Challenge

Find as many solutions as possible to

$$x^2 + y^2 = x \oplus y$$

with $x, y \in \mathbb{Z}$.

Solutions with Given Number of Digits

Assume y has k digits. Then

$$x^2 + y^2 = x \oplus y = 10^k x + y$$

Rearranging and solving for x

$$x^2 - 10^k x + y^2 - y = 0$$

And

$$\begin{aligned} x &= \frac{1}{2}\left(10^k \pm \sqrt{10^{2k} - 4(y^2 - y)}\right) \\ &= \frac{1}{2}\left(10^k \pm \sqrt{10^{2k} + 1 - (2y-1)^2}\right) \end{aligned}$$

Generating Solutions

Solution needs $10^{2k} + 1 - (2y - 1)^2$ to be square.

So decompose as

$$10^{2k} + 1 = (2y - 1)^2 + T^2$$

If y has k digits, get a solution.

Example: $k = 2$

Factorization $10^{2\times 2} + 1 = 10001 = 73 \times 137$.

Factors decompose as $73 = 3^2 + 8^2$, and $137 = 11^2 + 4^2$.

Get all decompositions:

$$\begin{aligned} 10^4 + 1 &= 100^2 + 1^2 \\ &= 76^2 + 65^2 \end{aligned}$$

Decomposition $10^4 + 1 = 100^2 + 1^2$

Get

$$y = \frac{1+1}{2} = 1$$

which has 1 digit. Don't get solution.

Would find

$$x = \frac{1}{2}\left(10^2 \pm 100\right) = 0, 100$$

giving 'solutions'

$$100^2 + 1^2 = 10001 = 100 \oplus 01$$

$$0^2 + 1^2 = 1 = 0 \oplus 1$$

Decomposition $10^4 + 1 = 76^2 + 65^2$

Get

$$y = \frac{1 + 65}{2} = 33$$

which has 2 digits.

Then

$$x = \frac{1}{2}\left(10^2 \pm 76\right) = 12, 88$$

.

Solutions $(x, y) = (12, 33), (88, 33)$.

Patterns

A solution is

$$x = 88321167\ 88321167\ 88321167\ 88321167\ 88321167\ 883212$$

$$y = 32116788\ 32116788\ 32116788\ 32116788\ 32116788\ 321168$$

An Infinite Family?

Having n repeating blocks

$$
\begin{aligned}
x &= \underbrace{88321167 \cdots 88321167}_{n \text{ blocks}} \; 883212 \\
&= 88321167 \sum_{i=0}^{n-1} 10^{8i+6} + 883212 \\
&= 88321167 \frac{10^6(10^{8n} - 1)}{10^8 - 1} + 883212 \\
&= \frac{88321167}{10^8 - 1} 10^{8n+6} + 883212 - \frac{88321167}{10^8 - 1} 10^6 \\
&= \frac{121}{137} 10^{8n+6} + \frac{44}{137}
\end{aligned}
$$

An Infinite Family?

So

$$x = \underbrace{88321167 \cdots 88321167}_{n \text{ blocks}}\ 883212$$
$$= \frac{121}{137} 10^{8n+6} + \frac{44}{137}$$

Similarly

$$y = \underbrace{32116788 \cdots 32116788}_{n \text{ blocks}}\ 321168$$
$$= \frac{44}{137} 10^{8n+6} + \frac{16}{137}$$

Does $x^2 + y^2 = x \oplus y$?

An Infinite Family?

$$
\begin{aligned}
& x^2 + y^2 \\
&= \left(\frac{121}{137}10^{8n+6} + \frac{44}{137}\right)^2 + \left(\frac{44}{137}10^{8n+6} + \frac{16}{137}\right)^2 \\
&= \frac{121^2 + 44^2}{137^2}10^{16n+12} + \frac{2(121 \times 44 + 44 \times 16)}{137^2}10^{8n+6} + \frac{44^2 + 16^2}{137^2} \\
&= \frac{121}{137}10^{16n+12} + \frac{88}{137}10^{8n+6} + \frac{16}{137}
\end{aligned}
$$

$$
\begin{aligned}
& x \oplus y \\
&= \left(\frac{121}{137}10^{8n+6} + \frac{44}{137}\right)10^{8n+6} + \left(\frac{44}{137}10^{8n+6} + \frac{16}{137}\right) \\
&= \frac{121}{137}10^{16n+12} + \frac{44+44}{137}10^{8n+6} + \frac{16}{137} \\
&= \frac{121}{137}10^{16n+12} + \frac{88}{137}10^{8n+6} + \frac{16}{137}
\end{aligned}
$$

Infinitely Many Solutions

So

$$x = \underbrace{88321167 \cdots 88321167}_{n \text{ blocks}}\ 883212$$
$$y = \underbrace{32116788 \cdots 32116788}_{n \text{ blocks}}\ 321168$$

is a solution.

There are infinitely many solutions!