continuity in any topological space. However, the converse is not true in general. Still, for most of the spaces you will deal with during the master’s, the following holds:

Proposition 18. Given metric spaces (X, d_X) , (Y, d_Y) , $f : X \rightarrow Y$, $a \in X$ and $b \in Y$,

$$\left(\lim_{x \rightarrow a}^{net} f(x) = b \right) \iff \left(\lim_{x \rightarrow a} f(x) = b \right)$$

♦

Corollary 6. Given metric spaces (X, d_X) , (Y, d_Y) ,

$$f : X \rightarrow Y \text{ is continuous} \iff \text{it is sequentially continuous.}$$

♦

Now, the continuity in terms of open sets seems unmotivated, but it becomes a very handy tool to prove when a function is continuous everywhere in X :

Theorem 9. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. $f : X \rightarrow Y$ is continuous everywhere in X if and only if, the pre-image of an open set is open:

$$\forall U \in \mathcal{T}_Y, \quad f^{-1}(U) \in \mathcal{T}_X.$$

♦

Note that when one says that a function is continuous (without specifying where), one means it is continuous everywhere.

Now, although not as practical for proving continuity, in terms of nets as well, asking for continuity everywhere looks simpler:

Lemma 23. Given topological spaces X, Y , a function $f : X \rightarrow Y$ is continuous (resp. sequentially continuous) if and only if for any convergent net (resp. sequence) $(x_j)_{j \in D}$,

$$\lim_{x \in D} f(x_j) = f\left(\lim_{x \in D} x_j\right).$$

In words, when we can commute f with taking a limit.^[b]

♦

^[b]This notation must be handled and interpreted with care in non-Hausdorff spaces, where there are nets with multiple possible limit points.

This will be the main rigorous element we will use to justify the exchange of a derivative and an integral. We will see that later. Let us now talk about some example continuous functions.

Example 25. 1. In a metric space (X, d) the distance function to a point $b \in X$,

$$d_b : X \rightarrow [0, \infty), \quad x \mapsto d_b(x) := d(x, b) \quad (4.1)$$

is continuous. Also the full distance function $d : X \times X \rightarrow [0, \infty)$ is continuous on $(X \times X, d_p^2)$ $\forall p \in \mathbb{N}$ —or by simply equipping $X \times X$ the product topology. (It better be, because d is what induces the topology.)

2. In any normed vector space $(V, \|\cdot\|)$ the norm itself:

$$\|\cdot\| : V \rightarrow [0, \infty), \quad (4.2)$$

the vector addition:

$$+ : V \times V \rightarrow V, \quad (x, y) \mapsto x + y, \quad (4.3)$$

and the multiplication by scalars:

$$\cdot : \mathbb{K} \times V \rightarrow V, \quad (\lambda, v) \mapsto \lambda \cdot v \quad (4.4)$$

are all continuous (both, fixing one argument and taking the product topology). Hence, in particular they are net/sequentially continuous, such that one can take the limit operation out of them.

3. If X is equipped with the discrete topology, then every map $f : X \rightarrow Y$ is continuous. On the other hand, if X is equipped with the trivial topology, then every map $f : Y \rightarrow X$ is continuous.

4. In general, the composition of continuous functions is continuous, i.e., if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then also $g \circ f : X \rightarrow Z$ is continuous. $\blacklozenge$

4.1.1 Homeomorphism

Now that we have defined when a function preserves the topological structure, as we did with mappings preserving the linear structure, we will be able to define an isomorphism between structures:

Definition 51. Given topological spaces X, Y , a mapping $f : X \rightarrow Y$ that is (i) continuous, (ii) bijective and (iii) has a continuous inverse, is called an *homeomorphism* or *topological isomorphism*. If such a function exists, X and Y are said to be *homeomorphic* or *topologically isomorphic*. In the sense of Remark 1, *homeomorphic spaces are identical for all purposes of topology*. $\blacklozenge$

Notice that a homeomorphism $f : X \rightarrow Y$ not only identifies points of X with points of Y , but also bijectively identifies open sets of X with those of Y preserving the inclusion relations and in particular, bijectively identifies convergent nets of those spaces.

Recall that if X, Y are isomorphic, when it is clear from the context which type of isomorphism holds (say, a group-, a linear-, a metric- etc isomorphism), one writes $X \cong Y$.

Importantly, norm or metric space isomorphisms are particular cases of homeomorphisms:

Lemma 24. Norm isomorphism implies metric isomorphism and the latter implies homeomorphism. $\blacklozenge$

Example 26. 1. For all $a, b \in \mathbb{R}$, $a < b$, the interval $(a, b) \subseteq \mathbb{R}$ is homeomorphic to (but not isometric to) $\mathbb{R}$. The map

$$f : (a, b) \rightarrow \mathbb{R}, \quad x \mapsto f(x) = \frac{1}{a-x} + \frac{1}{b-x}$$

is an example of a homeomorphism. So, surprisingly, a bounded open interval is in topological terms equivalent to the whole $\mathbb{R}$. (Yet, this might not be that surprising if we remember that topology is just talking about the local “stitching”/contiguity of the elements in the set discarding any quantitative scale/direction/magnitude notion.)

2. The sphere of dimension $n \in \mathbb{N}$, when we remove a point from it becomes homeomorphic to $\mathbb{R}^n$ as we proceed to see. Given

$$S^n := \left\{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \right\},$$

we saw that rotations are norm isometries, so $S^n \setminus \{a\}$ and $S^n \setminus \{b\}$ are homeomorphic for every $a, b \in S^n$. Thus, we can center on removing a particular point to prove the claim. We choose the north pole $N := (0, \dots, 0, 1)$. Then, the so-called *stereographic projection*:

$$\begin{aligned} \varphi : S^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) &\longmapsto \frac{1}{1-x_{n+1}}(x_1, \dots, x_n) \end{aligned}$$

is an homeomorphism, of inverse:

$$\varphi^{-1}(x_1, \dots, x_n) = \frac{1}{1+x_1^2+\dots+x_n^2}(2x_1, \dots, 2x_n, x_1^2+\dots+x_n^2-1) \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n.$$

As a picture of an S^2 sphere cut in the equator by the $\mathbb{R}^2$ plane shows, φ takes a $p \in S^n$, draws the line joining p and N , and then $\varphi(p)$ is the point in the plane intersected by this line. The equator stands still, while the points in the upper hemisphere are mapped outside the unit disk (mapping the north pole to the “infinite horizon”), and the points in the southern hemisphere are mapped to the interior of the disk.

3. For all $n \in \{2, 3, \dots\}$, $\mathbb{R}^n \setminus \{\vec{0}\}$ is homeomorphic to the cylinder $S^{n-1} \times (0, +\infty)$ (with product topology). Just take

$$\begin{aligned} \varphi : \mathbb{R}^n \setminus \{\vec{0}\} &\longrightarrow S^{n-1} \times (0, +\infty) \\ x &\longmapsto \left(\frac{x}{\|x\|_2}, \|x\|_2 \right) \end{aligned}$$

of inverse:

$$\varphi^{-1}((y, t)) = ty \quad \forall t \in (0, +\infty), y \in S^{n-1}.$$

4. The torus (the “doughnut”), is homeomorphic to a mug and to $S^1 \times S^1$ with the product topology. ♦

4.1.2 Lipschitz continuity

There is a particular type of continuity that is relevant in the analysis of ordinary differential equations at later chapters:

Definition 52. Let X, Y be metric spaces. A function $f : X \rightarrow Y$ is called *Lipschitz-continuous* when there exists $L \in [0, +\infty)$ such that

$$d_Y(f(x_1), f(x_2)) \leq L \cdot d_X(x_1, x_2) \quad \forall x_1, x_2 \in X. \quad (4.5)$$

If so, L is called a *Lipschitz-constant* for f and if $L < 1$, f is called a *contraction*. ♦

Note that the Lipschitz condition says that there is a uniform upper bound to the rate of change of the distance:

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq L \quad \forall x_1, x_2 \in X : x_1 \neq x_2.$$

Hence for differentiable functions between normed spaces, the Lipschitz condition will give a uniform upper bound for the derivative’s norm.

Example 27. 1. $f(x) = ax + b$ is Lipschitz continuous with $L = |a|$.

2. If $f \in \mathcal{C}^1(\mathbb{R})$ and it is Lipschitz, then $L = \sup_{x \in \mathbb{R}} |f'(x)|$.

3. $f(x) = x^2$ is continuous but not Lipschitz continuous in $\mathbb{R}$.

4. $f(x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous in $\mathbb{R}$, as its derivative around 0 diverges. ♦

There is a very important theorem related to the contraction maps: the so-called *Banach fixed point theorem*:

Theorem 10. Let (X, d) be a Cauchy complete (non-empty) metric space and $F : X \rightarrow X$ a contraction. Then, there exists a unique point $x_0 \in X$ such that $F(x_0) = x_0$, which called a *fixed point*. In particular, x_0 can be found as follows: given an arbitrary $x_1 \in X$ define $x_{n+1} := F(x_n) \forall n \in \mathbb{N}$. Then, $\lim_{n \rightarrow \infty} x_n = x_0$ ♦

Among others, it is a key step in the proof of the Picard-Lindelöf theorem for existence and uniqueness of ODEs (which we will see in a later lecture).^[c]

An interesting “real world application”: given the map of a city, if you place the map in the ground somewhere in that city, it represents a function from the actual city to the points in the map, which is clearly a contraction. Then, by the fixed point theorem, there must exist a point of the true city which has its corresponding point of the map exactly over it.

4.2 Point-wise and Uniform Limits

Definition 53. Let X be a set, Y a topological space and let there be a sequence of functions

$$f_n : X \rightarrow Y, \quad \text{for each } n \in \mathbb{N} \quad \text{and} \quad f : X \rightarrow Y. \quad (4.6)$$

1. We say that the sequence $(f_n)_{n \in \mathbb{N}}$ *converges pointwise* to f , when

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in X. \quad (4.7)$$

2. When Y is moreover a metric space, pointwise convergence is equivalently that

$$\lim_{n \rightarrow \infty} d_Y(f_n(x), f(x)) = 0 \quad \forall x \in X.$$

Then, when this convergence occurs uniformly, i.e., when

$$\lim_{n \rightarrow \infty} \sup_{x \in X} d_Y(f_n(x), f(x)) = 0, \quad (4.8)$$

we say that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f .

3. If $(Y, \|\cdot\|)$ is furthermore a normed vector space, then for bounded functions $B_\infty(X, Y)$ (see Example 14.2), uniform convergence is just regular norm-topology convergence. That is, for $(f_n)_{n \in \mathbb{N}} \subseteq B_\infty(X, Y)$, $\lim_{n \rightarrow \infty} f_n = f$ in $\|\cdot\|_\infty$ -norm precisely when

$$\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0 \quad \Longleftrightarrow \quad \lim_{n \rightarrow \infty} \sup_{x \in X} \|f_n(x) - f(x)\| = 0. \quad (4.9)$$

♦

Example 28. For each $n \in \mathbb{N}$,

$$x^n \xrightarrow{n \rightarrow +\infty} \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}. \quad (4.10)$$

Thus, the sequence of functions $f_n : [0, 1] \rightarrow [0, 1]$, $f_n(x) := x^n$ converges pointwise to the function $f : [0, 1] \rightarrow [0, 1]$

$$f(x) := \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}.$$

However, $(f_n)_{n \in \mathbb{N}}$ does not converge uniformly to f because $\sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1$. After all, by continuity, each $f_n(x)$ gets arbitrarily close to 1 for $x \in [0, 1)$, where $f(x) \equiv 0$. ♦

^[c]One expresses the solution of the ODE as a fixed point of an integral operator on the space of continuous functions with the ∞ -norm. One then checks the integral operator is a contraction and uses the Banach fixed point theorem to prove there is a unique solution.

Proposition 19. Let $(X, \mathcal{T})$ a topological and (Y, d) a metric space. Consider a sequence $f_n : X \rightarrow Y$, $n \in \mathbb{N}$, of continuous functions. If for some $f : X \rightarrow Y$, uniformly $f_n \xrightarrow{n \rightarrow \infty} f$, then f is continuous. ♦

By the sequential characterization of closed sets in metric spaces, this implies that:

Corollary 7. Let $(X, \mathcal{T})$ a topological and (Y, d) a normed vector space. Define the set of continuous bounded functions,

$$\mathcal{C}_b^0(X, Y) := \left\{ f \in B_\infty(X, Y) \mid f \text{ continuous} \right\}. \quad (4.11)$$

It is a closed subspace of $(B_\infty(X, Y), \|\cdot\|_\infty)$. Thus, if Y is Banach (because then, $B_\infty(X, Y)$ is Banach), this implies that $(\mathcal{C}_b^0(X, Y), \|\cdot\|_\infty)$ is itself a Banach space. ♦

4.3 Compactness

Although in a topological space there is no quantitative way to say that a set is “bounded” and “closed”, one can find ways to suggest it. For instance, that there is no way to choose infinitely many points on the set without accumulating them arbitrarily close to some point, i.e. without their sequence necessarily coming infinitely often arbitrarily close to some point. This is going to be called “sequential compactness”. It would be even stronger if we ask that also for arbitrary nets.

Definition 54. Given a topological space $(X, \mathcal{T})$ and a net $(x_j)_{j \in D} \subseteq X$, $a \in X$ is said to be a *cluster point of the net* (recall in Def. 38 when a sequence has a cluster point) when the net comes back to any neighborhood of a arbitrarily late in the net, i.e.,

$$\forall U \in \mathcal{T} : a \in U, \quad \forall j_0 \in D, \quad \exists j \in D : j \geq j_0 \quad \text{with} \quad x_j \in U. \quad \diamond$$

Lemma 25. In a topological space (resp. metric space), a net (resp. a sequence) has a cluster point *if and only if* it has a convergent subnet (resp. sub-sequence). ♦

Definition 55. In a topological space X , a subset $K \subseteq X$ is said to be *net-compact* (resp. *sequentially compact*) when every net in K (resp. sequence in K) has a cluster point in K . ♦

Now, because the usual way to give a topology is using open sets, the notion of compactness defined in terms of them (which will be equivalent to net-compactness) is what is called *compactness* (instead of say, “open-compactness”).

Definition 56. Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$. A choice $U_j \in \mathcal{T} \forall j \in I$, is called an *open cover* of K , when

$$K \subseteq \bigcup_{j \in I} U_j. \quad (4.12)$$

Then, a set $K \subset X$ is called *compact*, when for any open cover of K , $(U_j)_{j \in I} \subseteq \mathcal{T}$, there exists a finite subcover, i.e. there exists a finite $J \subseteq I$, such that:

$$K \subseteq \bigcup_{j \in J} U_j. \quad \diamond$$

Luckily, one proves that this notion is exactly net-compactness:

Proposition 20. In an arbitrary topological space X ,

$$K \subseteq X \text{ is compact} \iff K \text{ is net-compact}.$$

Hence, in particular “compact $\implies$ sequentially compact” (this is the so-called Bolzano-Weierstraß theorem). The reverse implication also holds in metric spaces (i.e., in metric spaces “compact $\iff$ sequentially compact”). ♦

Although less intuitive, the characterization in terms of open covers is very powerful to check when a set is compact.

Example 29. 1. Every finite subset $K = \{x_1, \dots, x_n\}$ of a topological space is compact.

2. $(0, 1] \subset \mathbb{R}$ is not a compact set: the open cover $(0, 1] \subseteq \bigcup_{n=2}^{\infty} (\frac{1}{n}, 2)$ clearly admits no finite subcover. ♦

The net characterization clearly tells that compactness must be a property preserved by continuous maps:

Proposition 21. Let X, Y topological spaces and $f : X \rightarrow Y$ continuous. Then, if $K \subseteq X$ is compact, also $f(K) \subseteq Y$ is compact. ♦

Now, let us see how close we can get to having that a compact set is literally a closed and bounded set. Although first we need to define what bounded means (for which we need a distance function):

Definition 57. Let (X, d) be a metric space. A non-empty subset $A \subseteq X$ is said to be *bounded* when

$$\exists C \in \mathbb{R} : \quad d(x, y) \leq C \quad \forall x, y \in A. \quad (4.13)$$

The *diameter* of the set $A \subset X$ is the least such C if any exists, and otherwise ∞ , i.e.,

$$\text{diam}(A) = \sup \left\{ d(x, y) \mid x, y \in A \right\} \in [0, \infty].$$

♦

Theorem 11. Let X be a topological space and $K \subseteq X$ be compact.

- (i) Then any closed subset $A \subset K$ is also compact.
- (ii) If X is Hausdorff, then K is necessarily closed.
- (iii) If X is a metric space, then K is necessarily closed and bounded. ♦

We obtain the reverse of item (iii) only in so-called Heine-Borel spaces. In particular, it is a phenomenon exclusive to finite dimensional spaces:

Theorem 12. In a normed vector space V ,

$$V \text{ is finite dimensional} \iff \text{the closed unit ball is compact.}$$

Hence,

$$\text{all closed and bounded sets of } V \text{ are compact} \iff V \text{ is finite dimensional.} \quad \blacklozenge$$

The key is that in infinite dimensions, we can build a sequence in the unit ball that goes from “axis to axis” without ever accumulating anywhere, because there is always another “axis” where to go to.

4.3.1 A Few Results Concerning Compactness

One of the reasons why compactness of a set is relevant is the so-called Weierstraß theorem, important for optimization theory:

Theorem 13. Let X be a topological space, $K \subseteq X$ non-empty and compact, and $f : K \rightarrow \mathbb{R}$ continuous. Then, f is bounded (i.e., $f(K) \subseteq \mathbb{R}$ is bounded—even compact) and attains its supremum and its infimum (i.e., f has maximum and minimum). ♦

There is another train of results in the direction of geometry, for which a glimpse can be obtained by the following fixed point theorem:

Definition 58. Given a $\mathbb{K}$ -vector space V , a subset $C \subseteq V$ is said to be *convex* when the line segment connecting any pair $x, y \in C$ is fully contained in C , i.e.,

$$tx + (1 - t)y \in C \quad \forall t \in [0, 1] \quad \forall x, y \in C.$$

♦

For example an open (or closed) ball in a normed vector space is always convex. With that, the *Brouwer-Schauder fixed point theorem* says:

Theorem 14. Given a Banach space V and a convex compact (non-empty) set $K \subseteq V$, *every* continuous function $F : K \rightarrow K$ has at least one fixed point.

♦

As a “real world application”, consider a pot or coffee mug initially at rest. Then, no matter how wildly you stir it, at each time, at least one point of the fluid (possibly a different one at each time) will occupy the exact same position as it did at the initial time.

Finally, often it is hard to find if a set is compact or not using the general characterizations, especially in function spaces. That is why there exist results like the following one designed to prove when certain function sets are compact, the so-called Arzelà-Ascoli theorem.

Definition 59. Let X, Y be metric spaces and A a set of continuous functions $f : X \rightarrow Y$. Then A is called *equicontinuous* at $x \in X$ when

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall f \in A \quad f(B_\delta(x)) \subset B_\varepsilon(f(x)). \quad (4.14)$$

So, the same δ gives the continuity of every function in A .

♦

Theorem 15. Let X be a compact metric space and consider $C_b^0(X, \mathbb{C})$ (as in Corollary 7). A subset $K \subseteq C_b^0(X, \mathbb{C})$ is compact *if and only if* it is all the following:

- (i) closed,
- (ii) equicontinuous at every point $x \in X$,
- (ii) bounded pointwise, i.e., across K the functions are point-wise bounded

$$\sup_{f \in K} |f(x)| < +\infty \quad \forall x \in X.$$

♦

4.4 Connectedness

Since topology was about how we “sew” a set, we could also characterize when that sewing breaks or gets interrupted. Again, there is a way to characterize it in terms of the “threads” (namely, nets) and a way in terms of open sets. Both will be equivalent.

Definition 60. Let X be a topological space. If X is the union of two disjoint, open, non-empty sets, we say that X is *disconnected*. Otherwise, we say it is *connected*.

♦

Proposition 22. A topological space X is connected *if and only if* any of the following equivalent conditions holds:

- (a) the only simultaneously open and closed sets are X and $\emptyset$.
- (b) for each non-empty $A \subsetneq X$, at least one of the following holds:
 - (i) there is a net in A converging to a point outside A (i.e., in $X \setminus A$).
 - (ii) there is a net outside A converging to a point in A .

That is, there is no proper subset of X that is isolated from the rest of X . Proper, because certainly X in itself is isolated (makes no sense to say “outside X ”).

♦

Hence, a set X is disconnected when there is a proper subset $A \subseteq X$ that has no “thread” linking it to the rest of X nor vice-versa.

Now, there is an even more intuitive notion of connectedness:

Definition 61. Let X be a topological space. We say that X is *path-connected*, when any two points $x, y \in X$ can be connected by a continuous path, i.e. for all $x, y \in X$ there exists a continuous $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. $\diamond$

In general, it is not the same as path connected:

Proposition 23. Given X is a topological space,

$$X \text{ is path-connected} \implies X \text{ is connected.}$$

The converse is false in general (i.e., there exist exceptions). $\diamond$

Yet, such exceptions will not be relevant for our purposes:

Proposition 24. Let V be a normed vector space and $U \subseteq V$ open. Then

$$U \text{ is connected} \iff \text{it is path connected.}$$

As is obvious, mappings preserving the sewing must preserve its connectedness:

Proposition 25. Let X, Y topological spaces and $f : X \rightarrow Y$ continuous. Then, if $A \subseteq X$ is connected (resp. path-connected), $f(A) \subseteq Y$ is also connected (resp. path-connected). $\diamond$

4.4.1 Connected Components

If the topological space is disconnected, it seems obvious that it should be expressible as a disjoint union of connected sets. This is indeed the case as we proceed to see.

Definition 62. Given a topological space X , a subset $A \subseteq X$ is said to be connected when it is connected in its relative topology. $\diamond$

Proposition 26. Given a topological space X and an arbitrary family $(A_j)_{j \in I}$ of connected subsets of X with non-empty intersection $\bigcap_{j \in I} A_j \neq \emptyset$, then their union $\bigcup_{j \in I} A_j$ is also connected. $\diamond$

Definition 63. Given a topological space X , we define the *connected component* to which a point $x \in X$ belongs as

$$\mathcal{C}[x] := \bigcup_{\substack{Y \subseteq X: \\ x \in Y \\ Y \text{ connected}}} Y$$

since by the last Proposition it is still connected, this is the largest connected subset of X that contains x . $\diamond$

Proposition 27. Given a topological space X , the family of its connected components:

$$\left\{ \mathcal{C}[x] : x \in X \right\}$$

is a partition of X . Hence, “belonging to a connected component” is an *equivalence relation*. X is connected exactly when there is a single class of equivalence. $\diamond$

4.5 More on Normed Vector Spaces

4.5.1 Equivalent Norms

Definition 64. Given a vector space V and two norms $\|\cdot\|_a, \|\cdot\|_b$, we say they are *equivalent norms* when $\exists c, C \in (0, +\infty)$:

$$c\|v\|_a \leq \|v\|_b \leq C\|v\|_a \quad \forall v \in V. \quad \blacklozenge$$

Geometrically, this means that for any open ball B in one of the norms, say in $\|\cdot\|_a$, there are two open balls in the other norm, $\tilde{b}, \tilde{B}$, such that $\tilde{b} \subseteq B \subseteq \tilde{B}$.

Proposition 28. Given a vector space V , if two norms $\|\cdot\|_a, \|\cdot\|_b$ are equivalent then, they induce the same topology (hence, the same notion of limits etc). $\blacklozenge$

In finite dimensions we never really cared about the choice of norm because of the following:

Theorem 16. If V is a finite dimensional vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then all norms are equivalent. $\blacklozenge$

But then, because every finite dimensional vector space V is linearly isomorphic to $\mathbb{K}^n$ (with $n := \dim(V)$) and we can induce a norm in V by pulling back the Euclidean norm:

Corollary 8. All finite dimensional normed vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are Banach spaces. $\blacklozenge$

4.5.2 Arbitrary Sums of Vectors

Retaking the notion of infinite sum of vectors of Def. 31 and the remark on the order independence that followed, if we now had an arbitrary set of vectors, the natural way to sum them together would be to consider all possible finite partial sums and check when the resulting net converges.

Recall that given an arbitrary set I , we denote by $\mathcal{P}_0(I)$ the *family of finite subsets of I* , which we now consider to be directed by the set-inclusion $\subseteq$.

Definition 65. Given a normed vector space V and an arbitrary index set I , a family of vectors $(v_j)_{j \in I} \subseteq V$ is said to have a well-defined sum $\sum_{j \in I} v_j$ (or that the sum exists) and it equals v , in symbols, $\sum_{j \in I} v_j = v$, when

$$\lim_{J \in \mathcal{P}_0(I)} \sum_{j \in J} v_j = v. \quad \blacklozenge$$

One can rephrase this without the net theoretic jargon in two different ways:

Lemma 26. Let $(V, \|\cdot\|)$ be a normed vector space and let $v, (v_j)_{j \in I} \subseteq V$ be arbitrary. Then, $\sum_{j \in I} v_j = v$ if and only if $\forall \varepsilon > 0$ there exists an $I^\varepsilon \in \mathcal{P}_0(I)$ such that for all finite super-set $J \in \mathcal{P}_0(I)$, $I^\varepsilon \subseteq J$, one has that

$$\left\| v - \sum_{j \in J} v_j \right\| \leq \varepsilon. \quad (4.15)$$

(Informally when finite partial sums become arbitrarily close to some fixed vector.) $\blacklozenge$

Proposition 29. Let $(V, \|\cdot\|)$ be a Banach space and let $(v_j)_{j \in I} \subseteq V$ be arbitrary. Then, $\sum_{j \in I} v_j$ exists if and only if $\forall \varepsilon > 0$ there exists an $I^\varepsilon \in \mathcal{P}_0(I)$ such that for all $J \in \mathcal{P}_0(I)$: $J \cap I^\varepsilon = \emptyset$, one has that $\left\| \sum_{j \in J} v_j \right\| \leq \varepsilon$. $\blacklozenge$

Following the use of these notions of arbitrary sum (in the “ ε -form”), one can prove the next proposition.

Proposition 30. Let V be a Banach space, I an arbitrary index set, $(v_j)_{j \in I}, (u_j)_{j \in I} \subseteq V$ arbitrary families of vectors and $v, u \in V$.

(i) If I is *finite*, the notion in Def. 65 coincides with the usual finite sum: $\sum_{j \in \{1, \dots, n\}} v_j = v_1 + \dots + v_n$.

(ii) If I is *countable*, say, if $I = \{j_n\}_{n \in \mathbb{N}}$, then

$$\left(\sum_{j \in I} v_j = v \right) \iff \left(\sum_{n=1}^{\infty} v_{j_n} = v \quad \text{unconditionally} \right).$$

If so, then $\sum_{j \in I} v_j = \sum_{n=1}^{\infty} v_{j_n} \equiv \lim_{N \rightarrow \infty} \sum_{n=1}^N v_{j_n}$.

(iii) If I is *uncountable*,

$$\left(\sum_{j \in I} v_j = v \right) \iff \left(\exists \text{ a finite or countable } J \subseteq I : \sum_{j \in J} v_j = v \quad \text{and} \quad v_j = 0 \quad \forall j \in I \setminus J \right).$$

(Thus, the uncountable sums that converge are always “secretly” finite or countable sums!)

(iv) If $\sum_{j \in I} v_j = v$ and $\sum_{j \in I} u_j = u$ then, for all $\alpha \in \mathbb{K}$,

$$(a) \quad \sum_{j \in I} \alpha v_j = \alpha v \qquad (b) \quad \sum_{j \in I} (v_j + u_j) = v + u.$$

♦

4.5.3 Bounded Functions vs Bounded Operators

A function is bounded when it does not range an arbitrarily large set.

Definition 66. Given a set X and a metric space (Y, d) , $f : X \rightarrow Y$ is said to be bounded when $f(X) \subseteq Y$ is bounded (i.e., it has a finite diameter). ♦

However, in a normed vector space, no linear map (other than the zero map) is bounded because the image is always a vector subspace and by homogeneity of the norm, this means that there will be arbitrarily long vectors in the image (hence, the diameter of the image is going to be infinite). Instead, to talk about linear maps that do not allow an arbitrary growth of distances, one defines ad hoc for linear maps, the following notion of *boundedness*.

Definition 67. Given normed vector spaces V, W , a linear map $A : V \rightarrow W$ is said to be a *bounded operator* or simply *bounded* (if clear we are talking about a linear map), when the unit closed ball is maps to a bounded set, i.e., $A(B_1(\vec{0}))$ is *bounded*:

$$\exists C \in [0, +\infty) : \quad \|Ax\|_W \leq C \quad \forall x \in \overline{B_1(\vec{0})}.$$

For a linear map, this is equivalent to

$$\exists C \in [0, +\infty) : \quad \|Ax\|_W \leq C\|x\|_V \quad \forall x \in V \tag{4.16}$$

and also to

$$\exists C \in [0, +\infty) : \quad \|Ax\|_W \leq C \quad \forall x \in V : \|x\| = 1.$$

The smallest such constant C is called the *operator norm* of A and we denote it as:

$$\|A\|_{op} := \sup_{x \in V : \|x\| \leq 1} \|Ax\|_W \stackrel{V \neq \{\vec{0}\}}{=} \sup_{x \in V \setminus \{\vec{0}\}} \frac{\|Ax\|_W}{\|x\|_V} = \sup_{x \in V : \|x\| = 1} \|Ax\|_W. \tag{4.17}$$

The space of bounded linear maps $V \rightarrow W$ is denoted by

$$\mathcal{L}(V, W) := \left\{ A \in \text{Lin}(V, W) \mid \|A\|_{op} < +\infty \right\}.$$

♦

Proposition 31. For normed spaces V, W , if W is Banach, then $(\mathcal{L}(V, W), \|\cdot\|_{op})$ is a Banach space. ♦

Bounded operators are distinguished among linear maps because they are exactly the continuous ones:

Proposition 32. Given normed vector spaces V, W and $A \in \text{Lin}(V, W)$

$$A \text{ is bounded} \iff A \text{ is continuous.} \quad \blacklozenge$$

In finite dimensional vector spaces we did not need any such distinction because in fact it was a characteristic of finite dimensions that:

Proposition 33. Given normed vector spaces V, W , with $\dim(W) \geq 1$,

$$V \text{ is finite dimensional} \iff \text{every linear } A : V \rightarrow W \text{ is a bounded operator.} \quad \blacklozenge$$

4.5.4 Extra: Bounded Multilinear Operators

Definition 68. Given vector spaces $V_1, \dots, V_n, W$ ($n \in \mathbb{N}$) over the same field $\mathbb{K}$, a map $A : V_1 \times \dots \times V_n \rightarrow W$ is said to be *n-multi-linear* when it is linear in each argument separately, i.e.,

$$A(v_1, \dots, \lambda v_j + \mu w_j, \dots, v_n) = \lambda A(v_1, \dots, v_n) + \mu A(v_1, \dots, w_j, \dots, v_n) \quad \forall v_k, w_k \in V_k, \lambda, \mu \in \mathbb{K}, j \in \{1, \dots, n\}.$$

The case $W = \mathbb{K}$ is called a *tensor*. ♦

Proposition 34. Given normed vector spaces $V_1, \dots, V_n, W$ ($n \in \mathbb{N}$) over the same field $\mathbb{K}$, an *n*-multilinear map is said to be bounded and called a *bounded multi-linear operator*, when $\exists C \in [0, +\infty)$:

$$\|A(v_1, \dots, v_n)\| \leq C \|v_1\| \cdots \|v_n\| \quad \forall v_j \in V_j.$$

If so, the infimum of such constants is called the *operator norm* and equals

$$\|A\|_{op} := \sup_{\substack{v_j \in V_j \\ \|v_j\| \leq 1}} \|A(v_1, \dots, v_n)\| \stackrel{V_j \neq \{\vec{0}\}}{=} \sup_{\substack{v_j \in V_j \\ \|v_j\|=1}} \|A(v_1, \dots, v_n)\| = \sup_{v_j \in V_j \setminus \{\vec{0}\} \forall j} \frac{\|A(v_1, \dots, v_n)\|}{\|v_1\| \cdots \|v_n\|}.$$

The set of all such bounded multilinear operators:

$$\mathcal{L}(V_1, \dots, V_n; W) = \left\{ A : V_1 \times \dots \times V_n \rightarrow W \mid A \text{ n-multilinear, } \|A\|_{op} < +\infty \right\}$$

is a normed vector space when equipped with $\|\cdot\|_{op}$ and is Banach if W is so. ♦

Exercise Sheet 4

1. In metric spaces, one often uses the so-called “ $\varepsilon - \delta$ characterization” of continuity. Prove that it is equal to the notion of topological continuity we defined, i.e., prove the following lemma:

Lemma 27. Given metric spaces (X, d_X) , (Y, d_Y) , a function $f : X \rightarrow Y$ is continuous at a point $a \in X$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad d_X(x, a) \leq \delta \implies d_Y(f(x), f(a)) \leq \varepsilon. \quad \blacklozenge$$

2. Let X, Y be topological spaces. Prove that if $f : X \rightarrow Y$ is continuous at $a \in X$, then it is sequentially continuous at a .

Bonus: Prove it is also net-continuous at a .

3. Let X, Y be metric spaces. Prove that if $f : X \rightarrow Y$ is sequentially continuous at $a \in X$, then it is continuous at a .

Bonus: Prove that net-continuous implies continuous in general topological spaces X, Y .

4. Prove Proposition 8.

5. Prove proposition 12.

6. Given a topological space $(X, \mathcal{T})$, prove that the family of all its closed sets $\mathcal{C} := \{X \setminus U \mid U \in \mathcal{T}\}$ satisfies (i),(ii), (iii) of Definition 42 (i.e., that $\mathcal{C}$ is a “clausurology”).

6. Given X a topological space, $(Y, \|\cdot\|)$ a Banach space, you already proved that $(B_\infty(X, Y), \|\cdot\|_\infty)$ (of Example 14.2) is a normed vector space. Given that, prove that:

(i) $B_\infty(X, Y)$ is a Banach space.

(ii) A closed subspace of an arbitrary Banach space is itself a Banach space when equipped with the restricted norm.

(iii) $C_b^0(X, Y)$ (as in Eq. (4.11)) is a closed subspace of $B_\infty(X, Y)$ (you can use Proposition 18). By (ii), this proves it is a Banach space.

7. Consider the set of *square summable* complex sequences

$$\ell^2(\mathbb{N}, \mathbb{C}) := \left\{ (z_1, z_2, \dots) \in \prod_{j \in \mathbb{N}} \mathbb{C} \mid \sum_{j=1}^{\infty} |z_j|^2 < +\infty \right\}.$$

(Recall the definition of an infinite sum of vectors Def. 31 where now the vectors are $\mathbb{C}$ -numbers.) Assume you know it is a vector subspace of $\prod_{j \in \mathbb{N}} \mathbb{C}$ and a normed vector space when equipped with the generalized Euclidean norm:

$$\|(z_1, z_2, \dots)\|_2 := \sqrt{\sum_{j=1}^{\infty} |z_j|^2}.$$

Prove that the unit closed ball is not compact.

8. Prove the Bolzano-Weierstrass theorem:

Given a topological space X , if $K \subseteq X$ is compact, then any sequence in K has a cluster point in K .

9. Given X, Y are topological spaces with Y Hausdorff, let there be $f, g : X \rightarrow Y$ continuous. Prove that if $A \subseteq X$ is dense (in X), then

$$\left(f|_A = g|_A \right) \iff \left(f = g \right).$$