Algebraic topology

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1 Set topology

The first part of the lecture consists of a short introduction to or a repetition of set topology.

1.1 Topological spaces

Recall that a **topological space** is a pair $(X, \mathcal{O})$ consisting of a set X and a system $\mathcal{O} \subset \mathcal{P}(X)$ of subsets of X such that

- $\emptyset, X \in \mathcal{O}$.
- $A, B \in \mathcal{O} \Rightarrow A \cap B \in \mathcal{O}$,
- $A_i \in \mathcal{O} \, \forall_{j \in J} \Rightarrow \bigcup_{j \in J} A_j \in \mathcal{O}$.

The system $\mathcal{O}$ is called a **topology** on X and the elements of $\mathcal{O}$ are called **open sets**.

Examples 1.1.1.

- The trivial topology: $\mathcal{O} = \{\emptyset, X\}$.
- The **discrete topology**: $\mathcal{O} = \mathcal{P}(X)$. Here every set is open.

Lemma 1.1.2. *Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Then there is a coarsest topology $\mathcal{O}_{\mathcal{E}}$, which contains $\mathcal{E}$. This is called the* **topology generated by** $\mathcal{E}$*.*

There is a way to construct $\mathcal{O}_{\mathcal{E}}$ as follows: Let $\mathcal{S}$ be the set of all finite intersections of sets from $\mathcal{E}$, together with $\emptyset$ and X, so

$$\mathcal{S} = \left\{A_1 \cap \cdots \cap A_n : n \in \mathbb{N},\ A_1, \ldots A_n \in \mathcal{E}\right\} \cup \left\{\emptyset, X\right\}.$$

Then $\mathcal{O}_{\mathcal{E}}$ is the system consisting of all unions of sets in $\mathcal{S}$.

Proof. Set

$$\mathcal{O}_{\mathcal{E}} = \bigcap_{\substack{\mathcal{O} \supset \mathcal{E} \\ \mathcal{O} \text{ topology}}} \mathcal{O}.$$

Then this is a topology, which is contained in every topology which contains $\mathcal{E}$.

Now for the construction. Let $\mathcal{O}'$ be the system given in the lemma. Since $\mathcal{O}_{\mathcal{E}}$ is a topology containing $\mathcal{E}$, it contains $\mathcal{S}$ and so it contains $\mathcal{O}'$. If we can show that $\mathcal{O}'$ is a topology, too, we get $\mathcal{O}' \supset \mathcal{O}_{\mathcal{E}}$ and so the two agree.

The only point, which is not obvious, is stability under intersections. So let $A, B \in \mathcal{O}'$, for instance

$$A = \bigcup_{j \in J} \underbrace{\left({}_{j,1} \cap \cdots \cap A_{j,n_j}\right)}_{=A_j} = \bigcup_{j \in J} A_j \qquad B = \bigcup_{i \in I} \underbrace{\left(B_{i,1} \cap \cdots \cap B_{i,n_i}\right)}_{=B_i} = \bigcup_{i \in I} B_i.$$

with $A_{i,j}, B_{i,j} \in \mathcal{E}$. Define A_j and B_j as in the displayed formula. Then we have

$$A \cap B = \left(\bigcup_{j \in J} A_j \right) \cap \left(\bigcup_{i \in I} B_i \right) = \bigcup_{j \in J} \bigcup_{i \in I} A_j \cap B_i$$

Since $A_j \cap B_i$ is in $\mathcal{S}$ for all i, j, the claim follows. □

Examples 1.1.3.

- The standard topology on $\mathbb{R}$ is generated by the system of open intervals.
- The topology of a metric space is generated by the system of all open balls.
- The discrete topology on any set X is generated by the system of

singletons $\mathcal{E} = \big\{\{x\} : x \in X\big\}$.

Definition 1.1.4. Let X be a topological space and $A \subset X$ a subset. On A we instal the **subspace topology**:

$$\mathcal{O}_A := \big\{U \cap A : U \text{ open in } X\big\}.$$

Example 1.1.5. The topology on $\mathbb{R}$ is the subspace topology of $\mathbb{R} \subset \mathbb{C}$.

Definition 1.1.6. Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by all **open products**, i.e. all sets of the form $U \times V$, where $U \subset X$ and $V \subset Y$ are open sets.

Lemma 1.1.7. *The open sets in $X \times Y$ are exactly the unions of open products.*

Proof. This follows from Lemma 1.1.2 and the fact that an intersection of two open products is an open product itself. □

Definition 1.1.8. A map $f : X \to Y$ between topological spaces is called **continuous**, if for every open set $U \subset Y$ the pre-image $f^{-1}(U) \subset X$ is an open set.

Example 1.1.9. For a map $f : \mathbb{R} \to \mathbb{R}$ this definition coincides with the definition of Analysis 1, as one sees using the $\varepsilon - \delta$-definition of continuity.

If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f$.

Lemma 1.1.10. (a) *Let Y be a space with a topology generated by a system $\mathcal{E} \subset \mathcal{P}(Y)$. Then a map $f : X \to Y$ from a topological space X is continuous if and only if $f^{-1}(E)$ is open for every $E \in \mathcal{E}$.*

(b) *Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ denote the projections. A map $f : Z \to X \times Y$ from a topological space Z to a product space is continuous if and only if the maps $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to Y$ are continuous.*

(c) *Let X be a topological space and $A \subset X$ a subspace. A map $f\colon Z \to A$ from a topological space Z is continuous if and only if the composition $Z \xrightarrow{f} A \xrightarrow{i} X$ is continuous, where $i\colon A \to X$ is the inclusion map.*

Proof. Analysis Book. □

Definition 1.1.11. A bijective map $f\colon X \to Y$ between topological spaces such that f and $f^{-1}\colon Y \to X$ are continuous, is called a **homeomorphism**. If a homeomorphism exists between X and Y, then the spaces X and Y are called **homeomorphic**.

Definition 1.1.12. Let X be a topological space. An **open neighbourhood** of a point x is an open set $U \subset X$, containing the point x.

A **neighbourhood** of x is an arbitrary subset $A \subset X$, which contains an open neighbourhood.

A **neighbourhood basis** $\mathcal{U}$ is a set of neighbourhoods U, such that for every neighbourhood V of x there is some $U \in \mathcal{U}$, such that $U \subset V$ holds. An **open neighbourhood basis** is one, which consists of open sets only.

Example 1.1.13. For $x \in \mathbb{R}$ the set of all intervals of the form $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, $n \in \mathbb{N}$ is an open neighbourhood basis.

Definition 1.1.14. A subset $C \subset X$ of a topological space is called **closed**, if its complement $C^c = X \setminus C$ is open. For an arbitrary set $A \subset X$ the **closure**;

$$\overline{A} := \bigcap_{\substack{S \supset A \\ S \text{ closed}}} S$$

is the smallest closed set containing A.

A subset $D \subset X$ of a space X is called **dense**, if $\overline{D} = X$.

Definition 1.1.15. Let $S \subset X$ be a subset of a topological space X. An open set $U \subset X$, containing S is called an **open neighbourhood** of S. A **neighbourhood** of S is a set which contains an open neighbourhood of S.

Lemma 1.1.16. *Let $A \subset X$. A given $x \in X$ lies in $\overline{A}$, if and only if for every open neighbourhood U of x one has $U \cap A \neq \emptyset$.*

Proof. Let $x \in \overline{A}$ and let V be an open set with $V \cap A = \emptyset$. Then $S = X \smallsetminus V$ is a closed set containing A, so $\overline{A} \subset S$ and thus $x \in S$, which means that V is not a neighbourhood of x. Therefore, $U \cap A \neq \emptyset$ for every open neighbourhood U of x.

The other way round, let $x \notin \overline{A}$. Then $V = X \smallsetminus \overline{A}$ is an open neighbourhood of x which does not meet A. So, if every open neighbourhood of x meets A, we get $x \in \overline{A}$. □

* * *

1.2 Hausdorff spaces and compactness

Definition 1.2.1. A topological space X is called a **Hausdorff space**, if any two points in X can be separated by disjoint open neighbourhoods, i.e., if

$$x \neq y \;\Rightarrow\; \exists_{U,V \subset X} \text{ open with } {}^{x \in U}_{y \in V} \text{ and } \; U \cap V = \emptyset.$$

Examples 1.2.2. (a) A metric space is a Hausdorff space, since for $x \neq y$ in X one has $r = d(x, y) > 0$, so $U = B_{r/2}(x)$ and $V = B_{r/2}(y)$ are disjoint open neighbourhoods of x and y.

(b) If X has at least 2 elements, then the trivial topology on X is not hausdorff.

Compactness

Definition 1.2.3. Let X be a topological space and let $K \subset X$ be a subset. A family $(U_i)_{i\in I}$ of subsets of X is called a **cover** of K, if $K \subset \bigcup_{i\in I} U_i$. It is called an **open cover**, if every U_i is open in X.

Here the case $K = X$ is included.

Example 1.2.4. The family of all open intervals $(k, k+2)$, $k \in \mathbb{Z}$ is an open cover of $\mathbb{R}$.

Definition 1.2.5. A **subcover** of $(U_i)_{i\in I}$ is a cover $(U_j)_{j\in J}$, where J is a subset of I. The set K is called **compact**, if every open cover has a finite subcover.

Example 1.2.6. A subset $K \subset \mathbb{R}^n$ is compact if and only if it is bounded and closed.

Definition 1.2.7. A family $(A_i)_{i\in I}$ is said to have the **finite intersection property**, if for every finite subset $E \subset I$ one has $\bigcap_{i\in E} A_i \neq \emptyset$.

Lemma 1.2.8. *A topological space X is compact if and only if for every family $(A_i)_{i\in I}$ of closed subsets with the finite intersection property one has*

$$\bigcap_{i\in I} A_i \neq \emptyset.$$

Proof. Take the definition of compactness and take complements everywhere. You end up with this lemma. □

Lemma 1.2.9. (a) *A closed subset of a compact space is compact itself.*

(b) *If X is hausdorff, then every compact subset is closed.*

Proof. (a) Let X be compact and $A \subset X$ a closed subset. Let $(U_i)_{i\in I}$ be an open cover of A. Then $X = \left(\bigcup_{i\in I} U_i\right) \cup A^c$ is an open cover of X, hence has a finite subcover, so There is a finite set of indices $E \subset I$ with $X = A^c \cup \bigcup_{i \in E} U_i$, which implies that $A \subset \bigcup_{i \in E} U_i$.

(b) Let X be hausdorff and $K \subset X$ compact. Let $a \in X \setminus K$. By the Hausdorff property, for any $x \in K$ there are two open sets U_x, V_x with $x \in U_x$, $a \in V_x$ and $U_x \cap V_x = \emptyset$. The family $(U_x)_{x \in K}$ is a cover of K, so there is a finite subcover, i.e., there are $x_1, \ldots, x_n \in K$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_n} =: U$. Let $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then V is an open neighbourhood of a such that $K \cap V = \emptyset$. Therefore, a does not lie in the closure $\overline{K}$ of K. Since this holds for any $a \notin K$, we have $K = \overline{K}$, so K is indeed closed. □

Proposition 1.2.10. *Continuous images of compact sets are compact.*

More precisely, let $f \colon X \to Y$ be a continuous map. If $K \subset X$ is compact, then the image $f(K) \subset Y$ is compact, too.

Proof. Let $(U_i)_{i \in I}$ be an open cover of $f(K)$. Then $\left(f^{-1}(U_i)\right)_{i \in I}$ is a cover of K. As K is compact, there is a finite set $E \subset I$ such that $K \subset \bigcup_{i \in E} f^{-1}(U_i)$. Applying f yields $f(K) \subset \bigcup_{i \in E} U_i$. □

* * *

1.3 Final topologies, collapsing and glueing

Lemma 1.3.1. *Let X be a set and $(f_i)_{i \in I}$ a family of maps $f_i \colon T_i \to X$, where each $(T_i, \mathcal{T}_i)$ is a topological space. Then there is a finest topology $\mathcal{O}$ on X, making all the maps f_i continuous. This is called the* **final topology** *induced by the f_i, $i \in I$.*

A subset U of X is open in the final topology if and only if all pre-images $f_i^{-1}(U)$ are open.

Proof. Define

$$\mathcal{O} = \left\{ U \subset X : f_i^{-1}(U) \text{ is open in } T_i \ \forall_{i \in I} \right\}.$$

Then $\mathcal{O}$ is a topology on X, since

- $\emptyset = f_i^{-1}(\emptyset) \in \mathcal{T}_i$, so $\emptyset \in \mathcal{O}$,
- $T_i = f_i^{-1}(X) \in \mathcal{T}_i$, so $X \in \mathcal{O}$,
- $A, B \in \mathcal{O} \ \Rightarrow \ f_i^{-1}(A \cap B) = f_i^{-1}(A) \cap f_i^{-1}(B) \in \mathcal{T}_i$ hence $A \cap B \in \mathcal{O}$.

The case of unions is treated in the same way.

Let $\mathcal{O}'$ be a topology on X, such that all f_i are continuous, so $f_i^{-1}(U) \in \mathcal{T}_i$ holds for every $U \in \mathcal{O}'$ and every $i \in I$, then it follows $\mathcal{O}' \subset \mathcal{O}$. □

Proposition 1.3.2. *Equip X with the final topology induced by $f_i \colon T_i \to X$. For a map $\phi \colon X \to Y$ one has:*

$$\phi \text{ is continuous} \quad \Leftrightarrow \quad \text{for every } i \in I \text{ the map } \phi \circ f_i \colon T_i \to Y \text{ is continuous.}$$

Proof. Clear by the last lemma. □

Definition 1.3.3. An important example of final topologies is given by quotients, i.e., sets of equivalence classes. Let T be a topological space and let $\sim$ be an equivalence relation on T. Let $X = T/\sim$ be the set of equivalence classes, so

$$X = \left\{ [t] : t \in T \right\},$$

where $[t] = \left\{ s \in T : s \sim t \right\}$ is the equivalence class of $t \in T$. One equips X with the final topology of the projection $p \colon T \to X$, $p(t) = [t]$ and calls this the **quotient topology**.

Collapsing

Example 1.3.4. On the disk

$$\mathbb{D}^2 := \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\}$$

one defines the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad \begin{cases} x = y \text{ or} \\ \|x\| = \|y\| = 1. \end{cases}$$

Then there are the equivalence classes $[x] = \left\{x\right\}$ if $\|x\| < 1$ and the class $S^1 = \left\{x \in \mathbb{R}^2 : \|x\| = 1\right\}$. The space $\mathbb{D}^2/\sim$ then is homeomorphic to the 2-sphere $S^2 = \left\{x \in \mathbb{R}^3 : \|x\| = 1\right\}$.

Definition 1.3.5. Let more generally $A \subset X$ be a subset of a topological space and define an equivalence relation on X by

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } x, y \in A.$$

Write X/A for the quotient space $X/\sim$. One says that the subspace A is **collapsed** to a point in X/A.

Glueing

Definition 1.3.6. Let $A \subset X$ and $f : A \to Y$ continuous. The **glueing** of X and Y along f is defined to be

$$X \sqcup Y \big/ \sim,$$

where $\sim$ is the equivalence relation generated by $a \sim f(a)$ for every $a \in A$.

Examples 1.3.7. (a) Let Z be a point and let $X = Y = S^1$, then $X \sqcup_Z Y$ is an eight-shaped figure, i.e., two circles with a common point.

(b) Let $X = [0,1] \times S^1$ be a cylinder, $A = \left(\{0\} \times S^1\right) \cup \left(\{1\} \times S^1\right)$ the union of the two ends, i.e., A is the disjoint union of two circles. Let $Y = \mathbb{C} \smallsetminus \left(D \cup E\right)$, where

$$D = \{z \in \mathbb{C} : |z| < 1\}$$
$$E = \{z \in \mathbb{C} : |z - 2| < 1\}.$$

Define $f : A \to Y$ by sending the first circle homeomorphically to the boundary of D and the second to the boundary of E. Then $X \sqcup_f Y$ is the plane with a handle attached.

* * *

2 Fundamental groups and coverings

2.1 Fundamental groups

Definition 2.1.1. We shall write

$$I = [0,1]$$

for the unit interval. A **path** in a topological space X is a continuous map $\gamma\colon I \to X$.

Definition 2.1.2. (Composition of paths) Let $\gamma\colon I \to X$ and $\eta\colon I \to X$ be paths with $\gamma(1) = \eta(0)$, then the **composition** $\gamma.\eta\colon I \to X$ is defined by

$$\gamma.\eta(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2}, \\ \eta(2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

This means that the path γ is run along first, then η. Further, the **reverse path** to $\gamma\colon I \to X$ is defined by

$$\check{\gamma}(t) = \gamma\big(1-t\big),$$

this path runs through γ, only backwards.

Definition 2.1.3. Let X, Y be topological spaces. Two continuous maps $f, g\colon X \to Y$ are called **freely homotopic**, if there is a continuous map $h\colon I \times X \to Y$, such that for every $x \in X$ one has $h(0,x) = f(x)$ and $h(1,x) = g(x)$.

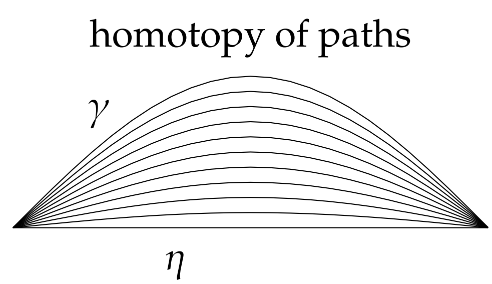
This means that two maps are freely homotopic if one can be deformed continuously into the other. Every map h as above is called **homotopy** from f to g.

We also write $h_s(x) = h(s,x)$, thus we consider $s \mapsto h_s$ as a path in the space of continuous maps.

Definition 2.1.4. Two paths $\gamma, \eta\colon I \to X$ in a topological space X are called **homotopic with fixed ends** or just **homotopic**, if no confusion arises, if there is a homotopy h from γ to η, such that it fixes the ends, i.e., for every $s \in I$ one has

$$h(t,0) = \gamma(0), \quad h(t,1) = \gamma(1).$$

This in particular implies that γ and η share the same endpoints.



If a path γ in X is homotopic to a constant path $\eta(t) = p$, then one says that γ is **nullhomotopic**.

Proposition 2.1.5. *Let X be a topological space.*

(a) *Homotopy with fixed ends is an equivalence relation on the set of paths in X. We write this relation as "$\simeq$".*

(b) *If $\gamma \simeq \gamma'$ and $\eta \simeq \eta'$ and further $\gamma(1) = \eta(0)$, then we have $\gamma.\eta \simeq \gamma'.\eta'$.*

(c) *If γ, η, τ are paths in X with $\gamma(1) = \eta(0)$ and $\eta(1) = \tau(0)$, then one has*

$$(\gamma.\eta).\tau \;\simeq\; \gamma.(\eta.\tau).$$

(d) *If c is a constant path $c(t) = p$, then one has*

$$\eta.c \simeq \eta \quad \textit{und} \quad c.\tau \simeq \tau$$

for all paths η and τ with $\eta(1) = p = \tau(0)$.

(e) *If γ is an arbitrary path in X, then $\gamma.\check{\gamma}$ is nullhomotopic.*

Proof. (a) Every path γ is homotopic to itself via $h(t,s) = \gamma(s)$. If h is a homotopy from γ to η, then $\check{h}(t,s) = h(1-t,s)$ is a homotopy from η to γ. Finally, let h_1 be a homotopy from γ to η and h_2 one from η to τ, then

$$h(t,s) = \begin{cases} h_1(2t,s) & 0 \le t \le \frac{1}{2}, \\ h_2(2t-1,s) & \frac{1}{2} < t \le 1, \end{cases}$$

is a homotopy from γ to τ.

(b) Let h_1 and h_2 be the respective homotopies. Then

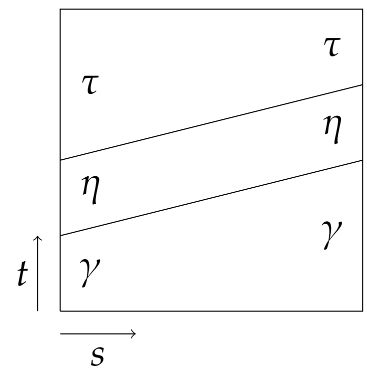
$$h(t,s) = \begin{cases} h_1(t,2s) & 0 \le s \le \frac{1}{2}, \\ h_2(t,2s-1) & \frac{1}{2} < s \le 1 \end{cases}$$

is a homotopy from $\gamma.\eta$ to $\gamma'.\eta'$.

(c) The path $(\gamma.\eta).\tau$ runs through γ in the first quarter of the unit interval, in the second, it runs through η and in the second half through τ. These intervals have to be moved. So one defines a homotopy h from $(\gamma.\eta).\tau$ to $\gamma.(\eta.\tau)$ by

$$h(t,s) = \begin{cases} \gamma\left(\frac{4s}{t+1}\right) & 0 \le s \le \frac{t+1}{4}, \\ \eta\left(4s-t-1\right) & \frac{t+1}{4} < s \le \frac{t+2}{4} \\ \tau\left(\frac{4s-t-2}{2-s}\right) & \frac{t+2}{4} < s \le 1. \end{cases}$$

The following picture shows the domain of h.



(d) The path $\eta.c$ is constant in the second half of the interval I. Then

$$h(t,s) = \begin{cases} \eta\left(\frac{2s}{t+1}\right) & 0 \le s \le \frac{t+1}{2}, \\ p & \frac{t+1}{2} < s \le 1 \end{cases}$$

is a homotopy from $\eta.c$ to η. The other case is treated in the same fashion.

(e) In this case one gets a homotopy by running through γ not completely to the end, but reversing earlier. More precisely we get a homotopy by

$$h(t,s) = \begin{cases} \gamma\Big(2(1-t)s\Big) & 0 \le s \le \frac{1}{2}, \\ \gamma\Big(2(1-t)s - 1\Big) & \frac{1}{2} < s \le 1. \end{cases}$$

□

Definition 2.1.6. If x_0 is a point in X and $G(x_0)$ is the set of all closed paths in X with end-point x_0. Further let

$$\pi_1(X, x_0) = G(x_0)/ \simeq$$

be the set of homotopy-classes (with fixed ends) in the endpoint x_0. The proposition implies, that the composition

$$[\gamma][\eta] := [\gamma.\eta]$$

on $\pi_1(X, x_0)$ is well-defined and turns $\pi_1(X, x_0)$ into a group. The neutral element is the class of the constant path and the inverse of a class $[\gamma]$ is $[\check{\gamma}]$. This group is called the **fundamental group** of X.

Definition 2.1.7. A space X is called **path-connected**, if any two points can be connected by a path, i.e., for any two $x, y \in X$ there exists a path γ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.1.8. *A path-connected space is* **connected***, i.e., If $X = U \sqcup V$ for two open sets U, V, then either $U = \emptyset$ or $V = \emptyset$.*

Proof. Analysis 2. □

Lemma 2.1.9. *Let X be a path-connected space. Then for any two $x_0, x_1 \in X$, the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Proof. Let γ be a path from x_0 to x_1. Then the map
$\phi_\gamma \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$, given by

$$\phi_\gamma([\eta]) = [\check{\gamma}.\eta.\gamma]$$

is a group homomorphism. The inverse map is $\phi_{\check{\gamma}}$. □

Notation. If the base-point x_0 is fixed, or if X is path-connected, we shall often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Note that in the path-connected case, the group does not depend upon the choice of a base point. So, as long as assertions about the group structure are concerned, like $\pi_1(X) \cong \mathbb{Z}$, the base-point doesn't play a role anyway, so it can very well be left out of the notation.

Lemma 2.1.10. *Let the spaces X, Y be homeomorphic. If X is path connected, then so is Y and the fundamental groups of X and Y are isomorphic.*

Proof. Let $\phi\colon X \to Y$ be a homeomorphism and let $y, y' \in Y$. Let γ be a path connecting $x = \phi^{-1}(y)$ and $x' = \phi^{-1}(y')$. Then the path $\phi \circ \gamma$ connects y and y'. Further, $\gamma \mapsto \phi \circ \gamma$ induces an isomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, when $y_0 = \phi(x_0)$. □

Example 2.1.11. Consider the space $X = \mathbb{T} = \left\{z \in \mathbb{C} : |z| = 1\right\} \cong S^1$. Then we have

$$\pi_1(X) \;\cong\; \mathbb{Z}.$$

For $z = 1$ an isomorphism $\mathbb{Z} \to \pi_1(X, 1)$ is given by

$$k \;\mapsto\; [\gamma_k],$$

where $\gamma_k(t) = e^{2\pi i k t}$.

We will only sketch a proof, as this assertion follows easily from coming sections. Consider the map $\pi\colon \mathbb{R} \to \mathbb{T}; t \mapsto e^{2\pi i t}$. A continuous map $\gamma\colon I \to \mathbb{T}$ with $\gamma(0) = \gamma(1) = 1$ can, in a unique way, be lifted to a continuous map $\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$, such that $\gamma = \pi \circ \tilde{\gamma}$. The map $\gamma \mapsto \tilde{\gamma}(1)$ is an inverse to $k \mapsto [\gamma_k]$.

Definition 2.1.12. A space $X \neq \emptyset$ is called **simply-connected**, if

- X is path connected and
- the fundamental group $\pi_1(X)$ is trivial.

Definition 2.1.13. A subset $S \subset \mathbb{R}^n$ is called **star shaped** if there exists a point $s_0 \in S$ such that for any point $s \in S$ the line segment

$$\left\{(1-t)s_0 + ts : t \in I\right\}$$

is contained in S. Any such s_0 is called a **central point** of S. This in particular means that S is path connected.

Lemma 2.1.14. *Let $n \in \mathbb{N}$. Any star shaped subset of $\mathbb{R}^n$ is simply-connected. If $n \geq 2$, then $S^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$ is simply-connected.*

Proof. Let $S \subset \mathbb{R}^n$ be star shaped. Translating S we can assume that $s_0 = 0$ is a central point. Let $\gamma : I \to S$ be continuous with $\gamma(0) = \gamma(1) = 0$. Then $h(s,t) = (1-s)\gamma(t)$ is a homotopy with fixed ends from γ to the constant path 0.

For S^n, let N be the north pole $N = (1, 0, \dots, 0)^t$ in S^n. WLOG we assume $p \neq N$. Every closed path γ with endpoint p is homotopic to a path avoiding N: To see this, let B be a small open ball around N, such that $p \notin \overline{B}$. Then $\gamma^{-1}(B)$ is an open subset of the interval $(0,1)$. Hence it is a disjoint union of its connected components, which are open intervals. Let $J = (a,b)$ be one such component. This means that $\gamma(t)$ enters B at $t = a$ and leaves it at $t = b$. The closure $\overline{B}$ is simply connected, hence $\gamma|_{[a,b]}$ can be homotoped with fixed ends to a path avoiding N. This construction can be done simultaneously with all connected components of $\gamma^{-1}(B)$ to end up with a path homotopic to γ, avoiding N. So we can replace γ with that graph and assume that γ avoids N.

The **stereographic projection** maps $S^n \setminus \{N\}$ homeomorphically to $\mathbb{R}^n$. In the latter, γ is homotopic to a constant path, so the same holds in S^n. □

Lemma 2.1.15. *Let X be simply-connected and let $p, q \in X$. Then any two paths γ and τ from p to q are homotopic with fixed ends.*

Proof. One has

$$\gamma \simeq \gamma.\check{\tau}.\tau \simeq p.\tau \simeq \tau. \qquad \square$$

* * *

2.2 Coverings

Definition 2.2.1. A **covering** of a space X is a continuous map

$$p : E \to X,$$

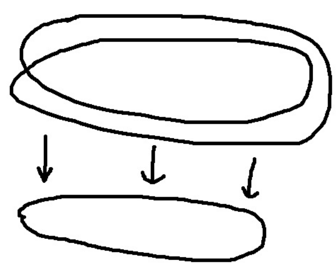
such that for every point $x \in X$ there exists an open neighbourhood U with the property that

$$p^{-1}(U) = \bigsqcup_{d \in D} V_d,$$

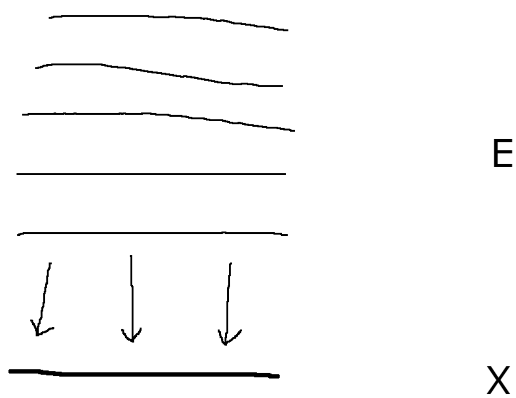
where D is a fixed index set and each V_d is mapped homeomorphically to U.

A neighbourhood U of x with this property is called a **trivialising neighbourhood**.

An example of a covering of the circle S^1 is depicted here:



Locally, a covering looks like this:



Definition 2.2.2. Let $p : E \to X$ be a covering and $x \in X$. The cardinality of $p^{-1}(x)$ is called the **local degree at** x. If the degree is the same for all $x \in X$, we call it the **degree of the covering** p.

Examples 2.2.3. (a) The trivial covering $p_1 : X \times D \to X$.

(b) The map $p : \mathbb{T} \to \mathbb{T}$ given by $p(z) = z^2$ is a non-trivial covering of degree 2.

(c) The map $p : \mathbb{R} \to \mathbb{T}$, given by $t \mapsto e^{2\pi i t}$ is a non-trivial covering of degree ∞.

Remark 2.2.4. A covering $p : E \to X$ is a **local homeomorphism**, i.e., for every $e_0 \in E$ there exists an open neighborhood V such that $p(V)$ is open in X and $P|_V : V \to p(V)$ is a homeomorphism.

Definition 2.2.5. A **homomorphism of coverings** from $E \to X$ to $F \to X$ is a continuous map $\psi : E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

commutes. If no confusion occurs, we call such a map a continuous map **over** X, or an X**-map**. An **isomorphism** is a homomorphism $\phi : E \to F$ which is bijective, such that the inverse map is a homomorphism, too.

Definition 2.2.6 (Lift)**.** Let $p : E \to X$ be a covering and let $f : S \to X$ be a continuous map. A continuous map $\tilde{f} : S \to E$ such that $f = p \circ \tilde{f}$ is called a **lift** of f to E. A lift does not necessarily exist.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ S \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Lemma 2.2.7 (Lifting of paths)**.** *Let $p : E \to X$ be a covering. Let $\gamma : I \to X$ be a continuous map and let $x = \gamma(0)$. Then to every $e \in p^{-1}(x)$ there is exactly one lift $\tilde{\gamma}_e : I \to E$ of γ, such that $\tilde{\gamma}_e(0) = e$.*

Proof. γ maps some small enough interval $[0, t_0)$ into a trivialising neighbourhood U of x_0. The homeomorphism $U \cong V_d$ with the component containing e allows for a lift of $\gamma|_{[0,t_0)}$ that maps 0 to e. This lift is unique, as the connected set $[0, t_0)$ can only be mapped into one component of the trivial covering $p^{-1}(U)$. Let s be the supremum of all t such that there exists a unique lift η of $\gamma|_{[0,t]}$ with $\eta(0) = e$. Then on $[0, s)$ there exists a unique lift. On a trivialising neighbourhood of $\gamma(s)$ this unique lift can be uniqely prolonged to $[0, s]$ and beyond s if $s < 1$, the latter case contradicting the maximality of s, whence $s = 1$ and the claim is proved. □

Lemma 2.2.8 (Lifting of homotopies)**.** *Let $h : I^2 \to X$ be continuous, let $x = h(0, 0)$ and $p : E \to X$ a covering. Then for each $e \in p^{-1}(x)$ there exists a unique lift $\tilde{h}$ of h with $h(0, 0) = e$.*

Proof. By the last lemma, there is a unique lift of $t \mapsto h(0, t)$. Let s_1 be the supremum of all $s \in I$ such that there is a unique lift on $[0, s] \times I$. We want to show that $s_1 = 1$, so we assume the contrary. Then there is a unique lift on $[0, s_1) \times [0, 1]$.xxx For each $0 \leq t \leq 1$ there exists an open ball $V_t \subset [0,1]^2$ around the point (s_1, t), which maps into a trivialising neighbourhood of $h(s_1, t)$, so that the lift can be extended to $[0, s_1) \times [0,1] \cup V_t$. The intersections of the V_t with the line $\{s_1\} \times [0,1]$ form an open cover of the compact line, hence finitely many $V_1, \dots, V_n$ suffice. The union $V_1 \cup \cdots \cup V_n$ contains a small strip around the line, and we can extend the unique lift to that strip and we get a contradiction to the maxilamity of s_1. Hence $s_1 = 1$ and the lemma is proven. □

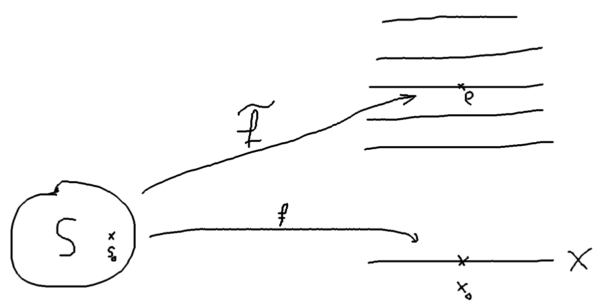
* * *

2.3 The universal covering, uniqueness

Theorem 2.3.1. (a) *Let $p : E \to X$ be a covering and let $f : S \to X$ a continuous map, where S is simply-connected. Let $s_0 \in S$ and let $x_0 = f(s_0) \in X$. Choose a point $e \in E$, with $p(e) = x_0$. Then there exists exactly one lift $\tilde{f}$ of f to E with $\tilde{f}(s_0) = e$. If f is a covering, then the map $\tilde{f}$ is a homomorphism of coverings.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ S \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

(b) *If X has a simply-connected covering, then it is uniquely determined up to isomorphy. We call it the* **universal covering** *and write it as $\tilde{X} \to X$.*



Proof. (a) Let $s \in S$ and fix a path γ_s in S from s_0 to s. Define

$$\tilde{f}(s) = (\widetilde{f \circ \gamma_s})_e(1).$$

That means, we first map γ_s to X, then lift it to E and then we evaluate at 1. We need to show that the value $\tilde{f}(s)$ does not depend on the choice of the path γ. Let η be another path from s_0 to s. As S is simply connected, there is a homotopy with fixed ends h between the two. This homotopy can uniquely be lifted and the lift gives a homotopy with fixed ends between the lifts $\tilde{\gamma}$ and $\tilde{\eta}$, so the endpoints of the lifts coincide.

Since p is a local homeomorphism, the map $\tilde{f}$ is continuous. The lifting property follows by construction. The uniqueness is clear as a given lift from S to E must map the path γ_s to the unique lift of $f \circ \gamma_s$.

(b) Let $p_E : E \to X$ and $p_F : F \to X$ be two simply-connected coverings, let $e \in E$ and $f \in F$. By part (a), there are uniquely determined X-homomorphisms $\psi : E \to F$ and $\phi : F \to E$ with $\psi(e) = f$ and $\phi(f) = e$. Then $\phi \circ \psi$ is the uniquely determined continuous X-map $E \to E$ mapping e to e. Therefore $\phi \circ \psi = \mathrm{Id}$. Similarly we get $\psi \circ \phi = \mathrm{Id}$. □

Corollary 2.3.2. *If X is simply-connected, then every covering is trivial.*

Proof. Let X be simply-connected and $p : E \to X$ be a covering. Let

$e_0 \in E$ and $x_0 = p(e_0)$. By the theorem, the identity map $X \to X$ has a unique lift to a continuous map $f_{e_0} : X \to E$ with $f_{e_0}(x_0) = e_0$. Let $D = p^{-1}(x_0)$ and define

$$\phi : X \times D \to E, \quad \phi(x, d) = f_d(x).$$

This map is surjective, since for $e_1 \in E$ we can apply the same construction and obtain a unique continuous map $f_{e_1} : X \to E$ with $f_{e_1}(x_1) = e_1$, where $x_1 = p(e_1)$. For $d_1 = f_{e_1}(x_0)$ the uniqueness implies that $f_{e_1} = f_{d_1}$ and so

$$\phi(x_1, d_1) = f_{d_1}(x_1) = f_{e_1}(x_1) = e_1.$$

Similarly, one sees that ϕ is injective. For each $d \in D$ the map $\phi(\cdot, d)$ is a homeomorphism onto its image, as p is an inverse. Since the topology on $X \times D$ is generated by the open sets in each sheet, the map ϕ^{-1} is continuous. Therefore, ϕ is an isomorphism of coverings. □

Definition 2.3.3. A space X is called **locally path-connected**, if every point possesses a path-connected neighbourhood.

Remark 2.3.4. Some authors would call a space locally path-connected, if every point possesses a neighbourhood base consisting of path-connected open neighbourhoods.

Lemma 2.3.5. *A connected space X, which is locally path-connected, is path-connected.*

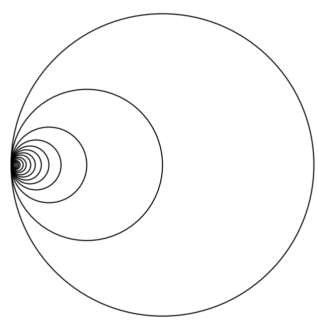
Proof. Let $x \in X$. Let $W(x)$ be the path-component of x, i.e. the set of all $y \in X$ such that there exist a path from x to y. If $y \in W(x)$, then $W(x) = W(y)$. We claim that $W(x)$ is open. So let $y \in W(x)$ and let U be a path-connected neighbourhood of y. Then $U \subset W(y) = W(x)$ and hence $W(x)$ is open. Now X decomposes into its pairwise disjoint path-components, which are open. As X is connected, there is only one path-component. $\square$

* * *

2.4 The universal covering, existence

Definition 2.4.1. A space X is called **locally simply-connected**, if every point has a simply-connected open neighbourhood.

Example 2.4.2. An example of a non-locally simply-connected space X is the **Hawaiian Earring**. Let $K_n \subset \mathbb{C}$ be the circle of radius $\frac{1}{n}$ around the point $\frac{1}{n}$ and let $X = \bigcup_{n=1}^{\infty} K_n$. Then no open neighbourhood on the point 0 is simply-connected.



Theorem 2.4.3. *Let X be connected and locally simply-connected. Then X has a universal covering $p : \tilde{X} \to X$.*

The fundamental group $\Gamma = \pi_1(X)$ acts by homeomorphisms on $\tilde{X}$, such that $X \cong \Gamma\backslash\tilde{X}$. For every connected covering $E \to X$ there is a subgroup Σ of Γ, such that $E \cong \Sigma\backslash\tilde{X}$.

Proof. The construction of $\tilde{X}$ is simple: choose a base point $x_0 \in X$ and define $\tilde{X}$ as the set of all paths τ with start point x_0 modulo homotopy with fixed ends. The projection $p : \tilde{X} \to X$ is given by

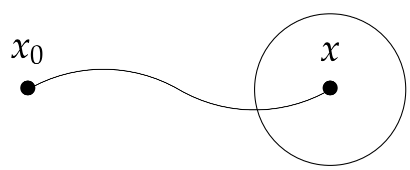
$$p([\tau]) = \tau(1).$$

The fundamental group $\Gamma = \pi_1(X, x_0)$ acts on $\tilde{X}$ by

$$[\gamma][\tau] = [\gamma.\tau]$$

for $[\gamma] \in \Gamma$ and $[\tau] \in \tilde{X}$. We give a topology on $\tilde{X}$ as follows: Let $[\tau] \in \tilde{X}$ and $x = \tau(1)$. Choose a simply-connected open neighbourhood $U \subset X$ of x. For every $y \in U$ choose a path σ_y from x to y, which runs inside U. Then σ_y is uniquely determined up to homotopy with fixed ends. Let

$$\tilde{U} = \tilde{U}_{[\tau]} = \left\{ [\tau.\sigma_y] : y \in U \right\}$$



Then $p|_{\tilde{U}}$ is a bijection $\tilde{U} \to U$. On $\tilde{U}$ we instal the topology induced by this bijection. Then we equip $\tilde{X}$ with the topology generated by all open sets in all $\tilde{U}$ found in this way. Then $p|_{\tilde{U}}$ is a homeomorphism. We claim that

$$p^{-1}(U) = \bigsqcup_{\substack{[\tau] \in \tilde{X} \\ \tau(1) = x}} \tilde{U}_{[\tau]}.$$

For this let $[\eta] \in p^{-1}(U)$ and write $u = \eta(1)$, then $[\eta.\check{\sigma}_u] \in p^{-1}(x)$ and the above construction applied to $[\eta.\check{\sigma}_u]$ yields copy $\tilde{U}_{[\eta.\check{\sigma}_u]}$ which is mapped to U homeomorphically. It remains to see that any two $\tilde{U}_{[\tau]}$ are disjoint. For this let $[\nu] \in \tilde{U}_{[\tau]} \cap \tilde{U}_{[\alpha]} = \emptyset$. Then $u = \nu(1) \in U$ and $\nu.\sigma_u^{-1}(1) = x$. Then $\nu.\sigma_u^{-1}$ is homotopic to τ and also to α, hence $[\alpha] = [\tau]$. Therefore p is a covering.

It remains to show that $\tilde{X}$ is simply-connected. For this let σ be a closed path in $\tilde{X}$ with start point $\tilde{x}_0$, where $\tilde{x}_0$ is the class of a constant path in X with endpoint $x_0 \in X$. Then σ is the unique lift of the path $p \circ \sigma$ starting at $\tilde{x}_0$. On the other hand, the class of $p \circ \sigma$ is an element of $\tilde{X}$ projecting onto x_0. Let τ_t be the path $s \mapsto p \circ \sigma(st)$. Then $t \mapsto [\tau_t]$ is a path, connecting $\tilde{x}_0$ to $[p \circ \sigma]$. This path $T : t \mapsto [\tau_t]$ is a lift of $p \circ \sigma$ with $T(0) = \tilde{x}_0$, so uniqueness implies $T = \sigma$. Therefore $\tilde{x}_0 = \sigma(1) = T(1) = [\tau_1] = [p \circ \sigma]$. This means that $p \circ \sigma$ is homotopic to the constant path and the corresponding homotopy lifts to a homotopy of σ to the constant path in $\tilde{X}$. □

Examples 2.4.4. (a) The map $\mathbb{R} \to \mathbb{T}$ given by $t \mapsto e^{2\pi i t}$ is the universal covering of $\mathbb{T} \cong S^1$.

(b) $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is the universal covering.

(c) We write $\mathbb{R}^\times$ for $\mathbb{R} \smallsetminus \{0\}$. Let $n \geq 2$ and let $\mathbb{P}^n(\mathbb{R})$ be the n-dimensional projective space, i.e.:

$$\mathbb{P}^n(\mathbb{R}) = \left(\mathbb{R}^{n+1} \smallsetminus \{0\}\right)\Big/\mathbb{R}^\times = S^n/\pm 1.$$

Then $p : S^n \to \mathbb{P}^n(\mathbb{R})$ is a double covering and since S^n is simply-connected, it is the universal covering.

* * *

2.5 Computation of the universal covering

Definition 2.5.1. A group action of a group Γ on a set M is **free**, if for every $m \in M$ and every $\gamma \in \Gamma$ one has

$$\gamma m = m \quad \Rightarrow \quad \gamma = 1,$$

This means that all stabiliser groups are trivial.

Definition 2.5.2. Let Y be a topological space. A group Γ acts **discontinuously** on Y, if every point $y \in Y$ has an open neighbourhood U such that

$$\gamma U \cap U \neq \emptyset \quad \Rightarrow \quad \gamma = 1.$$

If Γ acts discontinuously, then it acts freely.

Remark 2.5.3. Other sources use different definitions. Our definition of discontinuous action would, at most places, correspond to **properly discontinuous and free**.

Definition 2.5.4. We say that a group Γ acts **by continuous maps** on a space X, if for every $\gamma \in \Gamma$ the map $x \mapsto \gamma x$ is continuous.

Lemma 2.5.5. *Let G be a finite group acting by continuous maps and freely on a metric space (X, d). Then it also acts discontinuously.*

Proof. **Assume** not. Then there exists a point $x \in X$, such that for every $U_n = B_{1/n}(x)$, $n \in \mathbb{N}$ there is a $g_n \in G$ with $g_n \neq 1$ and $g_n U_n \cap U_n \neq \emptyset$. Since G is finite, one can assume $g_n = g$ for a fixed $g \in G$. This means that for every n there are $x_n, y_n \in U_n$ with $x_n = g y_n$. The sequences x_n and y_n both converge to x, so by continuity we get $gx = x$.
Contradiction! □

Definition 2.5.6. A group Γ acts **transitively** on a set M, if the set M consists of one orbit only, i.e., if

$$m, n \in M \quad \Rightarrow \quad \exists\, \gamma \in \Gamma \;:\; \gamma m = n.$$

Definition 2.5.7. Let X be path connected and $p : E \to X$ a covering. A **deck transformation** is a homeomorphism $d : E \to E$, such that the diagram

$$\begin{tikzcd} E \arrow[rr, "d"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. Let $\mathrm{D}(p)$ denote the group of all deck transformations.

Theorem 2.5.8. (a) *Let X be path-connected and $p : E \to X$ a covering. If E is path-connected and d a deck transformation with $d(e) = e$ for one $e \in E$, then $d = \mathrm{Id}_E$. In particular we get: If p is the universal covering, then* $\mathrm{D}(p) \cong \pi_1(X)$.

(b) *If a group Γ acts discontinuously by continuous maps on a space Y, then the map $\pi : Y \to \Gamma\backslash Y$ is a covering and the action induces an isomorphism $\Gamma \cong D(\pi)$.*

If moreover, Y is simply connected, then Y is the universal covering of $\Gamma\backslash Y$, i.e., we have $Y = \tilde{X}$ with $X = \Gamma\backslash Y$, as well as $\Gamma \cong \pi_1(X)$.

Proof. (a) Let $d(e) = e$ and let $f \in E$. Then there is a path α from e to f in E. Let $\gamma = p \circ \alpha$. Then α is the unique lift of γ to E with $\alpha(0) = e$, so $\alpha = \tilde{\gamma}_e$. On the other hand, $d \circ \alpha$ also is a lift of γ with $d \circ \alpha(0) = d(\alpha(0) = d(e) = e$. Therefore, $d \circ \alpha = \alpha$ and thus $d(f) = d \circ \alpha(1) = \alpha(1) = f$, this means that $d = \mathrm{Id}$.

If p is universal, then $\Gamma = \pi_1(X, x_0)$, with $x_0 = \pi(e)$, acts discontinuously on $\tilde{X}$ by deck transformations, so $\Gamma \hookrightarrow \mathrm{D}(p)$. The group Γ acts transitively on the fibre $F = p^{-1}(x_0)$. So let d be a deck transformation and $e \in F$. Then there is $\gamma \in \Gamma$ with $d(e) = \gamma(e)$, so $\gamma^{-1}d(e) = e$ and thus $\gamma^{-1}d = \mathrm{Id}$ or $\gamma = d$.

(b) Let Γ act discontinuously on Y. We show that the projection

$\pi : Y \to X := \Gamma\backslash Y$ is a covering. For this let $x \in \Gamma\backslash Y$, say $x = \Gamma y$. Let V be an open neighbourhood of y with $\gamma V \cap V \neq \emptyset \Rightarrow \gamma = 1$. Then $U = \pi(V)$ is an open neighbourhood of x and the diagram

$$\begin{tikzcd} \pi^{-1}(U) = \bigsqcup_{\gamma\in\Gamma} \gamma V \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & V \times \Gamma \cong U \times \Gamma \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

commutes. This means that π is a covering. As the action of Γ is free, it induces an injective group homomorphism $\Gamma \hookrightarrow D(\pi)$. Let $y_0 \in Y$ and let $x_0 = \pi(y_0)$ in X. The $D(\pi)$-orbit $D(\pi)y_0$ is a subset of the fibre $\pi^{-1}(x_0)$, which equals the Γ-orbit Γy_0. For given $d \in D(\pi)$, one thus gets $\gamma \in \Gamma$ with $dy_0 = \gamma y_0$, hence $d^{-1}\gamma y_0 = y_0$, so $d = \gamma$ by part (a). This means that Γ equals the group of deck transformations $D(\pi)$. □

* * *

2.6 The Seifert-Van Kampen Theorem

Proposition 2.6.1. *The fundamental groups of* $\mathbb{C}^\times$, $\mathbb{T}$ *and* $\mathbb{D}^\times = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$ *are isomorphic to* $\mathbb{Z}$*.*

Proof. Let γ be a closed path in $\mathbb{C}^\times$ with endpoint 1. Then $h(s,t) = \frac{\gamma(t)}{|\gamma(t)|^s}$ is a homotopy with fixed ends to a path in $\mathbb{T}$. We get isomorphisms $\pi_1(\mathbb{C}^\times) \cong \pi_1(\mathbb{T}) \cong \pi_1(\mathbb{D}^\times)$. One has $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, and since $\mathbb{Z}$ acts discontinuously on the simply-connected space $\mathbb{R}$, the fundamental group is $\mathbb{Z}$. □

Definition 2.6.2. Let G, H, L be groups and let $\phi : L \to G$, $\psi : L \to H$ group homomorphisms. Then the **amalgam**

$$G \star_L H$$

is defined as the set of all finite tuples or **words** of the form $(x_1, x_2, \dots, x_n)$ with $x_j \in G \sqcup H$ modulo the following reduction rules:

(a) $(\dots, x, y, \dots) = (\dots, xy, \dots)$,
if x, y both lie in G or both lie in H, and

(b) $(\dots, x, 1, y, \dots) = (\dots, x, y, \dots)$
where 1 is the neutral element of either G or H. Likewise

$$(1, x, \dots) = (x, \dots), \qquad (\dots, x, 1) = (\dots, x).$$

Finally,

(c) $(\dots, \phi(x), \dots) = (\dots, \psi(x), \dots)$
for every $x \in L$.

More precisely, we have

$$G \star_L H = \left(\bigcup_{k=1}^{\infty} (G \sqcup H)^k \right) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by (a),(b) and (c) above.

In the special case $L = \{1\}$ one writes this group as $G \star H$ and calls it the **free product** of G and H.

Remark 2.6.3. (i) The composition

$$(x_1, \dots, x_m)(y_1, \dots, y_n) = (x_1, \dots, x_m, y_1, \dots, y_n)$$

turns $G \star_L H$ into a group. The (class of) the tuple (1) is the neutral element.

(ii) The map $x \mapsto (x)$ is a group homomorphism $s_G : G \to G \star_L H$ and likewise for H. The group $G \star_L H$ is generated by the images if these two homomorphisms.

These homomorphisms are not necessarily injective. For instance, if $G = L$ and $\phi = \mathrm{Id}$, as well as $H = \{1\}$, then $G \star_L H = \{1\}$.

Examples 2.6.4. (a) $\mathbb{Z} \star \mathbb{Z} = F_2$ is called the free group in two generators. More gerenally, there is an obvious group-isomorphism $(G \star H) \star L \cong G \star (H \star L)$ for arbitrary groups G, H, L and the group $F_n = \mathbb{Z} \star \mathbb{Z} \star \cdots \star \mathbb{Z}$ with n copies of $\mathbb{Z}$ is called the **free group** in n generators.

(b) If $L = H$ and $\psi = \mathrm{Id}$, then $G \star_H H \cong G$, no matter, what ϕ looks like.

Proposition 2.6.5 (Universal property of the amalgam)**.** *The diagram*

$$\begin{tikzcd} & G \star_L H & \\ G \arrow[ur, "s_G"] & & H \arrow[ul, "s_H"'] \\ & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

is commutative and for every commutative diagram of group homomorphisms

$$\begin{tikzcd} & Z & \\ G \arrow[ur, "\alpha"] & & H \arrow[ul, "\beta"'] \\ & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

there is exactly one homomorphism $\eta : G \star_L H \to Z$, such that the diagram

$$\begin{tikzcd}
 & Z & \\
 & G \star_L H \arrow[u, "\eta"] & \\
G \arrow[ur, "s_G"] \arrow[uur, bend left, "\alpha"] & & H \arrow[ul, "s_H"'] \arrow[uul, bend right, "\beta"'] \\
 & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes.

Proof. Let $\eta(g_1, h_1, \dots, g_n, h_n) = \alpha(g_1)\beta(h_1)\cdots\alpha(g_n)\beta(h_n)$. The well-definedness and the universal property are easy to check. □

Theorem 2.6.6 (Seifert-van Kampen)**.** *Let $X = U \cup V$ with open sets U, V, such that $X, U, V, U \cap V \neq \emptyset$ are path connected. Then*

$$\pi_1(X) \cong \pi_1(U) \star_{\pi_1(U\cap V)} \pi_1(V).$$

Proof. Choose a base point $x_0 \in U \cap V$. The maps in the diagram

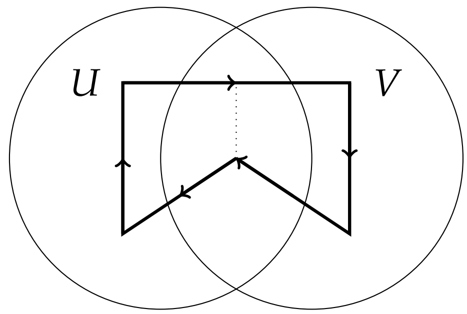
$$\begin{tikzcd}
 & \pi_1(X) & \\
\pi_1(U) \arrow[ur, "\alpha"] & & \pi_1(V) \arrow[ul, "\beta"'] \\
 & \pi_1(U \cap V) \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

are induced by inclusions. By the universal property of the amalgam there is a homomorphism $\eta : \pi_1(U) \star_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X)$ such that α

and β factors through η. We want to show, that η is an isomorphism.

Surjectivity: We need to show, that every closed path γ in X is homotopic to a composition of paths lying entirely inside U or V respectively.

The following picture shows the idea.



We cover the unit-interval $[0,1]$ with the connected components of $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. As $[0,1]$ is compact, finitely many suffice. This means, that there are numbers $0 = t_0 < \cdots < t_n = 1$, such that, say, $\gamma([t_{2k}, t_{2k+1}]) \subset U$ and $\gamma([t_{2k+1}, t_{2k+2}]) \subset V$ holds for all k. This means that γ is homotopic to a path $\gamma = \gamma_1.\gamma_2.\cdots.\gamma_n$, such that $\gamma_1 \subset U$, $\gamma_2 \subset V$ and so on, where all endpoints lie in $U \cap V$. Let η_1 be a path in $U \cap V$, connecting x_0 to $\gamma_1(1)$. Then

$$\gamma \cong \gamma_1.\eta_1^{-1}.\eta_1.\gamma_2.\dots.\gamma_n$$

and $\gamma_1.\eta^{-1}$ is a closed path with endpoint x_0, which lies in U entirely. Next let η_2 be a path connecting x_0 to $\gamma_2(1)$ inside $U \cap V$. Then

$$\gamma \cong \gamma_1.\eta_1^{-1}.\eta_1.\gamma_2.\eta_2^{-1}.\eta_2.\gamma_3.\dots.\gamma_n$$

and $\eta.\gamma_2.\eta^{-1}$ is a closed path in V. One repeats this construction to get $\gamma \cong \sigma_1.\dots.\sigma_n$ with closed paths σ_j, each of which lies in U or in V.

Injectivity: We write $G = \pi_1(U)$, $H = \pi_1(V)$ and $L = \pi_1(U \cap V)$. We further write $\overline{G}$ and $\overline{H}$ for the images in $\pi_1(X)$. Let $f = [(f_1, \dots, f_n)]$ be in $G \star_L H$ and let $[\gamma]$ be its image in $\pi_1(X)$. We need to show that

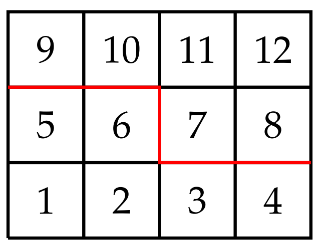
$$\gamma \simeq 1 \quad \Rightarrow \quad f = 1.$$

So let's suppose that γ is nullhomotopic in X. Let $h : [0,1] \times [0,1] \to X$ be a homotopy from γ to the constant path x_0.

Covering the compact square $[0,1] \times [0,1]$ by connected components of $h^{-1}(U)$ and $h^{-1}(V)$, one sees that there are $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$, such that every product

$$R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_i]$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$ either lies in $h^{-1}(U)$ or in $h^{-1}(V)$.



We number these products as indicated in the picture, where we assume that h maps the right and the left side of the square to x_0, which means that the paths run from the left to the right. If σ is any path inside $[0,1] \times [0,1]$, running from the left side to the right, then $h \circ \sigma$ is a closed path in X with endpoint x_0. The path γ is given by h restricted to the bottom of the square and the top of the square gives the constant path. Note that $\gamma = (h \circ \alpha_1). \dots .(h \circ \alpha_n)$, where the α_j are the lower sides of the boxes in the bottom row. Each $h \circ \alpha_j$ lies in $\overline{G}$ or in $\overline{H}$.

Let σ_r be the path in $[0,1] \times [0,1]$, which divides the first r rectangles from the others. In the picture you see σ_6 in red. Then σ_0 is the ground line and σ_{mn} the top line. We have $h \circ \sigma_0 = \gamma$ and $h \circ \sigma_{mn} = x_0$. Now

$h \circ \sigma_{r+1}$ differs from $h \circ \sigma_r$ by a homotopy on a sub-path which lies entirely in U or entirely in V, which accordingly means a corresponding relation in G or H. We extend this homotpy by constants to X to get $h \circ \gamma_r \sim h \circ \gamma_{r+1}$ and the theorem follows. □

Theorem 2.6.7. *Let $a_1, \dots, a_n$ be distinct points in $\mathbb{R}^2$. Then*

$$\pi_1\Big(\mathbb{R}^2 \smallsetminus \{a_1, \dots, a_n\}\Big) \ \cong\ F_n,$$

the free group in n generators.

Proof. The space $\mathbb{R}^2$ has a covering by open sets $U_1, \dots, U_n$ such that every U_j contains exactly one of the points $a_1, \dots, a_n$, the set U_j is simply-connected and $(U_1 \cup \cdots \cup U_k) \cap U_{k+1}$ is simply-connected. Let $V_j = U_j \smallsetminus \{a_1, \dots, a_n\}$. Then we have $\pi_1(V_j) \cong \mathbb{Z}$ and $\pi_1(V_1 \cup \cdots \cup V_{k+1}) \cong \pi_1(V_1 \cup \cdots \cup V_k) \star \pi_1(V_{k+1})$ and so, by induction

$$\pi_1(\mathbb{R}^2 \smallsetminus \{a_1, \dots, a_n\}) \cong \pi_1(V_1) \star \cdots \star \pi_1(V_n) \cong \mathbb{Z} \star \cdots \star \mathbb{Z} \cong F_n. \qquad \square$$

* * *

2.7 Homotopy Equivalence

Definition 2.7.1. Two continuous maps $f, g : X \to Y$ are called **freely homotopic** or just homotopic, if there is a continuous map

$$h : [0,1] \times X \to Y$$

with

$$h(0,x) = f(x), \quad h(1,x) = g(x)$$

for every $x \in X$. Wie write this as $f \sim g$.

For any two spaces X and Y write $[X, Y]$ for the set of homotopy classes of continuous maps $f : X \to Y$.

Examples 2.7.2. (a) Every path is freely homotopic to the constant path.

To see this, let $\gamma : [0,1] \to X$ be a path. Then $h(s,t) = \gamma((1-s)t)$ is a homotopy to the constant path with value $\gamma(0)$.

(b) The inclusion map $f : S^1 \to \mathbb{C}^\times$ is not homotopic to a constant map.

Definition 2.7.3. A continuous map $f : X \to Y$ is called a **homotopy equivalence**, if there exists a continuous map $g : Y \to X$ with

$$f \circ g \;\sim\; \mathrm{Id}_Y \quad \text{und} \quad g \circ f \;\sim\; \mathrm{Id}_X.$$

If there exists a homotopy equivalence between two spaces X and Y, then X and Y are called **homotopy equivalent** and we write $X \sim Y$.

Examples 2.7.4. (a) If X and Y are homeomorphic, then they are homotopy equivalent.

(b) $\mathbb{R}^n$ is homotopy equivalent to the point, as the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ is homotopic to the identity map by the homotopy

$$h(s,x) = sx.$$

Definition 2.7.5. A space X, which is homotopy equivalent to a point is called **contractible**. This means, there exists a point $p \in X$ and a homotopy $h : [0,1] \times X \to X$ such that

$$h(0,x) = x \qquad \text{and} \qquad h(1,x) = p$$

for every $x \in X$. This homotopy h is called a **contraction**.

Let $f : X \to Y$ be a continuous map of path connected spaces. We define the map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

by

$$f_*[\gamma] \coloneqq [f \circ \gamma].$$

This map is well-defined, since $\gamma \simeq \tau \implies f \circ \gamma \simeq f \circ \tau$. Further one has

$$[f \circ (\gamma.\tau)] = [(f \circ \gamma).(f \circ \tau)],$$

so that the map f_* is a group homomorphism.

Recall that every path η from y_0 to y_1 in a path-connected space Y induces an isomorphism

$$\begin{aligned} \theta_\eta : \pi_1(y_1, Y) &\xrightarrow{\cong} \pi_1(y_0, Y), \\ [\gamma] &\mapsto [\eta.\gamma.\check{\eta}]. \end{aligned}$$

Lemma 2.7.6. *Let X, Y be path-connected and $f, g : X \to Y$ be homotopic. Then there exists a path η from $f(x_0)$ to $g(x_0)$ such that*

$$f_* = \theta_\eta \circ g_*.$$

In particular, it g_* is an isomorphism, then so is f_*. The path η can be chosen as $\eta(t) = h(t, x_0)$ for any homotopy h from f to g.

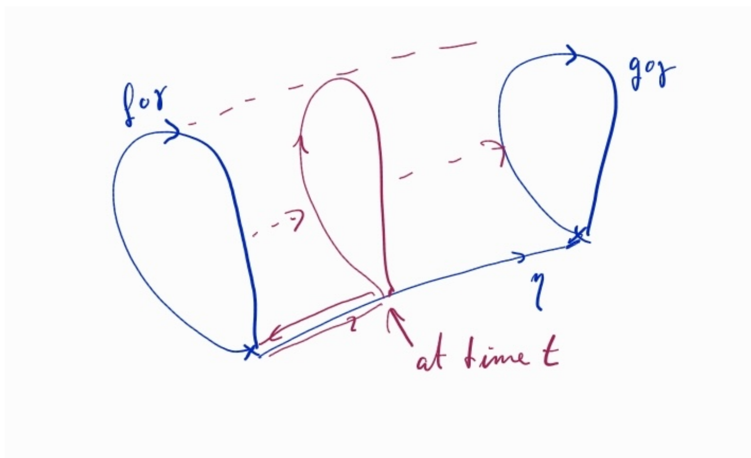
Proof. Let h be a homtopy from f to g and define $\eta(t) = h(t, x_0) \in Y$. Let

$\gamma \in G(x_0)$ be a closed path in X with endpoint x_0. Define

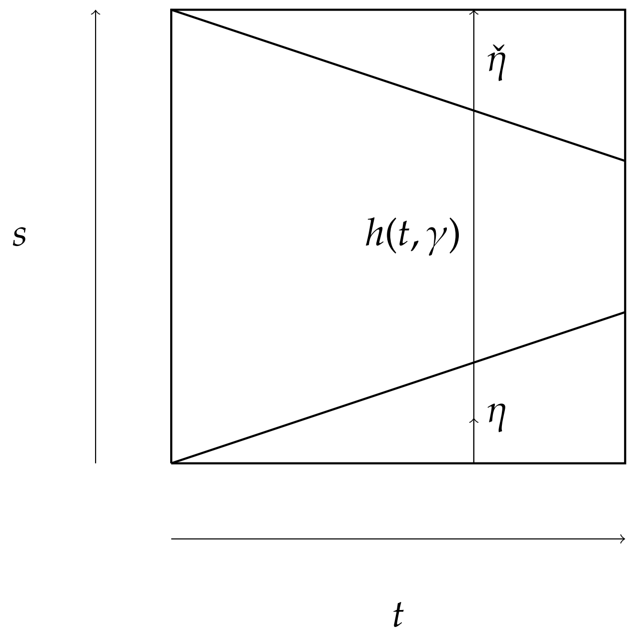
$$H : [0,1]^2 \to Y,$$
$$H(t,s) = \begin{cases} h\Big(3s, x_0\Big) & 0 \le s < \frac{t}{3} \\ h\Big(t, \gamma\Big(\frac{3s-t}{3-2t}\Big)\Big) & \frac{t}{3} \le s \le 1 - \frac{t}{3} \\ h\Big(3-3s, x_0\Big) & 1 - \frac{t}{3} < s \le 1. \end{cases}$$

One checks that H homotopes $f \circ \gamma$ to $\sigma.(g \circ \gamma).\check{\sigma}$ with fixed ends. $\square$

Below is a picture, that shows how H moves $f \circ \gamma$ to $\eta.(g \circ \gamma).\check{\eta}$.



Here's a sketch of the domain of $H(s,t)$:



Theorem 2.7.7. *If two path-connected spaces X and Y are homotopy equivalent, then*

$$\pi_1(X) \;\cong\; \pi_1(Y).$$

Proof. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then $(g \circ f)_* = g_* \circ f_*$ is homotopic to the identity. Since Id_* is the identity on the fundamental group, hence an isomorphism, the claim follows from the lemma. □

Definition 2.7.8. Let $A \subset X$ be a closed subset of a topological space X. The set A is called a **deformation retract** of X, if there exists a continuous map $h : [0,1] \times X \to X$, called a **deformation**, such that

$$\begin{aligned} h(0,x) &= x \qquad x \in X, \\ h(1,x) &\in A \qquad x \in X, \\ h(t,a) &= a \qquad a \in A. \end{aligned}$$

Note that the map $g : X \to A$, $g(x) = h(1, x)$ has the property $g|_A = \mathrm{Id}_A$. Such a map is called a **retraction**.

Examples 2.7.9. (a) S^1 is a deformation retract of $\mathbb{C} \setminus \{0\}$. A deformation is given by $h(s, z) = \frac{z}{|z|^s}$.

(b) The Sphere S^n is not a deformation retract of the closed unit Ball $B^{n+1} \subset \mathbb{R}^{n+1}$.

Proof. We use **Brouwer's Fixed Point Theorem** (Analysis): *Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.* So **assume** there is a retraction map $g : B^{n+1} \to S^n$. Let

$$f : B^{n+1} \to B^{n+1}, \qquad f(x) = -g(x).$$

Then f has a fixed point x_0. As $f(B^{n+1}) \subset S^n$, we get $x_0 \in S^n$. Therefore

$$x_0 = f(x_0) = -g(x_0) = -x_0,$$

which is a **contradiction!** □

Proposition 2.7.10. *If A is a deformation retract of X, then the inclusion map $f : A \hookrightarrow X$ is a homotopy equivalence. A homotopy inverse is any* **retraction map** *$g(x) = h(1, x)$, where h is a deformation.*

In particular, if X is path-connected, then so is A and we have

$$\pi_1(X) \cong \pi_1(A).$$

Proof. Let h be a deformation and set $g : X \to A$, $g(x) = h(1, x)$. Then $g \circ f = \mathrm{Id}_A$ and $f \circ g = g$ is homotopic to Id_X via h. □

* * *

3 Manifolds and cell complexes

3.1 Manifolds

Definition 3.1.1. A space X is called **locally euclidean** of dimension n, if every point $x \in X$ has an open neighbourhood U, which is homeomorphic to $\mathbb{R}^n$. A **manifold** is a locally euclidean Hausdorff space whose topology is countably generated.

Examples 3.1.2. (a) $\mathbb{R}^n$, S^n, $\mathbb{R}^n/\mathbb{Z}^n$.

(b) Möbius band.

Definition 3.1.3. A **surface** is a 2-dimensional connected manifold.

Examples 3.1.4. (a) The torus $\mathbb{R}^2/\mathbb{Z}^2$.

(b) The projective plane $\mathbb{P}^2(\mathbb{R}) = S^2/\pm 1$.

(c) There are locally euclidean spaces which are connected, but the topology is not countably generated.

The famous example is the **long line**. The construction uses transfinite induction and needs a bit of preparation.

Definition 3.1.5. A **well-ordering** on a set M is a partial order $\leq$ such that every subset $\emptyset \neq S \subset M$ possesses a smallest element, i.e., there exists $s_0 \in S$ with $s_0 \leq s$ for every $s \in S$.

The natural numbers with there natural order are well ordered, the set $\mathbb{Z}$ is not.

With Zorn's lemma one shows that every set X possesses a well-ordering.

A well ordering is always linear. For any two linearly ordered sets A, B let $A.B$ be the disjoint union $A \sqcup B$, which is linearly ordered by extending the orderings on A and B and insisting that $a < b$ for all $a \in A, b \in B$.

So let $\leq$ be a well ordering on some set Ω of the cardinality of $\mathcal{P}(\mathbb{R})$. For $\omega \in \Omega$ let $T_\omega = \{\eta \geq \omega\}$ and let A be the set of all $\omega \in \Omega$ such that T_ω is finite. Then A has a smallest element a_0 and by linearity of the ordering it follows that $A = T_{a_0}$ is finite. We remove this finite set from Ω and thus we can assume that Ω has no maximal element.

Let ω_0 be the smallest element of Ω. For every $\omega \in \Omega$ there is a smallest $\eta \in \Omega$ with $\eta > \omega$. In this case we write $\omega + 1 = \eta$ and we call η the **successor** of ω. We also write $\omega + 2 = \omega + 1 + 1$ and so on. Then the set $N = \{\omega + n : n \in \mathbb{N}\}$ is a copy of $\mathbb{N}$ with the natural order inside Ω. This copy N divides Ω in three parts $\Omega = A.N.B$.

If $\lambda \in \Omega$ is not the successor of any other element, we call λ a **limit number**. Note that if λ is a limit number and $\omega \in \Omega$ with $\omega < \lambda$, then $\omega + 1, \omega + 2, \dots < \lambda$ as well. We say that $\omega \in \Omega$ is a **countable element**, if the set $S_\omega = \{\gamma \in \Omega : \gamma < \omega\}$ is countable. There is a smallest uncountable element λ_1, which must be a limit number.

We construct a family of linearly ordered sets as follows. Let L_{ω_0} be the open unit interval. For every $\omega < \lambda_1$, for which L_ω is already defined, let $L_{\omega+1}$ be $L_\omega.[0,1)$. For every limit number $\lambda \leq \lambda_1$ one sets

$$L_\lambda = \bigcup_{\omega<\lambda} L_\omega.$$

Then the long line is defined to be $L = L_{\lambda_1}$. We equip L with the order topology, i.e., each open set is a union of open intervals (ω, τ) with $\omega < \tau$ in L. We can view S_{λ_1} as a subset of L by identifying ω with the element 0 in $L_{\omega+1} = L_\omega.[0,1)$. Then $I_\omega = (\omega, \omega + 1)$ is homeomorphic to $(0,1)$ and

$$L = S_{\lambda_1} \sqcup \bigsqcup_{\omega<\lambda_1} I_\omega.$$

We now claim:

(a) L is locally euclidean.

(b) L is connected.

(c) The topology of L is not countably generated.

(d) L is not separable.
A topological space is called **separable** if it contains a countable dense subset.

Proof. (a) For this we claim that for every countable ω the set L_ω is order-isomorphic (and hence homeomorphic) to $(0,1)$. For this, **assume** the contrary. Then let ω be the smallest countable element such that L_ω is not order-isomorphic to $(0,1)$. If ω is a limit number, then $L_\omega = \bigcup_{\tau<\omega} L_\tau$ is an increasing countable union of open intervals isomorphic to $(0,1)$, and thus it is isomorphic to $(0,1)$ itself, **contradiction!**. If otherwise $\omega = \eta + 1$, we have $L_\omega = L_\eta.[0,1)$ and this is order-isomorphic to $(0,1)$, too. **Contradiction!** Now let $x \in L$, then there is a smallest ω such that $x \in L_\omega$. This ω cannot be a limit number, because then ω would have appeared earlier. Hence $\omega \neq \lambda_1$ and so ω is countable. But then $L_\omega \cong (0,1)$ and so L is locally euclidean.

(b) We have that $L = \bigcup_{\omega<\lambda_1} L_\omega$ is an increasing union of spaces L_ω each of wich is homeomorphic to $(0,1)$ and thus connected. This implies that L is connected.

(c) and (d) Let $\mathcal{E}$ be any generating set for the topology on L. Then we show that $\mathcal{E}$ is uncountable. For this we can replace $\mathcal{E}$ by the set of finite intersections of elements of $\mathcal{E}$ and thus assume that $\mathcal{E}$ is closed under finite intersections. Then every open set is a union of elements of $\mathcal{E}$. So in particular, every interval of the form $(\omega, \omega+1)$, $\omega < \lambda_1$ contains a non-empty element E_ω of $\mathcal{E}$. Since the intervals $(\omega, \omega+1)$, $\omega \in S_{\lambda_1}$ are pairwise disjoint, we get an injective map $S_{\lambda_1} \to \mathcal{E}$, $\omega \mapsto E_\omega$. Since S_{λ_1} is uncountable, hence so is $\mathcal{E}$.

Next, any dense subset D must contain an element d_ω of $(\omega, \omega+1)$ and then $\omega \mapsto d_\omega$ is an injection $S_{\lambda_1} \hookrightarrow D$ and thus D is uncountable. □

Definition 3.1.6. Let S be a surface. You can **attach a crosscap** to S in the following way: You cut out an open disk and then you identify opposite points on the boundary.

Theorem 3.1.7. *Every compact surface S is exactly one of the following:*

(a) *A sphere S^2 with g handles attached. The number $g \geq 0$ is called the* **genus** *of S.*

(b) *A sphere with h crosscaps. The number $h \geq 1$ is called the (non-orientable) genus of S.*

the ones under (a) are the orientable surfaces, where the ones under (b) are non-orientable.

One can replace (b) with

(b′) (i) *If h is odd: A real projective plane $\mathbb{P}$ with $(h-1)/2$ handles attached to it, or*

(ii) *If h is even: A Klein bottle with $(h-2)/2$ handles attached to it.*

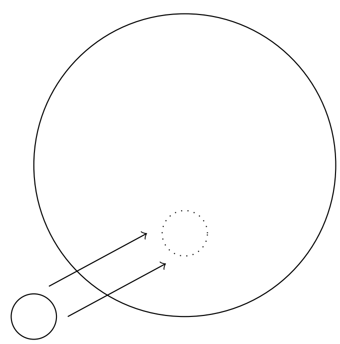
Proof. Omitted. □

* * *

3.2 Cell complexes

Definition 3.2.1. Let X be a topological space and let $f : S^{n-1} \to X$ be a continuous map. Let $\mathbb{D}^n = \left\{x \in \mathbb{R}^n : \|x\| \leq 1\right\}$. Then S^{n-1} is a subset of $\mathbb{D}^n$. Let $Y = X \sqcup_{S^{n-1}} \mathbb{D}^n$ the glueing of X and $\mathbb{D}^n$ along S^{n-1}.

In this case we say, that Y is obtained from X by glueing on an n-**cell**. This **cell** is the interior of $\mathbb{D}^n$.



Attaching a 2-cell is a bit like putting a lid on a hole, with the difference that there needn't be a hole.

Likewise, one can extend X by a family of n-cells simultaneously. For this let Ω be an index set and for every $\omega \in \Omega$ let $\mathbb{D}^n_\omega$ be a copy of $\mathbb{D}^n$. Further, let S^{n-1}_ω be the boundary of $\mathbb{D}^n_\omega$. Let $f_\omega : S^{n-1}_\omega \to X$ be continuous maps and let $D = \bigsqcup_{\omega\in\Omega} \mathbb{D}^n_\omega$ and $S = \bigsqcup_{\omega\in\Omega} S^{n-1}_\omega$. Then $X \sqcup_S D$ is the extension of X by the n-cells $\mathbb{D}^n_\omega$.

A **cell complex** is a topological space X together with a sequence of closed subsets $(X_n)_{n\geq 0}$ such that

- $X = \bigcup_{n\geq 0} X_n$,

- X_0 is discrete.
- X_n is an extension of X_{n-1} by a family of n-cells.
- X has the final topology of the embeddings of the $X_n \hookrightarrow X$.

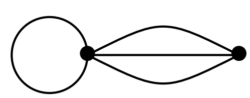
It follows that a map $X \to Y$ to a space Y is continuous if and only if the induced map $X_n \to Y$ is continuous for every n. Further, $U \subset X$ is open/closed if and only if $U \cap X_n$ is open/closed for every n.

If $X = X_n$ for some n and if n is minimal with this property, then we say that X has dimension n.

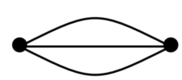
Examples 3.2.2.

(a) Manifolds are cell complexes.

(b) A 1-dimensional cell complex is a multigraph.



Definition 3.2.3. A cell complex X is called **regular**, if for every cell the glueing map $f : S^{n-1} \to X_{n-1}$ is a homeomorphism onto its image. Example: a multigraph without loops.



The set X_n is called the n**-dimensional skeleton** of X.

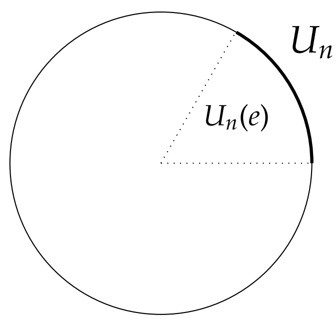
Proposition 3.2.4. *Every cell complex is a Hausdorff space.*

Proof. Let $X = \bigcup_n X_n$ be a cell complex and let $x \neq y$ in X. Let n be the smallest index with $x, y \in X_n$. So at least one of them is "new". Let's assume that $x \in X_n \smallsetminus X_{n-1}$. If x, y are in the same cell e, then there are open neighbourhoods in e, which separate x and y. If $x \in e$ and

$y \in X_n \setminus e$, then there are open neighbourhoods, which separate x and $X_n \setminus e$. In any case there are open subsets U_n, V_n of X_n with $x \in U_n$, $y \in V_n$ and $U_n \cap V_n = \emptyset$. We show that, when one attaches an $(n+1)$-cell e, then there are open sets $U_n(e), V_n(e) \subset X_n \cup e$ with $U_n(e) \cap X_n = U_n$, $V_n(e) \cap X_n = V_n$ and $U_n(e) \cap V_n(e) = \emptyset$. These sets are obtained as follows: Identify e with the interior of $\mathbb{D}^n$, and let $f : \partial\mathbb{D}^n \to X_n$ be the glueing map. Then set

$$U_n(e) = U_n \cup \left\{ x \in e : x \neq 0 \;\; f\left(\frac{x}{\|x\|}\right) \in U_n \right\}.$$

We call $U_n(e) \cap e$ the **pizza piece** attached to U_n.



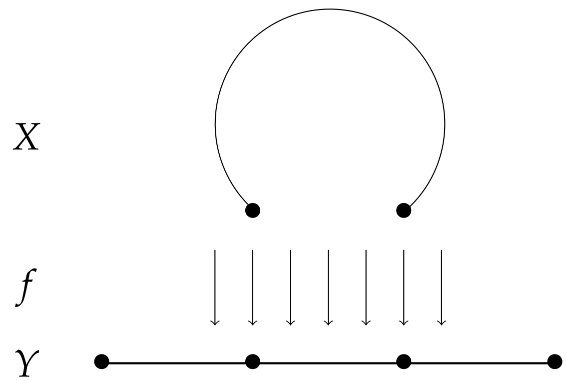
Taking the union of these sets for all $(n+1)$-cells one gets open sets $U_{n+1}, V_{n+1} \subset X_{n+1}$, which separate x and y and satisfy $U_{n+1} \cap X_n = U_n$, as well as $V_{n+1} \cap X_n = V_n$. Inductively, one gets a sequence of such sets and one defines

$$U = \bigcup_n U_n, \qquad V = \bigcup_n V_n.$$

By the definition of the topology on X, theses are open sets which separate x and y. $\square$

Definition 3.2.5. A continuous map $f : X \to Y$ between cell complexes is called **cellular**, if $f(X_n) \subset Y_n$ holds for every $n \geq 0$.

Note that a cellular map f does not need to send cells to cells. An example is given in the picture. Here, f is the orthogonal projection.



A **subcomplex** of a cell complex is a closed subset, which is a union of cells. Any subcomplex is a cell complex itself by the inherited structure.

Lemma 3.2.6. *Let $A \subset X$, where X is a cell complex, with $|A \cap e| \leq 1$ for every cell e. Then A is closed and discrete.*

Proof. Let $U = X \setminus A$ and $U_n = U \cap X_n$. For each cell e let $A_e = A \cap e$ which is empty or one point. Then U_0 is open in X_0. Next suppose we have shown that U_n is open in X_n. Then U_{n+1} equals $U_n \cup \bigcup_e (e \setminus A_e)$. Now $e \setminus A_e$ is open on X_{n+1} and its boundary equals the boundary of the whole cell e. Therefore, U_{n+1} is open in X_{n+1}. Therefore U is open, i.e., A is closed.

For each $a \in A$, let e_a be the unique cell containing a and let $n = \dim e_a$. Then $e_a \cup \bigcup_{\dim e > n} (e \setminus A)$ is an open neighbourhood of a which does not contain any other element of A, so A is discrete. □

Theorem 3.2.7. *A closed subset $K \subset X$ of a cell complex X is compact if and only if it only meets finitely many cells.*

Proof. Let $K \subset X$ be compact. For every cell e, which meets K, choose a point a_e in $K \cap e$. Let A be the set of all these points. By the lemma, A is closed and discrete. As K is compact, so is A and so A is compact and discrete, hence finite.

The other way around assume that K is closed and meets only finitely many cells $e_1, \dots, e_n$. Then $K \subset \bar{e}_1 \cup \dots \cup \bar{e}_n$ and the latter is compact, hence K is. □

Theorem 3.2.8. *A cell complex X is locally simply-connected.*

More sharply, to any open neighbourhood V of a point $x \in X$ there exists an open neighbourhood $U \subset V$ which is simply-connected.

Proof. Let $X = \bigcup_n X_n$ be a cell complex. Let $x \in V \subset X$ as in the theorem. Let n_0 be the smallest index with $x \in X_{n_0}$. Then x lies in an n_0-cell e and $V \cap e$ contains a simply-connected neighbourhood U_{n_0} in X_{n_0} of x.

Inductively, let U_n be a simply-connected open neighbourhood of x in $V \cap X_n$. We show that there is a simply-connected open neighbourhood U_{n+1} of x in $V \cap X_{n+1}$, with $U_{n+1} \cap X_n = U_n$. For this let e be an $(n+1)$-cell and let $f : S^n \to X_n$ be the corresponding glueing map. Let

$$U_{n+1}^e = \left\{ y \in \mathbb{D}^{n+1} \setminus S^n : y \neq 0,\ f\left(\frac{y}{\|y\|}\right) \in U_n \right\}.$$

There exists $0 < \varepsilon < 1$, such that

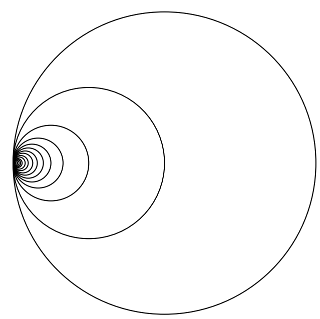
$$\tilde{U}_{n+1}^e := U_{n+1}^e \cap \left\{ x : \|x\| > 1 - \varepsilon \right\} \ \subset\ V.$$

Set

$$U_{n+1} := \bigcup_e \tilde{U}_{n+1}^e \cup U_n,$$

where the union runs over all $(n+1)$-cells e. Every closed path in U_{n+1} with ends in U_n is homotopic with fixed ends to a path in U_n. Since U_n is simply-connected and U_{n+1} path-connected, the set U_{n+1} is simply-connected. Any path γ in $U = \bigcup_n U_n$ meets only finitely many cells, as the image of γ is compact. So γ lies in U_n for some n and thus is null-homotopic. So U is simply-connected. □

Example 3.2.9. The Hawaiian Earring, see Example 2.4.2,



is not a cell complex, since the point 0 has no simply-connected neighbourhood.

With a different topology, however, it becomes a cell complex. For this, one has to draw all circles at an equal size.

* * *

3.3 Simplicial complexes

Definition 3.3.1. A **simplicial complex** over a set V is a system $\mathcal{S} \subset \mathcal{P}(V)$ of non-empty finite subsets, such that

(a) For every $v \in V$ one has $\{v\} \in \mathcal{S}$.

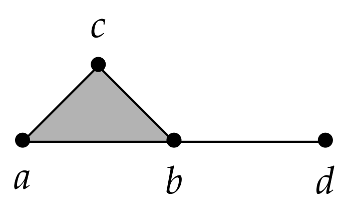
(b) $E \in \mathcal{S}$ and $\emptyset \neq F \subset E$ implies $F \in \mathcal{S}$. In this case, F is called a **face** of E.

Every element $E \in \mathcal{S}$ is called a **simplex** of the complex $\mathcal{S}$. An element of V is called a **vertex** of the complex.

If E is a simplex, then the **dimension** of E is by definition

$$\dim E = |E| - 1.$$

In the following picture, we find two maximal simplices $\{a, b, c\}$ and $\{b, d\}$.



Definition 3.3.2. Let $(V, \mathcal{S})$ be a simplicial complex. For every $E = \left\{v_0, \ldots, v_n\right\} \in \mathcal{S}$ with $(n+1)$ elements choose an n-simplex $\Delta_E \subset \mathbb{R}^n$ and use $v_0, \ldots, v_n$ to label the vertices of the simplex Δ_E. Set

$$X(V, \mathcal{S}) := \left(\bigsqcup_{E \in \mathcal{S}} \Delta_E \right) \Big/ \sim,$$

where $\sim$ is the equivalence relation, which for $F \subset E$ identifies the simplex Δ_F with the corresponding face of Δ_E. Then $X(V, \mathcal{S})$ is a cell complex with vertex set V and for a finite set $E \subset V$ one has

$$E \in \mathcal{S} \quad \Leftrightarrow \quad \text{there exists a cell } e, \text{ such that } E \text{ is the set of vertices of } e.$$

The cell complex $X(V, \mathcal{S})$ is called the **geometric realisation** of the simplicial complex $(V, \mathcal{S})$.

Remark 3.3.3. The geometric realisation X of a simplicial complex $(V, \mathcal{S})$ has the following properties:

- it is regular, i.e., each glueing map $f : S^n \to X$ is a homeomorphism onto its image.

- every n-cell has exactly $n + 1$ vertices and
- two cells with the same set of vertices coincide, i.e.:

$$\mathrm{vert}(e) = \mathrm{vert}(f) \;\Rightarrow\; e = f.$$

When speaking of simplicial complexes, we shall often identify the simplicial complex with its geometric realisation.

Definition 3.3.4. If X is a simplicial complex and e a cell of X, then its closure $\bar{e}$ is called a **simplex** of the complex X. It consists of the cell together with its faces.

Example 3.3.5. Let V be a set. The **full simplicial complex** over V is the set $\mathcal{S}$ of all finite subsets of V.

Definition 3.3.6. A simplicial complex X is called **star-shaped**, if there exists a vertex v_0 such that X is the union of all simplicies which contain v_0. Such a vertex v_0 is called a **central** vertex.

The full complex over a set V is always star-shaped. In the following picture, the first triangle is star shaped, the second is not.

Proposition 3.3.7. *A star-shaped simplicial complex X is contractible.*

Proof. Let $v_0 \in V$ be a central vertex. For every simplex Δ, there is a canonical homotopy $h_\Delta : [0, 1] \times \Delta \to \Delta$, which contracts Δ to v_0:

$$h_\Delta(t, x) = tv_0 + (1 - t)x.$$

In particular one has: If Δ' is a face of Δ containing v_0, then $h_{\Delta'} = h_\Delta|_{[0,1]\times\Delta'}$. This means that one can put these homotopies together to a homotopy, which contracts the space X to v_0. □

* * *

3.4 Classifying spaces

Definition 3.4.1. Let Γ be a group. A connected cell complex $B\Gamma$ is called a **classifying space** of Γ, if

- $\pi_1(B\Gamma) \cong \Gamma$ and
- the universal covering $E\Gamma$ of $B\Gamma$ is contractible.

Theorem 3.4.2. *For every group* Γ *there exists a classifying space.*

Proof. Let Γ be a group. Let $V = \mathbb{N}_0 \times \Gamma$ and let $\mathcal{S}$ be the system of all finite subsets E of V with the property that

$$\left| E \cap \left(\left\{ n \right\} \times \Gamma \right) \right| \ \le \ 1$$

for every $n \in \mathbb{N}_0$. Then $(V, \mathcal{S})$ is a simplicial complex over V. Let $V_0 = \{0\} \times \Gamma$.

The group Γ acts on $(V, \mathcal{S})$ by left translations, i.e., as

$$\gamma\Big\{(k_1, \gamma_1), \dots, (k_n, \gamma_n)\Big\} = \Big\{(k_1, \gamma\gamma_1), \dots, (k_n, \gamma\gamma_n)\Big\}.$$

By construction, for every $\gamma \neq 1$ we have

$$\gamma E \cap E = \emptyset.$$

Let X be the geometric realisation. Then Γ acts on X by cellular homeomorphisms. The elements of X can be written in the form

$$\sum_{j=0}^{\infty} \lambda_j(j, \gamma_j),$$

where $\lambda_j \geq 0$ only finitely many are $\neq 0$ and $\sum_{j=0}^{\infty} \lambda_j = 1$.

We claim that X is contractible and that Γ acts discontinuously on X.

For every $n \in \mathbb{N}_0$, let $f_n : X \to X$ be the map

$$f_n\left(\sum_{j=0}^{\infty} \lambda_j(j, \gamma_j)\right) = \sum_{j=0}^{n-1} \lambda_j(j, \gamma_j) + \sum_{j=n}^{\infty} \lambda_j(j+1, \gamma_j).$$

Note that in $f(x)$ there is no summand of the form $\lambda(n, \gamma_n)$. Then f_n is homotpic to f_{n+1} by a homotopy h_n as follows

$$\begin{aligned} & h_n\left(s, \sum_{j=0}^{\infty} \lambda_j(j, \gamma_j)\right) \\ & = \sum_{j=0}^{n-1} \lambda_j(j, \gamma_j) + (1-s)\lambda_n(n, \gamma_n) + s(\lambda_n(n+1, \gamma_n) + \sum_{j=n+1}^{\infty} \lambda_j(j+1, \gamma_j). \end{aligned}$$

We apply h_0 on the interval $[0, \frac{1}{2}]$, then h_1 on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4}]$ and so on. These fit together to give a homotopy on all of $[0, 1]$ from f_0 to Id, since for each element of X, only finitely many j matter.

Finally, the map f_0 is homotopic to the point map $x \mapsto (0, 1)$ via the homotopy

$$h(s, x) = (1-s)f_0(x) + s(0, 1).$$

Hence X is contractible.

In order to show that Γ acts discontinuously, we first note that for given

n the group Γ acts freely on the set of all n-simplices. Let $x \in X$ and let n_0 be the smallest index with $x \in X_{n_0}$. For every $n \geq n_0$ we construct an open neighbourhood U_n of x in X_n such that

$$\gamma U_n \cap U_n = \emptyset \text{ for } \gamma \neq 1, \quad \text{and} \quad U_{n+1} \cap X_n = U_n.$$

For $n = n_0$ let $U_n = e$ the n_0-cell, in which x lies. Since Γ acts freely on the set of all n_0-cells, the property follows. Let now $n \geq n_0$ and U_n already constructed. For every $(n+1)$-simplex Δ choose a fixed point x_Δ in the interior of Δ and set

$$V_\Delta = \left\{ t x_\Delta + (1-t) y : 0 < t < 1,\ y \in \Delta \cap U_n \right\}.$$

Then set $U_{n+1} = U_n \cup \bigcup_\Delta V_\Delta$, where the union runs over all $(n+1)$-simplices. The desired properties are clear. Finally set

$$U = \bigcup_n U_n.$$

Then U is an open neighbourhood of x in X with

$$\gamma U \cap U \neq \emptyset \ \Rightarrow\ \gamma = 1.$$

Hence Γ acts discontinuously and, since X is contractible, hence simply-connected, it follows that

$$\Gamma \ \cong\ \pi_1(\Gamma \backslash X),$$

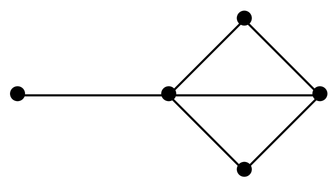
and X is the universal covering of $B\Gamma := \Gamma \backslash X$. $\square$

* * *

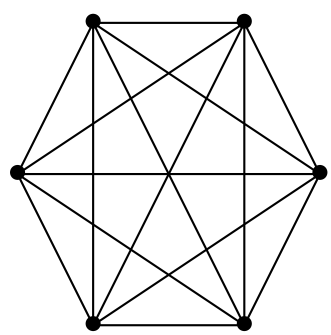
3.5 The fundamental group of a graph

Definition 3.5.1. A **graph** is a simplicial complex of dimension 1. The simplices of dimension 1 are called **edges**.

One usually draws a graph by indicating vertices by points and edges by line segments like this:



For any set of vertices, the **full graph** is the graph with E being the set of all subsets or two elements:



Definition 3.5.2. A simply connected graph is called a **tree**. In the following, let $T = (V, E)$ be a tree.

Definition 3.5.3. By a **path** we understand a sequence of vertices $x_0, \dots, x_n$ such that x_{j-1} is neighboured to x_j for each $j = 1, \dots, n$. For any two vertices x, y let $d(x, y)$ be the length of the shortest path from x to y. This path is unique, as otherwise there would exist a loop in T. We call this path the **geodesic** from x to y. Then $d(x, y)$ defines a metric on V.

Definition 3.5.4. For two nonempty sets $A, B \in V$ we define the **distance**

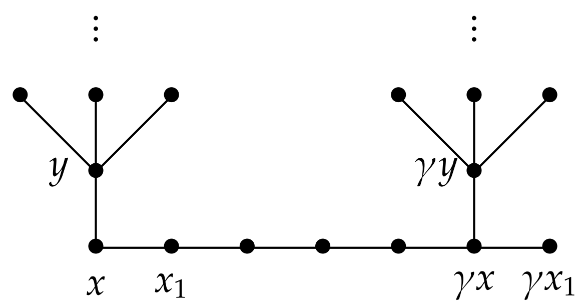
$$d(A, B) = \min_{a \in A, b \in B} d(a, b).$$

Definition 3.5.5. Let γ be an automorphism of a tree T, which has no fixed points. Let

$$D(\gamma) = \min\Big\{d(x, \gamma x) : x \in V\Big\}.$$

Let $\mathrm{Min}(\gamma)$ be the set of all vertices x, at which this minimum is taken.

Lemma 3.5.6. *Let γ be an automorphism of a tree T, which has no fixed points. Then the set $\mathrm{Min}(\gamma)$ is a straight line and γ acts by translating along this line. The line is also called the* **axis** *of γ.*



Proof. Let $x \in \mathrm{Min}(\gamma)$ and let $x = x_0, \dots, x_n = \gamma x$ be the path realising it. Then $x_1, x_2, \dots, x_n = \gamma x_0, \gamma x_1$ is another path realising $D(\gamma)$. Hence the points $\dots, \gamma^{-1}x, x, \gamma x, \gamma^2 x, \dots$ all lie on a single line L and, since the other vertices, like y in the picture, all have a bigger distance to their γ-translate and so $\mathrm{Min}(\gamma) = L$. $\square$

Theorem 3.5.7. (a) *A group acting freely by simplicial homomorphisms on a tree, is a free group.*

(b) *The fundamental group of a graph is a free group.*

(c) *If the graph is finite, its fundamental group is finitely generated.*

Proof. (a): Let $T = \big(V(T), E(T)\big)$ be a tree and Γ a group acting freely and cellularly on T. Note first that $\gamma \in \Gamma \smallsetminus \{1\}$ cannot fix an edge (a, b) for if it would, there would be a fixed point in this edge, but γ acts fixed point freely. Fix a base point $x_0 \in V$ and let

$$D = D(x_0) = \left\{ x \in V : d(x, x_0) \le d(\gamma x, x_0) \ \forall_{\gamma \in \Gamma} \right\}$$

be the **Dirichlet-domain** at x_0. Then $D(\gamma x_0) = \gamma D(x_0)$ for $\gamma \in \Gamma$. We also observe that

$$\gamma x_0 \notin D \text{ for every } \gamma \in \Gamma \setminus \{1\},$$

for otherwise $d(\gamma x_0, x_0) \le d(1\,x_0, x_0) = d(x_0, x_0) = 0$, so $\gamma x_0 = x_0$ which contradicts the fixed point freeness. We claim that

$$V = \bigcup_{\gamma \in \Gamma} \gamma D, \tag{$\star$}$$

$$\gamma D = \eta D \Rightarrow \gamma = \eta. \tag{$\star\star$}$$

($\star$): Let $v \in V$. Then in the orbit Γv there is a vertex $v_0 = \gamma v$ closest to x_0. This then lies in D, so $v = \gamma^{-1} v_0 \in \gamma^{-1} D$.
($\star\star$): If $\gamma D = \eta D$, then $\eta^{-1}\gamma D = D$, so $\eta^{-1}\gamma x_0 \in D$, hence $\eta^{-1}\gamma = 1$ and ($\star\star$) follows.

Let

$$S = S(D) = \left\{ \gamma \in \Gamma : d(\gamma D, D) \le 1 \right\}.$$

We claim that S generates the group Γ. Let $s \in S$, then, as Γ acts by isometries, we get $d(s^{-1}D, D) = d(D, sD) = d(sD, D) \le 1$. Hence s^{-1} lies in S again, so $S = S^{-1}$.

So let Σ be the subgroup generated by S and let $W = \bigcup_{\sigma \in \Sigma} \sigma D$. We claim that $W = V$. For this call two domains γD and ηD **neighboured**, if $d(\gamma D, \eta D) \le 1$. Now if $\gamma D \subset W$ and ηD is neighboured to γD, then $\eta D \subset W$, because $\sigma^{-1}\gamma D$ is neighboured to D, hence belongs to W, and so does $\sigma(\sigma^{-1}\gamma D) = \gamma D$. Finally, let $v \in V$ and assume $v \notin W$. Then the path from v to x_0 at one stage enters W for the first time, but then through a domain τD, which is not in W, yet neighboured to one in W, which cannot be true.

We now conclude that S generates Γ, i.e., $\Gamma = \Sigma$: As $W = V$, for given $\gamma \in \Gamma$ there exists $\sigma \in \Sigma$ with $\sigma D = \gamma D$, hence $\gamma = \sigma$. So S generates Γ.

If $s \in S$, then $s^{-1} \in S$, too. Of each set $\{s, s^{-1}\}$ in S, pick one element and let F denote the set of all those. Then we have $S = F \sqcup F^{-1}$ and F still generates the group Γ. We claim that it is a free generator set.
A **reduced word** in the generator S is a tuple $f_1 f_2 \cdots f_n$ in which $f_j = 1$ or $f_j f_{j+1} = 1$ do not occur. Assume, two reduced words are equal in Γ, i.e., $\gamma = f_1 \cdots f_n = g_1 \cdots g_m$. Then the path from x_0 to γx_0 leaves D at a unique neighbour sD. As s is unique, it follows $f_1 = s = g_1$. Replacing γ with $s^{-1}\gamma$, this reasoning repeats, ending in the quality of the words, i.e., Γ is freely generated by F. This proves (a) and hence (b). For (c) assume X finite. Then D is finite, hence D has only finitely many neighbours. $\square$

* * *

4 Higher Homotopy groups

4.1 Commutativity

Definition 4.1.1. Let I denote the unit interval and let $k \in \mathbb{N}$, $k \geq 1$. Let X a topological space and $x_0 \in X$ a point. A **k-spheroid** at the point x_0 is a continuous map $\gamma : I^k \to X$ such that $\gamma(\partial I^k) = \{x_0\}$. Here ∂I^k is the boundary of I^k in $\mathbb{R}^k$, i.e., the set of all $t \in I^k$ such that at least one coordinate is either 0 or 1.

A 1-spheroid is the same thing as a closed path.

Note that, if we collapse ∂I^k to a point, we get the sphere, i.e.

$$S^k \cong I^k/\partial I^k.$$

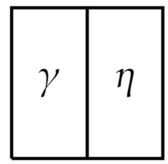
Therefore, a spheroid may be viewed as a pointed continuous map from S^k to X. Here pointed means that the special point of the sphere (say $N = (1, 0, \ldots, 0)$) is mapped to the special point x_0 of X.

Definition 4.1.2. Two k-spheroids γ, η at x_0 are called **homotopic**, if there exists a continuous map $h : I \times I^k \to X$ with $h(0, x) = \gamma(x)$, $h(1, x) = \eta(x)$ and $h\Big(I \times (\partial I^k)\Big) = \{x_0\}$.

Definition 4.1.3. Let γ, η be two spheroids at the point x_0. Define their **composition** to be the spheroid

$$\gamma.\eta(t) = \begin{cases} \gamma(2t_1, t_2, \ldots, t_k) & 0 \leq t_1 \leq \frac{1}{2}, \\ \eta(2t_1 - 1, t_2, \ldots, t_k) & \frac{1}{2} < t_1 \leq 1. \end{cases}$$

We basically compose them at the first variable. For $k = 2$ we can draw a picture of the domain of definition I^2.



We also define the **reverse spheroid** along the first coordinate as

$$\check{\gamma}(t) = \gamma(1 - t_1, t_2, \dots, t_n).$$

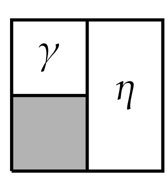
Theorem 4.1.4. *For $n \in \mathbb{N}$, the set $\pi_n(X, x_0)$ of all closed n-spheroids modulo homotopy with fixed boundary is a group with group law*

$$[\gamma][\eta] = [\gamma.\eta], \qquad [\gamma]^{-1} = [\check{\gamma}].$$

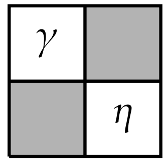
This group is called n-th **homotopy group** *of the pointed space (X, x_0). This group is commutative for $n \geq 2$.*

Proof. The only surprising part is the commutativity for $n \geq 2$ which is completely false for $n = 1$. All other assertions are proven similarly to the case $n = 1$ in Section 2.1.

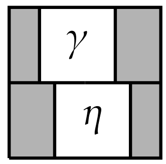
We indicate the proof of the commutativity by some pictures for $n = 2$. The reader will then be able to fill in the details. We start with the definition of $\gamma.\eta$. we then shrink the domain of definition of γ to the domain $t_2 \geq \frac{1}{2}$:



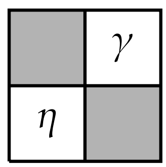
In the grey area, the value is constant equal to x_0. We next shrink the domain of η to $t_2 \leq \frac{1}{2}$ getting this picture



where again, the grey area is constant. We next move η left and γ right, so they can pass each other.



leading to



Finally, extend the domains to get $\eta.\gamma$ and the commutativity is proven. □

Definition 4.1.5. For any two spaces X, Y let

$$[X, Y] = C(X, Y)/\text{homotopy}$$

A **pointed space** is a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. A map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ is a continuous map with $f(x_0) = y_0$.

Two pointed maps $f, g : (X, x_0) \to (Y, y_0)$ are **pointed homotopic** if there exists a homotopy h fixing the points, i.e., $h(s, x_0) = y_0$ for all $s \in I$. The set of pointed homotopy classes is denoted by

$$[X, Y]_*.$$

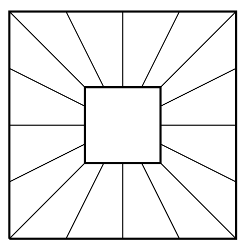
Here the marked points are left out in the notation as they are considered fixed.

The space $I^n/\partial I^n$ is homeomorphic to S^n and it has a marked point, say $N = (1, 0, \dots, 0)$. So we have a natural identification

$$\pi_n(X, x_0) \cong [S^n, X]_*.$$

Proposition 4.1.6. *If X is path-connected, then for any two $x_0, x_1 \in X$ the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.*

Proof. Fix a path γ from x_0 to x_1. Given an n-spheroid f at x_0, shrink the area of definition of f to a smaller cube inside I^n and connect this smaller cube to the boundary by copies of γ, which is followed outward:



This new spheroid is centered at x_1. Call it f^γ. Then $f \mapsto f^\gamma$ induces a group homomorphism from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$ and $\check{\gamma}$ induces the inverse homomorphism. □

In the last proposition, we have seen that a path γ from x_0 to x_1 induces an isomorphism $\phi_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$. It is easy to see that $\phi_{\gamma\eta} = \phi_\gamma \phi_\eta$, so, for $x_0 = x_1$ we get an action of the group $\pi_1(X, x_0)$ on the group $\pi_n(X, x_0)$. For $n = 1$, this is the action of $\pi_1(X, x_1)$ by conjugation.

Proposition 4.1.7. *A covering space projection $p : E \to X$ for every $n \geq 2$ induces an isomorphism*

$$\pi_n(E, e) \cong \pi_n(X, x_0),$$

where $x_0 = p(e)$*.*

Proof. Since S^n is simply-connected, every continuous map $f : S^n \to X$ with $f(N) = x_0$ lifts to a map $f_E : S^n \to E$ with $f_E(N) = e$. This gives surjectivity. As for injectivity, let $f : S^n \to E$ be continuous with $f(N) = e$ such that $p(f)$ is nullhomotopic with homotopy $h : I \times S^n \to X$. Then the homotopy h can be lifted to a homotopy $h_E : I \times S^n \to E$, so f itself is nullhomotopic already. □

Corollary 4.1.8. *If the universal covering space* $\tilde{X}$ *of* X *is contractible, then*

$$\pi_n(X) = 0$$

for every $n \geq 2$*. This holds in particular for the* 1*-sphere* S^1*.*

* * *

4.2 The long exact sequence of homotopy groups

We continue to write $I = [0, 1]$.

Definition 4.2.1. In the following, let A be a closed subset of a space X and let $a_0 \in A$ be a base point. In that case, we call (X, A, a_0) a **triplet**. Consider I^{n-1} as the face of I^n with the last coordinate t_n equals zero. Let J^{n-1} be the union of the remaining faces. Let $G_n(X, A, a_0)$ denote the set of all continuous maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, a_0)$, i.e., the set of all $f : I^n \to X$ with

$$f(\partial I^n) \subset A, \qquad f(J^{n-1}) = \{a_0\}.$$

Define

$$\pi_n(X, A) = \pi_n(X, A, a_0) = G_n(X, A, a_0)/\sim,$$

where $\sim$ is homotopy in the class $G_n(X, A, a_0)$. This means that $f \sim g$ iff there exists a continuous map $h : I \times I^n \to X$ such that

(a) $h(0,t) = f(t), \qquad h(1,t) = g(t)$ and

(b) $h(s,.) \in G_n(X, A, a_0)$ holds for every $s \in I$.

Then in particular $\pi_n\Big(X, \{a_0\}, a_0\Big) = \pi_n\Big(X, a_0\Big)$.

Remark 4.2.2. One can compose elements of $G_n(X, A, a_0)$ along the last coordinate t_n. Note that this does **not** work for $n = 1$. If $n \geq 2$, then $\pi_n(X, A, a_0)$ is a group and is abelian for $n \geq 3$, which is seen in the same way as for $\pi_n(X, x_0)$. The set $\pi_1(X, A, a_0)$ still has a marked point given by the constant map a_0.

If one collapses J^{n-1} to a point, one gets $I^n/J^{n-1} \cong \mathbb{D}^n$, where $\mathbb{D}^n$ is the closed n-ball $\Big\{x \in \mathbb{R}^n : \|x\| \leq 1\Big\}$. We write s_0 for the point in $I^n/J^{n-1} \cong \mathbb{D}^n$ given by J^{n-1}. Then I^{n-1} becomes the boundary S^{n-1}. Hence $G_n(X, A, a_0)$ can also be described as the set of continuous maps $(\mathbb{D}^n, S^{n-1}, s_0) \to (X, A, a_0)$. Note that this also works for $n = 1$, where $\mathbb{D}^1 = [-1, 1]$ and thus $S^0 = \{-1, 1\}$.

Lemma 4.2.3. *A continuous map $f : (\mathbb{D}^n, S^{n-1}, s_0) \to (X, A, a_0)$ represents the trivial element in $\pi_n(X, A)$ if and only if it is $G(X, A, a_0)$-homotopic to a map with values in A through a homotopy with $h(s,z) = f(z)$ for every $z \in S^{n-1}$.*

Proof. If f represents the trivial element, then there is a homotopy $h : I \times \mathbb{D}^n \to X$ with

$$h(0,x) = f(x),$$
$$h(1,x) = a_0,$$
$$h(I \times S^{n-1}) \subset A,$$
$$h(I \times \{s_0\}) = \{a_0\}.$$

We define $h' : I \times \mathbb{D}^n \to X$ by

$$h'(s,x) = \begin{cases} h\left(s, \frac{2x}{2-s}\right) & \|x\| < 1 - \frac{s}{2} \\ h\left(s, \frac{x}{\|x\|}\right) & \|x\| \geq 1 - \frac{s}{2}. \end{cases}$$

Then

$$\begin{aligned} h'(0,x) &= h(0,x) = f(x), \\ \underbrace{h'(\{1\} \times \mathbb{D}^n)}_{\{a_0\}} &\subset A, \\ h'(I \times S^{n-1}) &\subset A, \\ h'(I \times \{s_0\}) &= \{a_0\}. \end{aligned}$$

so this is the desired homotopy.

The other way round, assume f can be homotoped in $G(X, A, a_0)$ to a map g with values in A. As $\mathbb{D}^n$ is contractible, g can, inside A, with fixed $s_0 \mapsto a_0$ be homotoped to the constant map a_0. □

Definition 4.2.4. A continuous map $f : (X, A, a_0) \to (Y, B, b_0)$ to another triplet induces, by post-composing, a map $f_* : \pi_n(X, A, a_0) \to \pi_n(Y, B, b_0)$. Restricting a map $(\mathbb{D}^n, S^{n-1}, s_0) \to (X, A, a_0)$ to S^{n-1} induces a map

$$\partial : \pi_n(X, A) \to \pi_{n-1}(A).$$

This map is called the **boundary map**. It is a group homomorphism, if $n \geq 2$.

Definition 4.2.5. Let $\pi_0(X)$ denote the set of all path-connected components of a space X. If X is pointed with base-point x_0, then $\pi_0(X)$ is a **pointed set**with base-point being the class of x_0.

A sequence of maps of pointed sets

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

is called **exact** at (Y, y_0) if the **kernel** of g, i.e., the set $g^{-1}(z_0)$ equals the image of f.

Theorem 4.2.6. *Let (X, A, a_0) be a triplet. and let $i : A \to X$, $j : (X, a_0, a_0) \to (X, A, a_0)$ be the inclusion maps. Then the sequence*

$$\begin{aligned} \ldots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \\ \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \xrightarrow{j_*} \pi_{n-1}(X, A) \to \ldots \end{aligned}$$

$$\ldots \to \pi_1(X, A) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

is exact, where all homotopy groups are taken with respect to the same base-point $a_0 \in A$.

Proof. We show $j_* i_* = 0$. So let $f : \mathbb{D}^n \to A$ with $f(s_0) = a_0$. The map $j \circ i \circ f$ is just f followed by the inclusion $A \subset X$. As f takes values in A, Lemma 4.2.3 shows that it represents the trivial element in $\pi_n(X, A)$.

Let $[f] \in \pi_n(X)$ lie in the kernel of j_*. Then Lemma 4.2.3 implies that $[f]$ lies in the image of i_*.

Recall that $\pi_n(X) = \pi_n(X, \{a_0\})$. This implies that $\partial j_* = 0$. Let $[g]$ be in the kernel of ∂. This means that on A it is nullhomotopic. Therefore, it lies in $\pi_n(X)$, i.e., in the image of j_*.

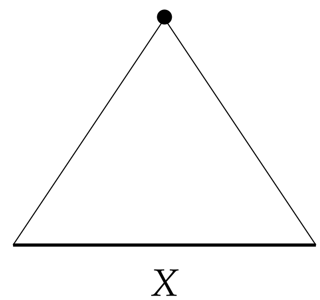
Next if $[f] \in \pi_{n-1}(A)$ lies in the image of ∂, then it is the restriction of a map from $\mathbb{D}^n$, hence nullhomotopic in X, so it lies in the kernel of i_*.

The other way round, if $[f]$ lies in the kernel of i_*, it becomes nullhomotopic in X, but a given homotopy then extends f to $\mathbb{D}^n$. □

Proposition 4.2.7. *For every group* Γ*, there exists a triplet* (X, A, a_0)*, such that* $\pi_2(X, A) \cong \Gamma$.

Proof. For a space X, define the **cone** over X as the space

$$CX = (I \times X)\,/\{1\} \times X.$$



The space CX is contractible via $h(s, x) = (s, x)$ and it contains X as a subspace $X \cong \{0\} \times X$. Let x_0 be the collapsed point $\{1\} \times X$. The long exact sequence for the triplet (CX, X, x_0) yields an isomorphism $\pi_2(CX, X) \cong \pi_1(X)$ (more generally $\pi_n(CX, X) \cong \pi_{n-1}(X)$) and since any group occurs as fundamental group, we are done. □

* * *

4.3 Whitehead's theorem

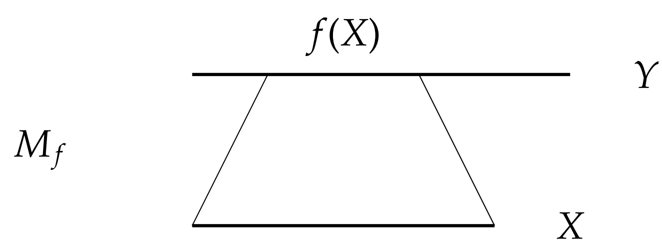
Definition 4.3.1. Let $f : X \to Y$ be a continuous map. The **mapping cylinder** of f is the space

$$M_f = ((I \times X) \sqcup Y)\Big/(1, x) \sim f(x).$$

It contains both X and Y as subspaces and Y is a deformation retract. We fix a deformation given by $h(s, y) = y$ and

$$h\big(s, (t, x)\big) = \big(s + (1 - s)t, x\big),$$

for $s \in I$ and $(t, x) \in [0, 1) \times X \subset M_f$. Then the map f is the composition of the inclusion $X \hookrightarrow M_f$ and the corresponding retraction map $M_f \to Y$.



Lemma 4.3.2. *Let X be a connected cell complex and let $A \neq \emptyset$ be a connected subcomplex.*

(a) *Let (Y, B) be a pair such that for every $n \geq 0$ one has $\pi_n(Y, B, b_0) = 0$ for every $b_0 \in B$, where $\pi_0(Y, B, b_0) = 0$ means that B meets every path-component of Y. Then every continuous map $f : (X, A) \to (Y, B)$ is A-pointwise-fixed homotopic to a continuous map $X \to B$.*

(b) *If $\pi_n(X, A) = 0$ for every $n \in \mathbb{N}$, then A is a deformation retract of X.*

Proof the lemma. Part (b) follows from (a) applied to the identity map $(X, A) \to (X, A)$.

So we prove part (a). We will show that for each $k \in \mathbb{N}_0$ there exists a homotopy $h_k : I \times X_k \to Y$ such that

$$h_{k+1}|_{I \times X_k} = h_k$$

and

$$h_k(0,x) = f(x),$$
$$h_k(1,x) \in B,$$
$$h_k(s,a) = f(a)$$

holds for all $x \in X_k$ and all $a \in A_k$. For $k = 0$ we can connect every $f(x)$, $x \in X_0 \setminus A_0$ by a path γ_{x_0} to some element of B and we set $h_0(s, x_0) = \gamma_{x_0}(s)$. For $a_0 \in A_0$ we set $h_0(s, a_0) = a_0$.

Next assume that $k \geq 0$ and h_k is constructed. Let $z \in X_{k+1} \setminus X_k$. Then z lies in the interior of a cell e which is the image of a map $\phi : \mathbb{D}^{k+1} \to X$. Set

$$h^1_{k+1}(s,z) = \begin{cases} f\left(\phi\left(\frac{z}{1-\frac{s}{2}}\right)\right) & |z| \leq 1 - \frac{s}{2}, \\ h_k\left(2|z| - (2-s), \phi\left(\frac{z}{|z|}\right)\right) & |z| > 1 - \frac{s}{2}. \end{cases}$$

Set $h^1_{k+1} \equiv h_k$ on $I \times X_k$. Then h^1_{k+1} is a homotopy from $f|_{X_{k+1}}$ to a map f^1 which maps X_k to B. Since $\pi_{k+1}(Y, B, b_0) = 0$, Lemma 4.2.3 implies that the composition $f^1 \circ \phi : (\mathbb{D}^{k+1}, S^k) \to (Y, B)$ is homotopic to a map with values in B through a homotopy h with $h(s,z) = f(\phi(z))$ if $z \in S^k$. Call this homotopy $h = h^2_{k+1}$.

We do this for all cells of dimension $k+1$ in $X \setminus A$, perform the homotopy h^2_{k+1} after h^1_{k+1} to get the homotopy h_{k+1} with the claimed properties.

Finally we define a homotopy $h : I \times X \to Y$ by $h(s,x) = h_k(s,x)$ if $x \in X_k$. This is the desired homotopy. □

Theorem 4.3.3 (Whitehead)**.** *Let $f : X \to Y$ be a continuous cellular map between connected cell complexes. We say that f has property (H), if*

$f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for every $n \in \mathbb{N}$.

(a) *If the inclusion $i_* : A \hookrightarrow X$ of a sub-complex has property (H), then A is a deformation retract of X.*

(b) *If f has property (H), then f is a homotopy equivalence.*

Proof. (a) Consider the long exact sequence of homotopy groups for the pair (X, A).

$$\begin{aligned} \ldots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \\ \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \xrightarrow{j_*} \pi_{n-1}(X, A) \to \ldots \end{aligned}$$

Since i_* is an isomorphism, the maps j_* and ∂ are zero, and therefore all relative homotopy groups $\pi_n(X, A)$ are zero. Then (a) follows by the lemma (4.3.2).

(b) Let M_f be the mapping cylinder of f. The retraction map $r : M_f \to Y$ is a homotopy equivalence. Let i denote the inclusion $X \hookrightarrow M_f$. As $f = X \xrightarrow{i} M_f \xrightarrow{r} Y$ is an isomorphism of homotopy groups, then so is i. By part (a), X is a deformation retract, so i is a homotopy equivalence and f is a composition of homotopy equivalences, hence a homotopy equivalence. □

* * *

4.4 Fibre bundles

Definition 4.4.1. A **fibre bundle** is a continuous map $p : E \to B$, for which there exists a topological space F, such that every point $b \in B$ has an open neighbourhood $U \subset B$ and a commutative diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

where the horizontal arrow is a homeomorphism and $U \times F$ has the product topology. The space F is called the **fibre** and B is called the **base space**.

Fixing a base-point $b_0 \in B$, one can identify F with $p^{-1}(b_0)$, this is why F is called the fibre. We use this identification and write $i : F \hookrightarrow E$ for the inclusion map. A fibre bundle is often denoted in the form $F \to E \xrightarrow{p} B$.

Examples 4.4.2.

- A covering is a fibre bundle with discrete fibre.
- The trivial fibre bundle $B \times F \xrightarrow{p_1} B$.
- The Moebius band: The base space B is $S^1 \cong \mathbb{R}/\mathbb{Z}$ and the fibre is $F = [-1, 1]$. The space E is defined as

  $$E = \mathbb{Z}\Big\backslash\Big(\mathbb{R} \times [-1, 1]\Big),$$

  where $\mathbb{Z}$ acts on the product via

  $$k(x, t) = (x + k, (-1)^k t).$$

  The quotient space can be obtained from $I \times [-1, 1]$ modulo the equivalence relation generated by

  $$(0, t) \sim (1, -t).$$

  The map p is derived from the projection to the first coordinate.

- The **Hopf bundle** is a fibre bundle $S^1 \to S^3 \to S^2$ that comes about as follows:

$$S^3 \hookrightarrow \mathbb{R}^4 \setminus \{0\} \cong \mathbb{C}^2 \setminus \{0\} \to \left(\mathbb{C}^2 \setminus \{0\}\right)\big/\mathbb{C}^\times = \mathbb{P}^1(\mathbb{C}) = \widehat{\mathbb{C}} \cong S^2.$$

Definition 4.4.3. For a homotopy $h : I \times X \to Y$ we also write $h_s(x) = h(s, x)$ and say that h is a homotopy from h_0 to h_1.

Theorem 4.4.4 (Homotopy lifting property)**.** *Let $F \to E \xrightarrow{p} B$ be a fibre bundle and let $h : I \times I^n \to B$ be a homotopy. Let $\tilde{h} : \{0\} \times I^n \to E$ be a lift of h, i.e., $p(\tilde{h}) = h|_{\{0\}\times I^n}$. Then $\tilde{h}$ can be extended to a homotopy $I \times I^n \to E$ lifting h.*

More generally, let $S \subset \partial I^{n+1}$ be a union of faces containing the face $\{0\} \times I^n$ but not the face $\{1\} \times I^n$. Let $\tilde{h} : S \to E$ be a lift of h, i.e., $p(\tilde{h}) = h|_S$. Then $\tilde{h}$ can be extended to a homotopy $I \times I^n \to E$ lifting h.

A map, having the homotopy lifting property is als called a **Serre fibration**.

Proof. To show this, note that the condition on S implies that S is a deformation retract of I^{n+1}. We first consider the case of a trivial bundle $E = B \times F$. The map $\tilde{h} : S \to B \times F$ has h as first coordinate. We write $\tilde{h}_2$ for the second coordinate. Let $f : I^{n+1} \to S$ be a retraction.

For $x \in I^{n+1} \setminus S$ we define $\tilde{h}(x)$ by

$$\tilde{h}(x) = (h(x), \tilde{h}_2(f(x))).$$

This does the job.

For the general case we cover I^{n+1} by closed cubes C which are disjoint up to the boundary and are so small that h maps each C into a trivialising open subset of B. We then apply the previous argument successively to these cubes. □

Theorem 4.4.5. *Given a fibre bundle $p : E \to B$, let $e_0 \in E$ be a base point and let $b_0 = p(e_0)$ as well as $F = p^{-1}(b_0)$. Then there is an exact sequence of homotopy groups*

$$\begin{aligned} \dots \pi_n(F) \xrightarrow{\delta} \pi_n(E) \xrightarrow{p} \pi_n(B) \xrightarrow{\delta} \pi_{n-1}(F) \to \dots \\ \dots \to \pi_1(B) \xrightarrow{i} \pi_0(F) \xrightarrow{p} \pi_0(E). \end{aligned}$$

Proof. Consider the exact sequence of Theorem 4.2.6.

$$\dots \pi_n(F) \xrightarrow{i} \pi_n(E) \xrightarrow{j} \pi_n(E, F) \xrightarrow{\delta} \pi_{n-1}(F) \to \dots$$

The projection map p yields a homomorphism

$$\hat{p} : \pi_n(E, F) \to \pi_n(B, \{b_0\}) = \pi_n(B).$$

This map is *injective:* Let $f : (\mathbb{D}^n, S^{n-1}, x_0) \to (E, F, e_0)$ be a continuous map, such that $p \circ f$ is nullhomotopic in B. Then there is a homotopy $h : I \times \mathbb{D}^n \to B$ such that $h(I \times S^{n-1}) = \{b_0\} = h(\{1\} \times \mathbb{D}^n)$ and $h(0, x) = p(f(x))$. By Theorem 4.4.4 there exists a homotopy $\tilde{h} : I \times \mathbb{D}^n \to E$ with

(a) $\quad p \circ \tilde{h} = h,$

(b) $\quad \tilde{h}(0, x) = f(x),$

(c) $\quad \tilde{h}(s, x) = f(x), \quad x \in S^{n-1}.$

By Lemma 4.2.3 we can conclude that f represents the trivial element in $\pi_n(E, F)$.

The map is also *surjective:* For this represent a given element of $\pi_n(B, b_0)$ by a map $f : (I^n, \partial I^n) \to (B, b_0)$. Considering the constant map e_0 as a lift and I^n as $I \times I^{n-1}$, we can use Theorem 4.4.4 with n replaced by $n-1$ and the last coordinate t_n of I^n by $1 - t_n$, to get a lift
$\tilde{f} : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$. □

* * *

4.5 Homotopy groups of spheres

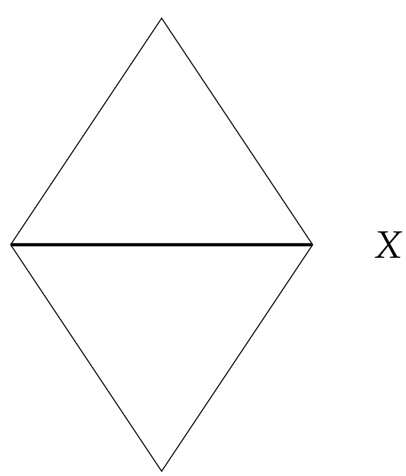
Definition 4.5.1. A space X is called n**-connected**, if $\pi_k(X) = 0$ for all $k \leq n$.

Likewise, a pair (X, A) is called n-connected if $\pi_k(X, A)$ is trivial for $k < n$.

Definition 4.5.2. Let X be a topological space. The **suspension** of X is the space

$$\Sigma X := \Big([-1,1] \times X\Big) \Big/ \Big[(-1, x) \sim (-1, y),\ \ (1, x) \sim (1, y)\Big].$$

Example: ΣS^n is homeomorphic to S^{n+1} for $n \geq 0$ (Exercise).



The suspension is like the space lying in a hammock.

If (X, x_0) is a pointed space, then ΣX becomes one with the marked point $(0, x_0)$. There is a natural continuous map $X \to \Sigma X$, $x \mapsto (0, x)$.

If $f : X \to Y$ is a continuous map, there is a natural induced continuous map

$$\Sigma f : \Sigma X \to \Sigma Y$$

given by $\Sigma f(t, x) = (t, f(x))$.

Proposition 4.5.3. *Let $k \in \mathbb{N}$, then the natural map*

$$\pi_k(X) = [S^k, X]_* \xrightarrow{\Sigma} [\Sigma S^k, \Sigma X]_* = [S^{k+1}, \Sigma X]_* = \pi_{k+1}(\Sigma X)$$

is a group homomorphism.

Proof. Simply follow the definition. □

In particular, we get a group homomorphism

$$\pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

Lemma 4.5.4. $\pi_2(S^2) = \mathbb{Z}$ *with the identity map as generator.*

Proof. = The long exact sequence of a fibre bundle in Theorem 4.4.5 applied to the Hopf bundle $S^1 \to S^3 \to S^2$ gives an exact sequence

$$\underbrace{\pi_2(S^3)}_{=0} \to \pi_2(S^2) \to \underbrace{\pi_1(S^1)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(S^3)}_{=0}.$$

The claim follows.

□

Theorem 4.5.5 (Freudenthal)**.** *The group homomorphism*

$$\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$$

is an isomorphism for $k < 2n-1$ and is surjective for $k = 2n-1$.

The Freudenthal Theorem will follow from

Theorem 4.5.6. *Let X be a CW-complex with sub-complexes A, B such that $X = A \cup B$ and write $C = A \cap B$. Suppose that (A, C) is r-connected and (B, C) is s-connected for some $r, s \in \mathbb{N}_0$. Then the map $\pi_j(A, C) \to \pi_j(X, B)$ induced by inclusion is an isomorphism for $j < r+s$ and surjective for $j = r+s$.*

Proof of Freundenthal's theorem using Theorem 4.5.6. Decompose the suspension ΣX as a union of two cones C_+X and C_-X intersecting in X. Since C_+X is contractible, the long exact sequence implies that the boundary map $\partial : \pi_k(C_+X, X) \to \pi_{k-1}(X)$ is an isomorphism. The suspension is the same as

$$\pi_k(X) \cong \pi_{k+1}(C_+X, X) \xrightarrow{\alpha} \pi_{k+1}(\Sigma X, C_-X) \cong \pi_{k+1}(\Sigma X).$$

Since $C_\pm X$ is contractible, all its homotopy groups vanish, so the long exact sequence tells us that the pair $(C_\pm X, X)$ is n-connected if X is $(n-1)$-connected.

At this point, we could infer Freudenthal's Theorem from 4.5.6, if we knew, that the sphere S^n is $(n-1)$ connected. Unfortunately, we cannot prove this (n-1)-connectivity without using Freudenthal. We solve this dilemma by an induction which proves both.

$n = 1$: Then $2n - 1 = 1$, so we only have to show that $\pi_1(S^1) \to \pi_2(S^2)$ is surjective. We apply Theorem 4.5.6 with $X = S^2 = \Sigma S^1$ and $A, B = C_\pm S^1$, so $C = S^1$. Further we consider $r = s = 1$. It follows that α is surjective.

$n \to n + 1$: We apply Theorem 4.5.6 with $C = S^n$, $A, B = C_\pm S^n$ and $X = \Sigma S^n \cong S^{n+1}$. By induction, we know that $C = S^n$ is $(n-1)$-connected, therefore, $(C_\pm S^n, S^n)$ is n-connected. Hence Theorem 4.5.6 implies that

$$\pi_{j-1}(S^n) \cong \pi_j(C_+S^n, S^n) \to \pi_j(S^{n+1}, C_-S^n) \cong \pi_j(S^{n+1})$$

is an isomorphism, if $j < 2n$ and surjective for $j = 2n$. In particular, as S^n is $(n-1)$-connected, it follows that S^{n+1} is n-connected and the induction is finished. $\square$

Proof of Theorem 4.5.6. Omitted. (The proof uses an intricate induction on the cells of A and B.) $\square$

Corollary 4.5.7. (a) *One has $\pi_k(S^n) = 0$ if $k < n$. This explains the zeros in the table below the diagonal.*

(b) *One has $\pi_n(S^n) \cong \mathbb{Z}$ with the identity map as a generator. This explains the diagonal.*

Proof. (a) Induction on k for $k = 1$ we know that $\pi_1(S^n) = 0$ for $n \geq 2$. Next for $k < n$ we have by Freudenthal's theorem, that $\pi_{k+1}(S^{n+1}) \cong \pi_k(S^n)$ and the latter vanishes by induction hypothesis.

(b) From Freudenthal's theorem we get that in the suspension sequence

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to \dots$$

the first map is surjective and all others are isomorphisms. Now

$\pi_1(S^1) \cong \mathbb{Z}$ and this group is generated by the identity map. Hence $\pi_n(S^n)$ is generated by the identity map for all n. In order to apply Freudenthal's theorem, it remains to show that $\pi_2(S^2) \cong \mathbb{Z}$. The long exact sequence of a fibre bundle in Theorem 4.4.5 applied to the Hopf bundle $S^1 \to S^3 \to S^2$ gives an exact sequence

$$\underbrace{\pi_2(S^3)}_{=0} \to \pi_2(S^2) \to \underbrace{\pi_1(S^1)}_{\cong\mathbb{Z}} \to \underbrace{\pi_1(S^3)}_{=0}.$$

The claim follows. □

Theorem 4.5.8 (Fixed point theorem of Brouwer)**.**
Every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

Proof. For $n = 1$ this follows from the intermediate value theorem, since from $-1 \le f(x) \le 1$ we get $-1 - x \le f(x) - x \le 1 - x$ and therefore the continuous function $h(x) = f(x) - x$ satisfies $h(-1) \le 0$ and $h(1) \ge 0$, therefore has to have a zero.

So let $n \ge 2$. **Assume**, f has no fixed point, i.e., $f(x) \ne x$ for every $x \in \mathbb{D}^n$. For $x \in \mathbb{D}^n$ let $h(x)$ be the point of S^{n-1}, at which the ray from $f(x)$ to x hits the boundary S^{n-1}. The map $h : \mathbb{D}^n \to S^{n-1}$ is continuous and one has $h|_{S^{n-1}} = \mathrm{Id}$. Let $i : S^{n-1} \hookrightarrow \mathbb{D}^n$ be the inclusion map. Then $h \circ i = \mathrm{Id}$, so $h_* \circ i_* = \mathrm{Id}|_{\pi_{n-1}(S^{n-1})}$. For $p = n - 1$ we have a commutative diagram

$$\begin{tikzcd} \pi_{n-1}(S^{n-1}) = \mathbb{Z} \arrow[r, "i_*"] \arrow[dr, "\mathrm{Id}"'] & \pi_{n-1}(\mathbb{D}^n) = 0 \arrow[d, "h_*"] \\ & \pi_{n-1}(S^{n-1}) = \mathbb{Z}, \end{tikzcd}$$

which means that the identity map on $\mathbb{Z}$ factors over zero, a **contradiction!** □

The following table lists some $\pi_k(S^n)$, where we have written $\mathbb{Z}_m$ for $\mathbb{Z}/m$.

	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$n=1$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

* * *

5 Homology

5.1 Simplicial homology

Definition 5.1.1. Let S be a set. The **free abelian group** in the generating set S is the group of all formal linear combinations

$$\sum_{s\in S} c_s s,$$

where $c_s \in \mathbb{Z}$ and almost all c_s are zero. The sum is defined by $(\sum_{s\in S} c_s s) + (\sum_{s\in S} d_s s) = \sum_{s\in S}(c_s + d_s)s$.

Let $(X_0, \mathcal{S})$ be a simplicial complex with geometric realisation X. We choose a linear ordering $<$ on the set X_0.

(This is a bit artificial, however, this trick helps to avoid introducing orientations.)

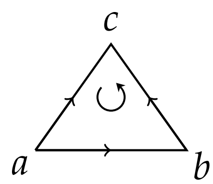
Let $\mathcal{S}_n$ be the set of all cells of dimension n, so the set of all $e \in \mathcal{S}$ with $|e| = n + 1$. Let $C_n(X)$ be the free abelian group over $\mathcal{S}_n$. We write an element of $\mathcal{S}_n$ as $[v_0, \ldots, v_n]$ with $v_0 < \cdots < v_n$. Define a $\mathbb{Z}$-linear map

$$\partial_n : C_n(X) \to C_{n-1}(X),$$

called the **boundary operator**, by

$$\partial_n[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots \hat{v}_i \ldots, v_n].$$

For instance, $\partial_n([a, b, c]) = [b, c] - [a, c] + [a, b]$.



The ordering of the vertices induces an orientation on the simplices and the minus sign occurs, where the orientations don't match.

Lemma 5.1.2. *One has $\partial_{n-1}\partial_n = 0$.*

Proof. We compute

$$\begin{aligned}
\partial_{n-1}\partial_n[v_0,\dots,v_n] &= \partial_{n-1}\sum_{i=0}^{n}(-1)^i[\dots\hat{v}_i\dots] \\
&= \sum_{j<i}(-1)^{i+j}[\dots\hat{v}_j\dots\hat{v}_i\dots] \\
&\quad + \sum_{j>i}(-1)^{i+j+1}[\dots\hat{v}_i\dots\hat{v}_j\dots] = 0.
\end{aligned}$$

□

This lemma implies

$$\operatorname{im}\partial_{n+1} \subset \ker\partial_n.$$

Definition 5.1.3. We define the **simplicial homology** of X as

$$H_n(X) := \ker\partial_n/\operatorname{im}\partial_{n+1}.$$

Here ∂_0 is the zero map.

Definition 5.1.4. More generally, a sequence of abelian groups and group homomorphisms $(A_k, \partial_k)_{k\in\mathbb{Z}}$

$$\dots \to A_{k+1} \xrightarrow{\partial_{k+1}} A_k \xrightarrow{\partial_k} A_{k-1} \to \dots$$

with the property that

$$\partial_k\partial_{k+1} = 0, \qquad k \in \mathbb{Z},$$

or equivalently $\mathrm{im}(\partial_{k+1}) \subset \ker(\partial_k)$, is called a **chain complex**. Its **homology** is defined to be the quotient group

$$H_k = \ker(\partial_k)/\operatorname{im}(\partial_{k+1}).$$

Lemma 5.1.5. *The simplicial homology does not depend on the choice of the linear ordering on the set X_0. If $<'$ is another linear ordering and if $H'_n(X)$ it its homology, then there exists a canonical isomorphism of abelian groups*

$$H_n(X) \xrightarrow{\cong} H'_n(X).$$

Proof. For the time being, we shall forget the ordering of the vertices. An n-simplex $\{v_0, \dots, v_n\}$ can now come in any ordering of its vertices. We call tow such orderings **equivalent**, if the ensuing permutation that changes one ordering to the other, has sign 1. An equivalence class of orderings is called an **orientation** of a the simplex σ. Thus every simplex of dimension ≥ 1 has exactly two orientations, whereas there's only one orientation on a point. We remedy this by artificially attaching a sign ω to a point and call it orientation. (The artificiality goes a way when you think of ± 1 being interchangeable generators of the cyclic group $\mathbb{Z}$.) A pair (σ, ω) of a simplex σ and an orientation ω on σ is called an **oriented simplex**. For a given orientation ω, we write $\overline{\omega}$ for its opposite orientation. Let $O_n(X)$ be the free group generated by all oriented n-simplices modulo the subgroup generated by all

$$(\sigma, \omega) + (\sigma, \overline{\omega}),$$

where (σ, ω) runs through all oriented simplices.

To a given oriented $(n+1)$-simplex σ and an oriented n-face η we attach a sign $I(\eta, \sigma)$ as follows: Write $\eta = \{v_0, \dots, v_n\}$, where the ordering belongs to the orientation of η. Let a be the missing vertex of σ. we define $I(\eta, \sigma) = 1$ if the ordering $(a, v_0, \ldots, v_n)$ belongs to ω and $I(\eta, \sigma) = -1$ otherwise.

We now define a boundary operator $\partial : O_n \to O_{n-1}$ as follows:

$$\partial(\sigma) = \sum_{\eta \subset \sigma} I(\eta, \sigma)\eta,$$

where the sum runs over all $(n-1)$-faces η of σ. This operator is well-defined and one realises that $\partial^2 = 0$.

Let $\phi : C_n(X) \to O_n(X)$ be the map that sends $[v_0, \ldots, v_n]$ to $\{v_0, \ldots, v_n\}$ with the indicated orientation. One easily checks that

$$I\Big([v_0, \ldots \widehat{v_j} \ldots, v_n], [v_0, \ldots, v_n]\Big) = (-1)^j.$$

Therefore, $\phi\partial = \partial\phi$ and since ϕ is a linear bijection, it induces an isomorphism on homology of the two chain complexes. □

Examples 5.1.6. (a) If $X = \{x_0\}$ is a one-point-space, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1. \end{cases}$$

(b) If X is a triangle graph, i.e., a simplicial complex of dimension 1 with vertices v_0, v_1, v_2, and all three possible edges (=1-cells), then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \geq 2. \end{cases}$$

Proof. The assertion (a) is trivial. We show (b). One has $C_n(X) = 0$ for

$n \geq 2$, so $H_n(X) = 0$ for $n \geq 2$. Further one has

$$C_0 = \mathbb{Z}[v_0] \oplus \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2]$$
$$C_1 = \mathbb{Z}[v_0, v_1] \oplus \mathbb{Z}[v_0, v_2] \oplus \mathbb{Z}[v_1, v_2]$$

and

$$\partial_1[v_i, v_j] = [v_j] - [v_i], \quad \text{for} \quad i < j.$$

This implies

$$\operatorname{im} \partial_1 = \mathbb{Z}([v_1] - [v_0]) \oplus \mathbb{Z}([v_2] - [v_0]).$$

This is exactly the kernel of the map $\delta : C_0(X) \to \mathbb{Z}$, given by $\delta(a[v_0] + b[v_1] + c[v_2]) = a + b + c$. Hence δ induces an Isomorphism $H_0(X) = C_0(X)/\operatorname{im}(\partial_1) \cong \mathbb{Z}$. Finally,

$$H_1(X) = \ker \partial_1 = \mathbb{Z}([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \cong \mathbb{Z}. \qquad \square$$

Proposition 5.1.7. *Let $X = (V, \mathcal{S})$ be a connected simplicial complex. (This means that the geometric realisation is connected.) Then $H_0(X) \cong \mathbb{Z}$.*

Proof. First note that, as a simplicial complex is glued of path-connected spaces, every connected component is path-connected, hence X is path-connected. Then, as simplices are simply connected, any path in X can be homotoped to a path inside X_1. Hence X_1 is path-connected.

Let $\operatorname{aug} : C_0(X) \to \mathbb{Z}$ be the group homomorphism defined by $\operatorname{aug}([v]) = 1$ for every $v \in V$. This is called the **augmentation map**. For every 1-simplex $[v_0, v_1]$ we have

$$\operatorname{aug}(\partial([v_0, v_1]) = \operatorname{aug}([v_1] - [v_0]) = 0.$$

So we have $\operatorname{im}(\partial_1) \subset \ker(\operatorname{aug})$. Therefore, aug is trivial on the image of

∂_1, so it factors through $H_0(X)$. We want to show that it yields an isomorphism $H_0(X) \to \mathbb{Z}$. It is clearly surjective. It remains to show that $\operatorname{im}(\partial_1) = \ker(\mathrm{aug})$. Fix a vertex $v_0 \in V$. As X_1 is path-connected, for any given vertex $v \in V$, there exist vertices $v_1, \dots, v_n$ such that $v_n = v$ and for every $j = 0, \dots, n-1$ the set $\{v_j, v_{j+1}\}$ is in $\mathcal{S}$, i.e., is an edge in X. That means that $[v_j] - [v_{j+1}]$ lies in the image of ∂_1, hence so does

$$[v_0] - [v] = ([v_0] - [v_1]) + ([v_1] - [v_2]) + \cdots + ([v_{n-1}] - [v_n]).$$

Now suppose that $\alpha \in \ker(\mathrm{aug})$. Then $\alpha = \sum_{j=1}^{m} k_j[w_j]$ for some $w_j \in V$ and $k_j \in \mathbb{Z}$ with $0 = \mathrm{aug}(\alpha) = \sum_{j=1}^{m} k_j$. Then

$$\alpha = \sum_{j=1}^{m} k_j[w_j] = \sum_{j=1}^{m} k_j[w_j] - \underbrace{\sum_{j=1}^{m} k_j}_{=0}[v_0] = \sum_{j=1}^{m} k_j \underbrace{\Big([w_j] - [v_0]\Big)}_{\in \operatorname{im}(\partial_1)}$$

lies in the image of ∂_1, whence the claim. □

Definition 5.1.8. A **chain complex** is a sequence of homomorphisms of abelian groups

$$\cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \dots$$

indexed over $\mathbb{Z}$, such that for all $k \in \mathbb{Z}$ we have

$$\partial_k \circ \partial_{k+1} = 0.$$

In that case we define the **homology** of the chain complex as

$$H_k(C) = \ker(\partial_k) / \operatorname{im}(\partial_{k+1}).$$

* * *

5.2 Singular homology

Definition 5.2.1. A subset $C \subset \mathbb{R}^N$, $N \in \mathbb{N}$ is called **convex**, if for any two $x, y \in C$ the line segment from x to y lies in C, i.e.,

$$\overline{xy} := \{(1-t)x + ty : 0 \le t \le 1\} \ \subset \ C.$$

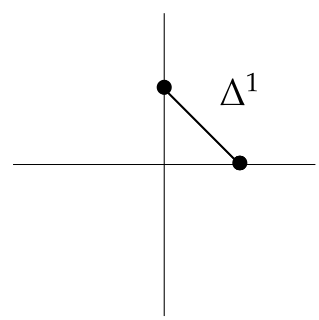
The **convex hull** of vectors $v_0, \dots, v_d \in \mathbb{R}^N$, $d, N \in \mathbb{N}$, is the smallest convex set containing all v_j. It can be described as

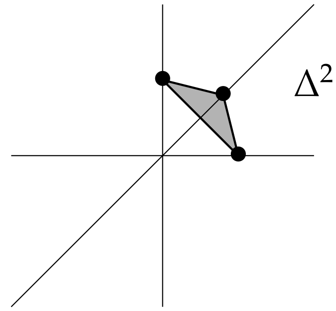
$$\mathrm{conv}(v_0, \dots, v_d) = \Big\{\lambda_0 v_0 + \cdots + \lambda_d v_d : \lambda_j \ge 0,\ \lambda_0 + \cdots + \lambda_d = 1\Big\}.$$

Definition 5.2.2. Let $n \in \mathbb{N}_0$. The **standard n-simplex** is the set

$$\Delta^n = \mathrm{conv}(e_0, \dots, e_n) \ \subset \ \mathbb{R}^{n+1},$$

where $e_0, \dots, e_n$ is the standard basis on $\mathbb{R}^{n+1}$.





Let X be a topological space. A **singular n-simplex** is a continuous map

$$\sigma : \Delta^n \to X.$$

If $v_0, \ldots, v_n$ are any linearly independent vectors in $\mathbb{R}^N$ for $N \geq n+1$, we write $[v_0, \ldots, v_n]$ for their convex hull. There is a unique affine bijection $f : [e_0, \ldots, e_n] \to [v_0, \ldots, v_n]$ mapping each e_j to v_j. It is given by

$$f\left(\sum_{j=0}^{n} \lambda_j e_j\right) = \sum_{j=0}^{n} \lambda_j v_j.$$

We use f to identify $[e_0, \ldots, e_n]$ with $[v_0, \ldots, v_n]$, where we keep track of the order. So we are quite happy to write down formulas like

$$\sigma|_{[e_0,\ldots,e_n]} = \alpha|_{[v_0,\ldots,v_n]} + \beta|_{[w_0,\ldots,w_n]}$$

where $\alpha : [v_0, \ldots, v_n] \to X$ and $\beta : [w_0, \ldots, w_n] \to X$. In such an expression, we use the canonical affine map to identify $[e_0, \ldots, e_n]$ with$[v_0, \ldots, v_n]$ and with $[w_0, \ldots, w_n]$. This notation gives us some flexibility in writing down calculations.

Let $C_n(X)$ be the free abelian group generated by the set of all singular n-simplices in X. Since a 0-simplex is a map of the one-point space to X, one can view $C_0(X)$ as the free group generated by X.

We can identify Δ_1 with the unit interval by $t \mapsto (1-t)e_0 + te_1$ and so a 1-simplex is the same as a path.

We want to identify $\Delta_n = [e_0, \ldots, e_n]$ with a subsimplex of $\Delta^{n+1} = [e_0, \ldots, e_{n+1}]$ in different ways. We consider the uniquely determined affine bijection as above

$$S_j^n = S_j : \Delta^n = [e_0, \ldots e_n] \to [e_0, \ldots, \hat{e}_j, \ldots e_{n+1}].$$

This map is called the j-th **face map**. In the following notation, we use S_j to identify $[e_0, \ldots, e_n]$ with $[e_0, \ldots, \hat{e}_j, \ldots, e_n]$. For a singular simplex

$\sigma \in C_{n+1}(X)$ we thus have

$$\sigma \circ S_j|_{[e_0,\dots,e_n]} = \sigma|_{[e_0,\dots,e_{j-1},\ \hat{e}_j,\ e_{j+1}\dots,e_n,e_{n+1}]}.$$

Let $\partial_{n+1} : C_{n+1}n(X) \to C_n(X)$ be the group homomorphism given by

$$\partial_n(\sigma) = \sum_{i=0}^{n+1} (-1)^i \sigma \circ S_j,$$

or

$$\partial_n(\sigma)|_{[e_0,\dots,e_n]} = \sum_{j=0}^{n+1} \sigma|_{[e_0,\dots,\hat{e}_j,\dots,e_{n+1}]}.$$

The map ∂_n is called the **boundary operator**.

Lemma 5.2.3. *One has $\partial_n \partial_{n+1} = 0$.*

Proof. We compute

$$\begin{aligned}
\partial_n \partial_{n+1}(\sigma)|_{[e_0,\dots,e_{n-1}]} &= \sum_{j=0}^{n} (-1)^j \partial_{n+1}(\sigma)|_{[e_0,\dots,\hat{e}_j,\dots,e_n]} \\
&= \sum_{j=0}^{n} \sum_{k \le j} (-1)^{j+k} \sigma|_{[e_0\dots,\hat{e}_k,\dots,\hat{e}_{j+1},\dots e_{n+1}]} + \sum_{j=0}^{n} \sum_{k>j} (-1)^{j+k} \sigma|_{[e_0\dots,\hat{e}_j,\dots,\hat{e}_k,\dots,e_{n+1}]} \\
&= \sum_{j=0}^{n} \sum_{k<j} (-1)^{j+k+1} \sigma|_{[e_0\dots,\hat{e}_k,\dots,\hat{e}_j,\dots,e_{n+1}]} + \sum_{j=0}^{n} \sum_{k>j} (-1)^{j+k} \sigma|_{[e_0\dots,\hat{e}_j,\dots,\hat{e}_k,\dots,e_{n+1}]} \\
&= 0
\end{aligned}$$

□

Definition 5.2.4. The **singular homology** of the space X is defined as

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Here ∂_0 is supposed to be the zero map.

Proposition 5.2.5. (a) *Let $X = \bigsqcup_\alpha X_\alpha$ be the decomposition in path* *connected components. Then one has*

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

(b) *If $X \neq \emptyset$ is path-connected, then $H_0(X) \cong \mathbb{Z}$. This means that $H_0(X) = \bigoplus_\alpha \mathbb{Z}$, where the direct sum runs over the set of all path-connected components of X.*

(c)

$$H_n\left(\left\{x_0\right\}\right) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

Proof. (a) A singular simplex is always contained in one path-component. Therefore $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. The map ∂_n sends the space $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$. Whence the claim.

(b) Consider the augmentation map $\mathrm{aug} : C_0(X) \to \mathbb{Z}$ given by

$$\mathrm{aug}\left(\sum_i k_i x_i\right) = \sum_i k_i.$$

Since $X \neq \emptyset$, the map aug is surjective. We show, that for a path-connected space X the kernel of aug coincides with the image of ∂_1. For this let $\sigma : [0,1] \to X$ be a singular 1-simplex. Let $x_0 = \sigma(0)$ and $x_1 = \sigma(1)$. Then one gets $\partial_1\sigma = x_1 - x_0 \in C_0(X)$. Therefore $\mathrm{aug}(\partial_1\sigma) = 1 - 1 = 0$, and thus $\mathrm{im}\,\partial_1 \subset \ker \mathrm{aug}$. Conversely let $f \in \ker \mathrm{aug}$. Then $f = \sum_{i=1}^N k_i x_i$ with $\sum_i k_i = 0$. One can write f in the form

$$f = \sum_{j=1}^M (a_j - b_j)$$

for suitable elements a_j, b_j of X. Let σ_j be a path from a_j to b_j. Then σ_j is a singular 1-simplex. So we get

$$f = \sum_j \partial_1(\sigma_j) = \partial_1\left(\sum_j \sigma_j\right) \in \operatorname{im}\partial_1.$$

(c) Let X be a one-point space. Then for every n there is exactly one singular n-simplex σ_n and one has

$$\partial_n\sigma_n = \sum_{i=0}^{n}(-1)^i\sigma_{n-1} = \begin{cases}\sigma_{n-1} & n \text{ even,}\\ 0 & n \text{ odd.}\end{cases}$$

So the $C_n(X)$ form the complex

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

The claim follows. ☐

Examples 5.2.6. (a) Let $n \geq 1$. Then $H_k(S^n) = \begin{cases}\mathbb{Z} & k = 0, n\\ 0 & \text{otherwise.}\end{cases}$

(b) $H_k(\mathbb{R}^n/\mathbb{Z}^n) = \begin{cases}\mathbb{Z}^{\binom{n}{k}} & 0 \leq k \leq n\\ 0 & \text{otherwise.}\end{cases}$

Proof: later.

* * *

5.3 Chain homotopy

Recall that a sequence of homomorphisms of abelian groups

$$\cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\cdots}$$

indexed by $k \in \mathbb{Z}$ is called a **chain complex**, if

$$\partial_k \partial_{k+1} = 0$$

holds for every $k \in \mathbb{Z}$. The operator ∂_k is called the k-th **boundary operator** of the complex. The **homology** of the complex is defined to be

$$H_k(C) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

The elements of $\ker \partial_k$ are called k**-cycles**. The elements of $\operatorname{im} \partial_{k+1}$ are called k**-boundaries**.

The complex is called **exact**, or an **exact sequence**, if $H_k(C) = 0$ for every k.

Definition 5.3.1. A **homomorphism of chain complexes**

$$\phi_\bullet : A_\bullet \to B_\bullet$$

is given by a family of group homomorphisms $\phi_k : A_k \to B_k$ such that the diagram

$$\begin{tikzcd} A_k \arrow[r, "\partial_k"] \arrow[d, "\phi_k"] & A_{k-1} \arrow[d, "\phi_{k-1}"] \\ B_k \arrow[r, "\partial_k"] & B_{k-1} \end{tikzcd}$$

commutes for every k. Strictly speaking we should have written ∂_k^A and ∂_k^B, but we will reduce notation whenever possible.

A homomorphism $\phi : A_\bullet \to B_\bullet$ maps the kernel of ∂^A into the kernel ∂^B and the same for the images. Therefore, it induces a homomorphism on the homology:

$$\phi_* : H_k(A) \to H_k(B).$$

Definition 5.3.2. A homomorphism of chain complexes $\phi_\bullet : A_\bullet \to B_\bullet$ is called **chain nullhomotopic**, if for every k there is a group homomorphism $h_k : A_k \to B_{k+1}$, such that

$$\phi_k = \partial h_k + h_{k-1}\partial$$

for every $k \in \mathbb{Z}$.

Two homomorphisms of complexes $\phi_\bullet, \psi_\bullet : A_\bullet \to B_\bullet$ are called **chain homotopic**, if the difference $\phi - \psi$ is chain nullhomotopic.

We illustrate this situation with the following (non-commutative!) diagram

$$\begin{tikzcd}
A_{k+1} \arrow[r, "\partial"] \arrow[d, "\phi_{k+1}"'] & A_k \arrow[r, "\partial"] \arrow[d, "\phi_k"] \arrow[dl, "h_k"] & A_{k-1} \arrow[d, "\phi_{k-1}"] \arrow[dl, "h_{k-1}"] \\
B_{k+1} \arrow[r, "\partial"'] & B_k \arrow[r, "\partial"'] & B_{k-1}.
\end{tikzcd}$$

Lemma 5.3.3. *Let $\phi_\bullet : A_\bullet \to B_\bullet$ be a homomorphism of chain complexes. If ϕ is nullhomotopic, then ϕ_* is the zero map on the homology.*

Proof. Suppose that $\phi_k = \partial h_k + h_{k-1}\partial$ and let $\alpha \in A_k$ with $\partial\alpha = 0$. Then

$$\phi_k(\alpha) = \partial(h_k\alpha) + h_{k-1}(\underbrace{\partial\alpha}_{=0}) = \partial(h_k\alpha).$$

Therefore $\phi_k(\alpha)$ is in the image of ∂, so it is zero in the homology. □

* * *

5.4 Homotopy

Definition 5.4.1. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Let $f_\# : C_n(X) \to C_n(Y)$ defined by

$$f_\#(\sigma) = f \circ \sigma.$$

Lemma 5.4.2. *One has*

$$f_\# \partial = \partial f_\#.$$

This means that $f_\#$ is a homomorphism of chain complexes and hence induces group homomorphisms

$$f_* : H_n(X) \to H_n(Y).$$

Proof. Look at $\partial : C_{n+1} \to C_n$,

$$\begin{aligned}
f_\#\big(\partial(\sigma)\big) &= f_\# \sum_{i=0}^{n+1} (-1)^i \sigma|_{[\dots \hat{e}_i \dots]} \\
&= \sum_i (-1)^i f_\#(\sigma|_{[\dots \hat{e}_i \dots]}) \\
&= \sum_i (-1)^i f \circ \sigma|_{[\dots \hat{e}_i \dots]} \\
&= \sum_i (-1)^i f_\#(\sigma)|_{[\dots \hat{e}_i \dots]} \\
&= \partial(f_\# \sigma).
\end{aligned}$$

□

The trivial observation $(fg)_\# = f_\# g_\#$ implies that

$$(fg)_* = f_* g_*.$$

We also have

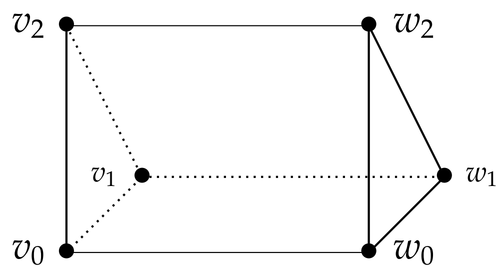
$$\mathrm{Id}_* = \mathrm{Id}.$$

Theorem 5.4.3. *If the continuous maps $f, g : X \to Y$ are homotopic, then*

$$f_* = g_*.$$

In particular, a homotopy equivalence induces an isomorphism on the homology.

Proof. We shall show that $f_\# - g_\#$ is chain-nullhomotopic. Let Δ be a k-simplex. The space $I \times \Delta$ is called a **prism**. The case $n = 2$ gives the idea:



We generally write v_j and w_j for the vertices, in the following way:

$$\left\{0\right\} \times \Delta = [v_0, \dots, v_n] \quad \text{and}$$
$$\left\{1\right\} \times \Delta = [w_0, \dots, w_n].$$

So the prism $I \times \Delta$ is the convex hull of $v_0, \dots, v_n, w_0, \dots, w_n$. Now let $h : I \times X \to Y$ be a homotopy from f to g, so

$$h(0, x) = f(x). \qquad h(1, x) = g(x).$$

Definition 5.4.4. Define the **prism operator**

$$P : C_n(X) \to C_{n+1}(Y)$$

by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i h \circ (1 \times \sigma)|_{[v_0, \dots, v_i, w_i, \dots, w_n]},$$

where $h \circ (1 \times \sigma)$ denotes the map

$$I \times \Delta \xrightarrow{1 \times \sigma} I \times X \xrightarrow{h} Y.$$

We have the following (non-commutative) diagram:

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[dl, "P"'] \\ C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \end{tikzcd}$$

We want to show, that

$$g_\# - f_\# = \partial P + P\partial.$$

For this we compute

$$\begin{aligned} \partial P(\sigma) &= \partial\left(\sum_{i=0}^{n} (-1)^i h \circ (1 \times \sigma)|_{[v_0,\dots,v_i,w_i,\dots,w_n]}\right) \\ &= \sum_{j\le i} (-1)^{i+j} h \circ (1 \times \sigma)|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j\ge i} (-1)^{i+j+1} h \circ (1 \times \sigma)|_{[v_0,\dots,v_i,w_i,\dots\hat{w}_j\dots,w_n]} \end{aligned}$$

and

$$\begin{aligned} P\partial(\sigma) &= P\left(\sum_{j=0}^{n} (-1)^j \sigma|_{[v_0,\dots\hat{v}_j\dots,v_n]}\right) \\ &= \sum_{j>i} (-1)^{i+j} h \circ (1 \times \sigma)|_{[v_0,\dots,v_i,w_i,\dots\hat{w}_j\dots,w_n]} \\ &\quad + \sum_{j<i} (-1)^{i+j+1} h \circ (1 \times \sigma)|_{[v_0,\dots\hat{v}_j\dots,v_i,w_i,\dots,w_n]}. \end{aligned}$$

It follows, that

$$\Big(\partial P + P\partial\Big)(\sigma) = \sum_j h \circ (1 \times \sigma)|_{[v_0,\dots,\hat{v}_j,w_j,\dots,w_n]} - \sum_j h \circ (1 \times \sigma)|_{[v_0,\dots,v_j,\hat{w}_j,\dots,w_n]}.$$

These terms cancel out, except for

$$h \circ (1 \times \sigma)|_{[w_1,\dots,w_n]} - h \circ (1 \times \sigma)|_{v_0,\dots,v_n]} = \Big(g_\# - f_\#\Big)(\sigma).$$

We have shown that the homomorphism of chain complexes $g_\# - f_\#$ is chain-nullhomotopic. By Lemma 5.3.3 the map $g_* - f_*$ on the homology is zero, so $f_* = g_*$ as desired. $\square$

Corollary 5.4.5. *If two spaces X and Y are homotopy-equivalent, then their homologies are isomorphic.*

Proof. Let $\phi : X \to Y$ be a homotopy equivalence with homotopy inverse $\psi : Y \to X$. Then $\phi \circ \psi$ is homotopic to the identity map, therefore $\phi_* \circ \psi_*$ is the identity map. The same goes the other way round. $\square$

Example 5.4.6.

$$H_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \geq 1. \end{cases}$$

This follows, as $\mathbb{R}^n$ is homotopy-equivalent to the point 0.

* * *

5.5 The Snake Lemma

Definition 5.5.1. A commutative diagram of abelian groups of the form

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "d'"] & B \arrow[r, "g"] \arrow[d, "d"] & C \arrow[r] \arrow[d, "d''"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f'"'] & Y \arrow[r, "g'"'] & Z & \end{tikzcd}$$

with exact rows is called a **snake diagram**.

Lemma 5.5.2 (Snake lemma)**.** *Let there be given a snake diagram. We add* *the kernels and cokernels so that now rows and columns are exact:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker d' \arrow[r] \arrow[d] & \ker d \arrow[r, "\tau"] \arrow[d, "k"] & \ker d'' \arrow[d, "k''"] & \\
 & A \arrow[r, "f"] \arrow[d, "d'"] & B \arrow[r, "g"] \arrow[d, "d"] & C \arrow[r] \arrow[d, "d''"] & 0 \\
0 \arrow[r] & X \arrow[r, "f'"'] \arrow[d, "\phi"'] & Y \arrow[r, "g'"'] \arrow[d, "\psi"] & Z \arrow[d] & \\
 & \operatorname{coker} d' \arrow[r, "\varepsilon"] \arrow[d] & \operatorname{coker} d \arrow[r] \arrow[d] & \operatorname{coker} d'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then the map

$$\delta = \phi \circ (f')^{-1} \circ d \circ g^{-1} \circ k''$$

is well-defined and induces an exact sequence

$$\ker d' \to \ker d \to \ker d'' \xrightarrow{\delta} \operatorname{coker} d' \to \operatorname{coker} d \to \operatorname{coker} d''$$

$$\begin{tikzcd}
 & \ker d' \arrow[r] \arrow[d] & \ker d \arrow[r] \arrow[d] & \ker d'' \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] & \\
 & \operatorname{coker} d' \arrow[r] & \operatorname{coker} d \arrow[r] & \operatorname{coker} d'' &
\end{tikzcd}$$

Proof. We use diagram chasing. In the claim the term $(f')^{-1}$ means that one chooses a pre-image. In order to be able to do that, the respective element must lie in the image of f' in the first place. This can be seen nicely in the diagram. So let $a \in \ker d''$. Map it with k'' to C and get some $c = k''(a)$. Choose a pre-image $b \in B$. Map it to Y to get a $y = d(b) \in Y$. The commutativity of the diagram implies $g'(y) = g'(d(b)) = d''(c) = d''(k''(a)) = 0$. Because of exactness, y has a pre-image $x \in X$ and one defines $\delta(a) = \phi(x)$.

Now for well-definedness. Let $b' \in B$ another pre-image of c. By exactness, there is $\alpha \in A$ with $b' = b + f(\alpha)$. Since $\phi(d'(\alpha)) = 0$, adding α does not change $\delta(a)$. The pre-image under f' is uniquely determined, so δ is well-defined.

Finally for the exactness of the sequence: First we show that $\delta\tau = 0$. For this let $z \in \ker d$ and $a = \tau(z)$. Then one can choose the element $k(z)$ as pre-image of c in B. But this element goes to zero under d.

Next we show $\varepsilon\delta = 0$. Let $a \in \ker d''$ and b, y, x as before. Then $\varepsilon(\delta(a)) = \varepsilon(\phi(x)) = \psi(y) = \psi(d(b)) = 0$.

Now let $a \in \ker(\delta)$. Then $\phi(x) = 0$, so there is $\alpha \in A$ with $x = d'(\alpha)$, or $y = f'(d'(\alpha)) = d(f(\alpha))$. This implies $b = f(\alpha) + \theta$ for some $\theta \in \ker d$. Then $c = g(f(\alpha) + \theta) = g(f(\alpha)) + g(\theta) = g(\theta) = k''(\tau(\theta))$, hence $a = \tau(\theta)$.

Now let $z \in \ker \varepsilon$. Choose a pre-image $x \in X$ and set $y = f'(x)$. Then $\psi(y) = \psi(f'(x)) = \varepsilon(\gamma'(x)) = \varepsilon(z) = 0$ and therefore there is $b \in B$ with $y = d(b)$. Let $c = g(b)$. Then it follows that $d''(c) = d''(g(b)) = g'(d(b)) = g'(y) = g'(f'(x)) = 0$, and so there is $a \in \ker d''$ with $c = k''(a)$. Pursuing the chain backwards, one sees that $z = \delta(a)$. $\square$

Theorem 5.5.3. *Let*

$$0 \to E \xrightarrow{f} F \xrightarrow{g} G \to 0$$

be an exact sequence of chain complexes. Then for every $k \in \mathbb{Z}$ the maps induced by f and g give an exact sequence

$$H_k(E) \to H_k(F) \to H_k(G).$$

There is a natural **connection homomorphism** $\delta : H_k(G) \to H_{k-1}(E)$ *which makes the sequence*

$$\begin{aligned} \ldots \to H_k(E) &\xrightarrow{f} H_k(F) \xrightarrow{g} H_k(G) \\ &\xrightarrow{\delta} H_{k-1}(E) \xrightarrow{f} H_{k-1}(F) \xrightarrow{g} H_{k-1}(G) \to \cdots \end{aligned}$$

$$\ldots \to H_1(G) \to H_0(E) \to H_0(F) \to H_0(G) \to 0.$$

exact.

Proof. We have commutative diagrams

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & E^{k+1} \arrow[r] \arrow[d, "d_E^{k+1}"] & F^{k+1} \arrow[r] \arrow[d, "d_F^{k+1}"] & G^{k+1} \arrow[r] \arrow[d, "d_G^{k+1}"] & 0 \\
0 \arrow[r] & E^{k} \arrow[r] \arrow[d, "d_E^{k}"] & F^{k} \arrow[r] \arrow[d, "d_F^{k}"] & G^{k} \arrow[r] \arrow[d, "d_G^{k}"] & 0 \\
0 \arrow[r] & E^{k-1} \arrow[r] \arrow[d, "d_E^{k-1}"] & F^{k-1} \arrow[r] \arrow[d, "d_F^{k-1}"] & G^{k-1} \arrow[r] \arrow[d, "d_G^{k-1}"] & 0 \\
0 \arrow[r] & E^{k-2} \arrow[r] \arrow[d, "d_E^{k-2}"] & F^{k-2} \arrow[r] \arrow[d, "d_E^{k-2}"] & G^{k-2} \arrow[r] \arrow[d, "d_E^{k-2}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The rows with indices k and $k-1$ form a snake diagram. The snake lemma yields a homomorphism

$$\delta : \ker d_G^k \;\to\; E^{k-1} / \operatorname{im} d_E^k.$$

By construction of the snake homomorphism δ, this factors through $H_k(G) = \ker d_G^k / \operatorname{im} d_G^{k+1}$ and maps to $\ker d_E^{k-1}$. The exactness of the resulting sequence is easy to see. □

Corollary 5.5.4 (Functoriality of the long exact sequence)**.** *The long exact sequence is functorial in the following sense. Suppose there is a commutative diagram of the form*

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "e"] & F \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & E' \arrow[r] & F' \arrow[r] & G' \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact sequences of chain complexes and e, f, g are homomorphisms of chain complexes.

Then the induced morphisms e_*, f_* and g_* commute with the long exact sequence, i.e., for each k, the ensuing diagram

$$\begin{tikzcd}
H_k(E) \arrow[r] \arrow[d, "e"] & H_k(F) \arrow[r] \arrow[d, "f"] & H_k(G) \arrow[r, "\delta"] \arrow[d, "g"] & H_{k-1}(E) \arrow[d, "e"] \\
H_k(E') \arrow[r] & H_k(F') \arrow[r] & H_k(G') \arrow[r, "\delta"] & H_{k-1}(E')
\end{tikzcd}$$

commutes.

Proof. The commutativity is clear for the first three squares by the commutativity of the diagram of the chain complexes above. The commutativity of the last square hinges on the construction of δ. This map is canonically constructed out of the morphisms in a snake diagram. Assume there is a second snake diagram and morphisms between snake diagrams, i.e., morphisms on each entry which commute with the morphisms of the two diagrams. Then it follows that the snake morphism also commutes with the morphism between the diagrams. This yields the claim. □

* * *

5.6 Barycentric decomposition

We shall show that a given simplex can be decomposed into smaller subsimplices without changing the homology.

Definition 5.6.1. (Barycentric subdivison) For a simplex $\Delta = [v_0, \dots, v_n]$ let

$$b_\Delta = b_{[v_0,\dots,v_n]} = \frac{1}{n+1}(v_0 + \dots + v_n)$$

be the **barycenter**.

We shall give Δ the structure of a simplicial complex with $(n+1)!$ maximal simplices, called the **barycentric subsimplices** as follows.

- If $n = 0$ then Δ is a point and Δ is the only subsimplex.
- Let $F_0, \dots, F_n$ be the $(n-1)$-dimensional faces of Δ. Inductively, assume each F_i has been given a simplicial structure with a number of $n!$ maximal simplices $S_{i,1}, \dots, S_{i,n!}$ Let $T_{i,j}$ denote the convex hull of $S_{i,j}$ and the barycenter b_Δ. Then these subsimplices make up the decomposition of Δ and we have $(n+1)!$ of them.

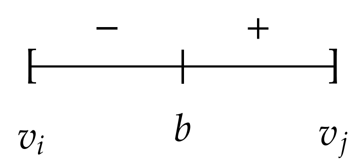
Definition 5.6.2. Let $\Delta = [v_0, \dots, v_n]$ be equipped with this simplicial structure. To every subsimplex $S \subset \Delta$ we attach a sign $\varepsilon(S) = \pm 1$ in the following way: $\varepsilon([v_j]) = 1$ for every $0 \le j \le n$. Let $1 \le k \le n$ be given and let $\varepsilon(T)$ be defined for all subsimplices of Δ of dimension $< k$. Let now S be a subsimplex of dimension k. Then there is a unique face

$F = [v_{i_0}, \dots, v_{i_k}]$ of Δ containing S, where we assume $i_0 < \cdots < i_k$. Then $S = [b_F, T]$ for some subsimplex T of F of dimension $k-1$. The simplex T lies in a unique face $[v_{i_0}, \dots, \hat{v}_{i_\nu}, \dots, v_{i_k}]$ and we set

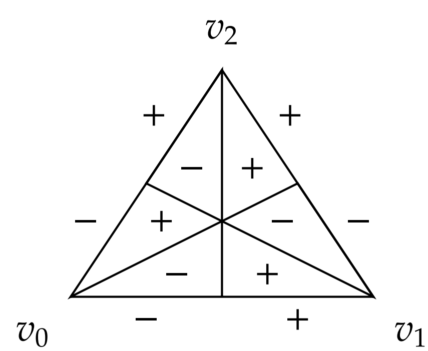
$$\varepsilon(S) := (-1)^\nu \varepsilon(T).$$

Let's look at these signs in dimension 1. For $i < j$ let $b = b_{[v_i, v_j]}$ be the barycenter of the line $[v_i, v_j]$. Then we have

$$\varepsilon([b, v_i]) = -1 \quad \text{and} \quad \varepsilon([b, v_j]) = 1$$



In the picture, you see the sign distributions of the simplices and the faces in dimension 2.



Definition 5.6.3. We define a sequence of maps $u_n : C_n(X) \to C_n(X)$ by

$$u_n(\sigma) = \sum_{S \subset \Delta} \varepsilon(S) \sigma|_S,$$

where the sum runs over all subsimplices S of dimension n.

Lemma 5.6.4. *Let $T, T' \subset \Delta = [v_1, \dots, v_n]$ be two barycentric subsimplices of* *dimension n, which share a common face of dimension $n-1$. Then $\varepsilon(T)+\varepsilon(T')=0$.*

Proof. We know this for $n=1$. So lets assume $n \geq 2$ and we know the claim for $n-1$. Then $T=[b_\Delta, S]$ and $T'=[b_\Delta, S']$ for some barycentric subsimplices S, S' of codimension 1 faces F, F' of Δ.

1. Case. $F=F'=F_j=[v_0, \ldots \widehat{v_j} \ldots, v_n]$. Inductively, we can assume $\varepsilon(S)=-\varepsilon(S')$. Then we have

$$\varepsilon(T)=\varepsilon([b_\Delta, S])=(-1)^j\varepsilon(S)=-(-1)^j\varepsilon(S')=-\varepsilon(T').$$

2. Case. We can assume $F=F_i$ and $F'=F_j$ for $i<j$. Then $S=[b_F, R]$ for some barycentric subcomplex R of $F \cap F'$ and $S'=[b_{F'}, R]$. One has $F \cap F'=F_{i,j}=[v_0, \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots, v_n]$. It follows

$$\begin{aligned}\varepsilon(T) &= \varepsilon([b_\Delta, S])=(-1)^i\varepsilon(S)\\ &= (-1)^i\varepsilon([b_F, R])=(-1)^{i+j+1}\varepsilon(R)\\ &= -(-1)^{i+j}\varepsilon(R)\\ &= -(-1)^j\varepsilon(S')=-\varepsilon(T').\end{aligned}$$

□

Lemma 5.6.5. *The maps u_n form a chain complex homomorphism, i.e., one has*

$$\partial u_n = u_{n-1}\partial.$$

Proof. We write a given subsimplex S of a face F_j in the form $S=[s_1, \ldots, s_n]$. We have

$$\begin{aligned}\partial u_n(\sigma) &= \partial\left(\sum_{j=0}^n (-1)^j \sum_{S \subset F_j} \varepsilon(S)\sigma|_{[b_\Delta, S]}\right)\\ &= \sum_{j=0}^n (-1)^j \sum_{S\subset F_j}\varepsilon(S)\left(\sigma|_S + \sum_{k=1}^n (-1)^k \sigma|_{[b_\Delta, s_1, \ldots, \widehat{s_k}, \ldots, s_n]}\right)\end{aligned}$$

and

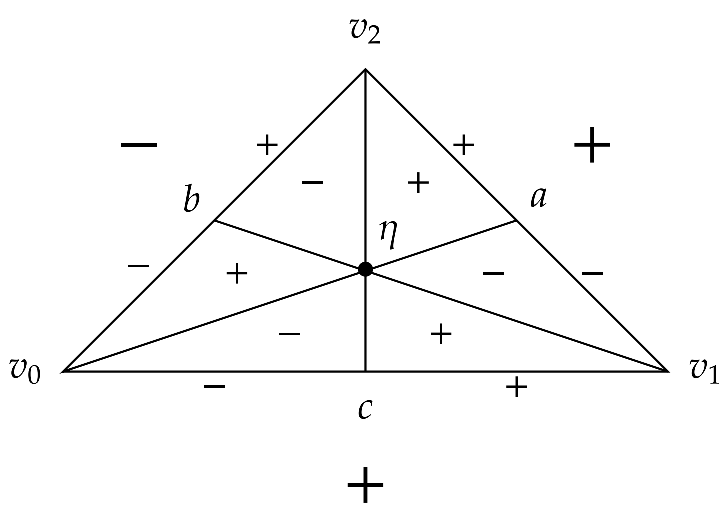
$$u_{n-1}\partial(\sigma) = u_{n-1}\left(\sum_{j=0}^{n}(-1)^j \underbrace{\sigma|_{[v_0,\ldots\widehat{v_j}\ldots,v_n]}}_{\sigma|_{F_j}}\right)$$
$$= \sum_{j=0}^{n}(-1)^j \sum_{S\subset F_j} \varepsilon(S)\sigma|_S$$

Therefore the difference $\partial u_n - u_{n-1}\partial$ is

$$\sum_{j=0}^{n}(-1)^j \sum_{S\subset F_j} \varepsilon(S)\left(\sum_{k=1}^{n}(-1)^k \sigma|_{[b_\Delta,s_1,\ldots,\widehat{s_k},\ldots,s_n]}\right)$$

For a given $S = [s_1, \ldots, s_n] \subset F_j$ and given k there exists exactly one $S' \subset F_{j'}$ sharing the face $[s_1, \ldots \widehat{s_k} \ldots, s_n]$. By Lemma 5.6.4, S and S' occur witg opposite signs, hence the sum above is zero.

Here is a picture of the sign distribution in the case $n = 2$.



The ordering of the vertices is $\eta < a < b < c < v_0 < v_1 < v_2$, the signs are given by the rule above. □

Definition 5.6.6. Let $\Delta = [v_0, \ldots, v_n]$ and $0 \le j \le n$. We define a projection $p_j : \Delta \to F_j = [v_0, \dots \widehat{v_j} \dots, v_n]$ as the uniquely determined affine map with

$$p_j(v_j) = b_{F_j}, \qquad p_j(v_k) = v_k, \text{ for } k \neq j.$$

For an $(n+1)$-simplex $\Delta^{n+1} = [v_0, \dots, v_{n+1}]$ we write $F_j = [v_0, \dots \widehat{v_j} \dots, v_{n+1}]$ and we define $P : C_n(X) \to C_{n+1}(X)$ by

$$P\sigma = \sum_{j=0}^{n+1} (-1)^j \sum_{S \subset F_j} \varepsilon(S)\, \sigma \circ p_j\big|_{p_j^{-1}(S)}.$$

Lemma 5.6.7. *One has*

$$P\partial + \partial P = u_n - \mathrm{Id}.$$

So (u_n) *is chain-homotopic to the identity.*

Proof. This is an involved computation. As this proof is neither entertaining nor of educational value, we omit it. □

* * *

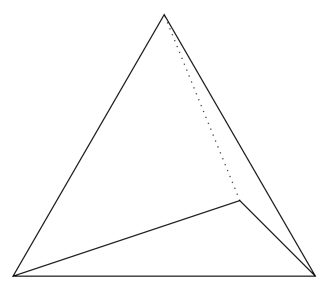
5.7 The Mayer-Vietoris sequence

The Mayer-Vietoris Theorem is similar to the Seifert-van Kampen Theorem, since a space X is considered, which is the union of two open subsets U and V. One then compares the homology groups nof X with the ones of U, V and $U \cap V$.

Lemma 5.7.1. *Let* $v_0, \dots, v_n \in \mathbb{R}^{n+1}$ *be linearly independent whith* $\left\|v_j\right\| \leq 1$ *for each* j. *Consider the simplex* $\Delta = [v_0, \dots, v_n]$. *Let* $b = \frac{v_0 + \cdots + v_n}{n+1}$ *be its barycenter. Then there is a constant* $0 < \theta < 1$ *only depending on the dimension* n *such that for every simplex* S *occurring in the barycentric* *subdivision of* Δ *we have*

$$\operatorname{diam}(S) \leq \theta \operatorname{diam}(\Delta).$$

Proof. We call a simplex **symmetric** if all edges have the same length. Let e denote an edge of Δ of maximal length $\ell(e)$. Then this length is the diameter of Δ. Let Δ' be a symmetric simplex having e for an edge and containing Δ. Inflate Δ inside Δ' by keeping e pointwise fixed, until Δ fills out δ'. During this process, all edge lengths and distances to the barycenter have been increased. As we have $\operatorname{diam}(\Delta) = \ell(e) = \operatorname{diam}(\Delta')$ we see that $\operatorname{diam}(S) \leq \operatorname{diam}(S')$ for some simplex S' in the barycentric subdivision of Δ'.



Therefore $\frac{\operatorname{diam}(S)}{\operatorname{diam}(\Delta)} \leq \frac{\operatorname{diam}(S')}{\operatorname{diam}(\Delta')}$, which means that it suffices to show the claim for a symmetric simplex, hence the standard simplex Δ_n. We have $\operatorname{diam}(\Delta_n) = \sqrt{2}$ and because of

$$\begin{aligned}
\|e_0 - b_\Delta\|^2 &= \left\| e_0 - \frac{1}{n+1}(e_0 + \cdots + e_n) \right\|^2 \\
&= \left(1 - \frac{1}{n+1}\right)^2 + n\frac{1}{(n+1)^2} \\
&= \frac{n^2}{(n+1)^2} + \frac{n}{(n+1)^2} = \frac{n}{n+1}.
\end{aligned}$$

we get $\frac{\operatorname{diam}(S)}{\operatorname{diam}(\Delta)} = \frac{\sqrt{\frac{n}{n+1}}}{\sqrt{2}} < 1$. □

Lemma 5.7.2. *Suppose* $X = U \cup V$ *for two open sets* U, V. *Let* $C_n(U + V)$ *denote the image of* $C_n(U) \oplus C_n(V)$ *in* $C_n(X)$.

Then $C_n(U+V)$, $n \in \mathbb{Z}$, forms a subcomplex of $C_n(X)$ The inclusion $C_n(U+V) \hookrightarrow C_n(X)$ induces an isomorphism

$$\eta : H_n\big(C_\bullet(U+V)\big) \xrightarrow{\cong} H_n(X).$$

Proof. Since $\partial(C_n(U)) \subset C_{n-1}(U)$ we get a sub-complex. By Lemma 5.7.1 we infer that every subsimplex S of $u^k\Delta$ satisfies $\operatorname{diam}(S) \leq \theta^k \operatorname{diam}(\delta)$ for some $\theta < 1$, so these diameters tend to zero, as $k \to \infty$. From this it follows that for every $\alpha \in C_n(X)$ there exists $r \in \mathbb{N}$ such that $u^r\alpha \in C_n(U+V)$.

Now we show that η is surjective. So let $\alpha \in C_n(X)$ with $\partial\alpha = 0$ and let $[\alpha]$ denote its homology class in $\ker\partial/\operatorname{im}\partial$. Let $r \in \mathbb{N}$ be so large that $u^r(\alpha) \in C_n(U+V)$. Since u is chain-homotopic to the identity, it induces the identity on the homology, hence $[u^r(\alpha)] = [\alpha]$ and so the map η is surjective.

For injectivity let $[\beta]$ be in its kernel. Then $\beta \in C_n(U+V)$ with $\partial\beta = 0$ and there exists $\gamma \in C_{n+1}(X)$ such that $\partial\gamma = \beta$. There is $r \in \mathbb{N}$ such that $u^r\gamma \in C_{n+1}(U+V)$. Then $\partial u^r\gamma = u^r\partial\gamma = u^r\beta$ Therefore, $[\beta] = [u^r\beta] = [\partial u^r\gamma] = 0$. □

Let $\phi : C_k(U \cap V) \to C_k(U) \oplus C_k(V)$ be defined by $\phi(\alpha) = \alpha \oplus \alpha$, where α is considered an element of $C_k(U)$ or $C_k(V)$ respectively. Let $\psi : C_k(U) \oplus C_k(V) \to C_k(X)$ be defined by $\psi(\alpha \oplus \beta) = \alpha - \beta$. Then ϕ and ψ are chain maps, i.e., one has $\partial\phi = \phi\partial$ and likewise for ψ, so they induce maps ϕ_* and ψ_* on the homology.

Theorem 5.7.3 (Mayer-Vietoris)**.** *Let $U, V \subset X$ open sets such that*

$X = U \cup V$. Then there is an exact sequence

$$\begin{aligned} \ldots \to H_k(U \cap V) &\xrightarrow{\phi_*} H_k(U) \oplus H_k(V) \xrightarrow{\psi_*} H_k(X) \\ &\xrightarrow{\delta} H_{k-1}(U \cap V) \xrightarrow{\phi_*} H_{k-1}(U) \oplus H_{k-1}(V) \xrightarrow{\psi_*} H_{k-1}(X) \to \ldots \end{aligned}$$

$$\ldots \to H_0(U \cap V) \xrightarrow{\phi_*} H_0(U) \oplus H_0(V) \xrightarrow{\psi_*} H_0(X) \to 0.$$

Proof. Consider the sequence

$$0 \to C_k(U \cap V) \xrightarrow{\phi} C_k(U) \oplus C_k(V) \xrightarrow{\psi} C_k(U + V) \to 0,$$

where $\phi(\alpha) = \alpha \oplus \alpha$ and $\psi(\alpha \oplus \beta) = \alpha - \beta$. One has $\phi\partial = \partial\phi$ and $\partial\psi = \psi\partial$, so this is an exact sequence of chain complexes and therefore Theorem 5.5.3 together with Lemma 5.7.2 implies the claim. □

Proposition 5.7.4. *As an application, we get the proof of the result announced earlier. For $n \geq 1$ we have*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \textit{otherwise.} \end{cases}$$

The case of dimension zero is

$$H_k(S^0) \cong \begin{cases} \mathbb{Z}^2 & k = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The zero-sphere is a two-point space, hence the last assertion follows by Proposition 5.2.5.

The n-dimensional sphere S^n is the union of two contractible open sets

U, V such that $U \cap V$ retracts to S^{n-1}. We start an induction using the Mayer-Vietoris sequence.

Let $n = 1$. The Mayer-Vietoris sequence yields

$$\underbrace{H_1(U) \oplus H_1(V)}_{0} \to H_1(S^1) \xrightarrow{\alpha} \underbrace{H_0(U \cap V)}_{\mathbb{Z}^2} \to \underbrace{H_0(U) \oplus H_0(V)}_{\mathbb{Z}^2} \to \underbrace{H_0(S^1)}_{\mathbb{Z}} \to 0$$

The map α is injective, hence $H_1(S^1)$ is either $0, \mathbb{Z}$ or $\mathbb{Z}^2$. In a finite exact sequence, which ends in zeros on both ends and all groups are finitely generated, the alternating sum of the ranks of the groups add up to zero, as one sees from the corresponding base change to $\mathbb{Q}$, where all groups are finite-dimensional $\mathbb{Q}$-vector spaces. Hence we get

$$\operatorname{rank}(H_1(S^1)) = \operatorname{rank}(\mathbb{Z}^2) - \operatorname{rank}(\mathbb{Z}^2) + \operatorname{rank}(\mathbb{Z}) = 1.$$

Hence $H_1(S^1) \cong \mathbb{Z}$. The rest of the proof is left as an exercise to the reader. □

Lemma 5.7.5. (a) *Let X be a space and $x_0 \in X$. Viewing a curve in X as a simplex yields a group homomorphism $\pi_1(X, x_0) \to H_1(X)$.*

(b) *The map $\sigma : [0,1] \to S^1$, $t \mapsto e^{2\pi i t}$ is a generator of $H_1(S^1)$.*

Proof. (a) Homotopic paths γ and τ yield the same element, since the homotopy induces a chain in C_2, the boundary of which is $\gamma - \tau$. To see this, let $h : I \times I \to S^1$ be a homotopy with fixed ends between γ and τ. Then h is constant on $A = I \times \{0\}$ and on $B = I \times \{1\}$. Let X be the space you get, when you collapse A to a point and collapse B to a point, so

$$X = \left(I^2/A\right)\Big/B$$

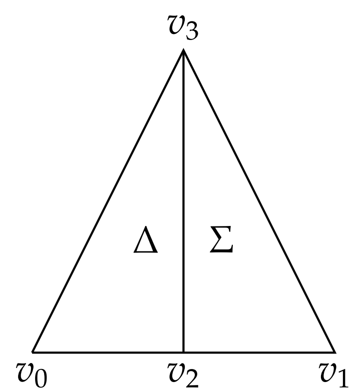
Then h factors through X in a unique way and X is the union of the closed sets Δ and Σ which intersect in a face:

$$\Delta = \left\{[(s,t)] \in X : t \leq \frac{1}{2}\right\},$$
$$\Sigma = \left\{[(s,t)] \in X : t \geq \frac{1}{2}\right\}.$$

The sets Δ and Σ are canonically homeomorphic to triangles with one common face. For instance, in the case of Δ the map

$$\eta : [(s,t)] \mapsto [(2st,t)]$$

is a homeomorphism from Δ to the triangle in $\mathbb{R}^2$ with vertices $(0,0), (0,\frac{1}{2}), (1,\frac{1}{2})$.



In the given ordering of the vertices the chain $\Delta - \Sigma$ has boundary $\gamma - \tau = \gamma + \check{\tau}$.

As a special case one gets that for any path γ the chain $\gamma + \check{\gamma}$ is the boundary of a 2-chain. Similarly, one sees that a composition of paths $\gamma.\eta$ yields the element $\gamma + \eta$ in $C_1(S^1)$ up to boundaries. Hence we get a group homomorphism $\pi_1(X, x_0) \to H_1(X)$.

(b) Let $\alpha = \sum_{j=1}^{n} k_j \sigma_j$ be an arbitrary element of $\ker(\partial)$. We can replace any σ_j with $k_j < 0$ by $\check{\sigma}_j$ and thus write $\alpha = \sum_{j=1}^{m} \gamma_j$ for paths γ_j. Since $\partial(\alpha) = 0$, the endpoints of the γ_j fit together in a way that the paths can be composed to form a single closed path γ, which shows that the group homomorphism $\phi : \pi_1(S^1) \to H_1(S^1)$ is surjective. Hence a generator of $\pi_1(S^1)$ maps to a generator of $H_1(S^1)$. □

Lemma 5.7.6. (a) *If the sequence* $(*)$ *splits, then* $B \cong A \oplus C$.

(b) *If* C *is a free abelian group, then the sequence* $(*)$ *splits.*

Proof. (a) Let $\phi : A \oplus C \to B$ be defined by $\phi(a, c) = \alpha(a) + s(c)$. Then ϕ is an injective group homomorphism since if (a, c) is the kernel, then $\alpha(a) = s(-c)$, also $-c = \beta s(-c) = \beta(\alpha(a)) = 0$. Since α is injective, we then also get $a = 0$.

The map ϕ is also surjective, since for $b \in B$ one has $\beta\Big(b - s(\beta(b))\Big) = \beta(b) - \beta s\beta(b) = \beta(b) - \beta(b) = 0$, so $b - s(\beta(b)) \in \ker(\beta) = \operatorname{im}(\alpha)$.

(b) Let $C = \bigoplus_{g\in G} \mathbb{Z}g$ for some generator set G. As β is surjective, for each $g \in G$ one can pick some $s(g) \in B$ with $\beta(s(g)) = g$. Then the map $s : G \to B$ extends to a uniquely determined group homomorphism $C \to B$ which splits the sequence. □

The following lemma is not part of the lecture.It explains the wording "split sequence".

Definition 5.7.7. We call an exact sequence $0 \to A \to B \to C \to 0$ **trivial**, if A or C is zero. In that case the sequence is just one isomorphism.

Lemma 5.7.8. *For an exact sequence* $0 \to A \xrightarrow{\alpha} B \to C \xrightarrow{\beta} 0$ *the following are equivalent:*

(a) *There exists* $t : B \to A$ *such that* $t\alpha = \mathrm{Id}_A$.

(b) *The sequence is isomorphic to a direct sum of trivial sequences.*

(c) *There exists* $s : C \to B$ *such that* $\beta s = \mathrm{Id}_C$.

Proof. (a)⇒(b): We need an isomorphism $\phi : B \to A \oplus C = A \times C$, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] & A \oplus C \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

commutes, where $\iota(a) = (a, 0)$ and p is the projection to the second coordinate. The choice is obvious: $\phi(b) := \big(t(b), \beta(b)\big)$.
Injectivity: If $\phi(b) = 0$, then $b \in \ker\beta = \operatorname{im}\alpha$, so $b = \alpha(a)$ for some $a \in A$, and then $a = t\alpha(a) = t(b) = 0$, hence $b = 0$.
Surjectivity: For given $(a, c) \in A \oplus B$ choose some $b' \in B$ with $\beta(b') = c$. Let $b = b' - \alpha t(b') + \alpha(a)$. Then $\beta(b) = \beta(b') = c$ and $t(b) = t(b') - t(b') + t\alpha(a) = a$.
Commutativity: We have $\phi(\alpha(a)) = \big(t\alpha(a), \beta\alpha(a)\big) = (a, 0) = \iota(a)$ and $p(\phi(b)) = \beta(b)$ by definition.

(b)⇒(c): Let ϕ be given. Then set $s : C \to B$, $s = \phi^{-1}\iota'$, where $\iota'(c) = (0, c)$. Then $\beta s = \mathrm{Id}_C$ by definition.

(c)⇒(a): For $b \in B$, the element $b - s\beta(b)$ lies in $\ker\beta = \operatorname{im}\alpha$ and so there is a unique $t(b) \in A$ with $\alpha t(b) = b - s\beta b$. It is easy to see that t is a homomorphism. For given $a \in A$ we have have

$$\alpha\big(t\alpha(a)\big) = \alpha(a) - \alpha s\beta(\alpha(a)) = \alpha(a)$$

and as α is injective, we get $t\alpha(a) = a$. □

Proposition 5.7.9. (a) *For any space X and $k \geq 0$ there is an exact sequence*

$$0 \to H_k(X) \to H_k(S^1 \times X) \to H_{k-1}(X) \to 0.$$

If $H_{k-1}(X)$ is a free abelian group, it follows

$$H_k(S^1 \times X) \cong H_k(X) \oplus H_{k-1}(X).$$

(b) *For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we have*

$$H_k(\mathbb{R}^n/\mathbb{Z}^n) \cong \mathbb{Z}^{\binom{n}{k}}.$$

Proof. (a) View S^1 as the unit circle in $\mathbb{C}$ and consider the open subsets $A = \left\{z \in S^1 : \operatorname{Re}(z) < \frac{1}{2}\right\}$ and $B = \left\{z \in S^1 : \operatorname{Re}(z) > -\frac{1}{2}\right\}$. Then A, B and the two components of $A \cap B$ are contractible and $S^1 = A \cup B$. Let $U = A \times X$ and $V = B \times X$. Then X is a retract of U and of V. Further, $U \cap V$ retracts to a disjoint union of two copies of X. The canonical map $\alpha : H^k(X)^2 \cong H^k(U \cap V) \to H^k(U) \oplus H^k(V) \cong H_k(X)^2$ can then be described as $(a, b) \mapsto (a + b, a + b)$. The kernel of this map, its image, and its cokernel are all isomorphic to $H_k(X)$.

This map appears in the exact Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to H_k(U \cap V) &\xrightarrow{\alpha_k} H_k(U) \oplus H_k(V) \xrightarrow{\beta} H_k(S^1 \times X) \\ &\xrightarrow{\gamma} H_{k-1}(U \cap V) \xrightarrow{\alpha_{k-1}} H_{k-1}(U) \oplus H_{k-1}(V) \to H_{k-1}(S^1 \times X) \to \ldots \end{aligned}$$

The cokernel of α_k is isomorphic to the image of β, so you get the exact sequence $0 \to H_k(X) \to H_k(S^1 \times X)$. Further, the cokernel of β is the image of γ is the kernel of α_{k-1}. This completes the claimed exact sequence. The second statement follows by Lemma 5.7.6.

(b) The case $n = 1$ is the case of S^1 which has been dealt with above. For the induction step, we apply part (a) with $X = (S^1)^n$. By induction hypothesis, all homology groups of X are free. Therefore, again using the induction hypothesis, we get

$$\begin{aligned} H_k((S^1)^{n+1}) &\cong H_k((S^1)^n) \oplus H_{k-1}((S^1)^n) \\ &\cong \mathbb{Z}^{\binom{n}{k}} \oplus \mathbb{Z}^{\binom{n}{k-1}} \\ &\cong \mathbb{Z}^{\binom{n}{k}+\binom{n}{k-1}} = \mathbb{Z}^{\binom{n+1}{k}}. \end{aligned}$$ □

Proposition 5.7.10. *For $n \in \mathbb{N}$ let $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ denote the n-dimensional real* *projective space. Then we have*

$$H_k(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z} & k = n \text{ and } n \text{ is odd}, \\ \mathbb{Z}/2 & k \text{ odd}, 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $n = 1$ we have $\mathbb{P}^n \cong S^1$ and so we know the assertion already.

For $n \geq 2$, the space $\mathbb{P}^n$ can be understood as the closed half-sphere modulo crosswise identification of the boundary points. Then $\mathbb{P}^n = U \cup V$ with open sets U and V, where U retracts to the boundary, i.e., to $\mathbb{P}^{n-1}$, the set V is simply connected and $U \cap V$ retracts to a sphere S^{n-1}.

Let $n = 2$. The map $f : S^1 = U \cap V \to U \to \mathbb{P}^1 \cong S^1$ is a double covering of the circle. But as the fundamental group $\mathbb{Z}$ has only one subgroup of index 2, namely $2\mathbb{Z}$, there is only one double covering of the circle. This map induces the multiplication by two on $H^1(S^1) \cong \mathbb{Z}$ as one sees from the fact that the generator of $H^1(S^1)$ (see Lemma 5.7.5) given by $\sigma : I \to S^1, t \mapsto e^{2\pi i t}$, is mapped to $e^{4\pi i t}$, which winds around the circle twice, hence can be written as the sum of two copies of σ. The Mayer-Vietoris sequence therefore yields the exact sequence

$$0 \to H_2(\mathbb{P}^2) \xrightarrow{\alpha} \underbrace{H_1(S^1)}_{\cong\mathbb{Z}} \xrightarrow{\cdot 2} \underbrace{H_1(S^1)}_{\cong\mathbb{Z}} \to H_1(\mathbb{P}^2)$$

$$\xrightarrow{\delta} \underbrace{H_0(S^1)}_{\cong\mathbb{Z}} \to \underbrace{H_0(U) \oplus H_0(V)}_{\mathbb{Z}^2} \to \underbrace{H_0\mathbb{P}^2)}_{\mathbb{Z}} \to 0.$$

Since multiplication by 2 is injective, it follows $H_2(\mathbb{P}^2) = 0$. Comparing the ranks in the last rows yields $\delta = 0$ and therefore $H_1(\mathbb{P}^2) = \mathbb{Z}/2$.

Next suppose $n \geq 3$ and the claim is proven up to $n-1$, we want to show it for n. We start with the case when n is odd. The Mayer-Vietoris sequence then reads

$$\begin{aligned} \ldots \to H_k(S^{n-1}) &\to H_k(\mathbb{P}^{n-1}) \to H_k(\mathbb{P}^n) \\ &\to H_{k-1}(S^{n-1}) \to H_{k-1}(\mathbb{P}^{n-1}) \to H_{k-1}(\mathbb{P}^n) \to \ldots \end{aligned}$$

For $k > n$ we conclude that $H_k(\mathbb{P}^n)$ is zero. For $k = n$ we get the exact sequence

$$\begin{aligned} \ldots \to \underbrace{H_n(S^{n-1})}_{0} &\to \underbrace{H_n(\mathbb{P}^{n-1})}_{0} \to H_n(\mathbb{P}^n) \\ &\to \underbrace{H_{n-1}(S^{n-1})}_{\mathbb{Z}} \to \underbrace{H_{n-1}(\mathbb{P}^{n-1})}_{0} \to H_{n-1}(\mathbb{P}^n) \to \underbrace{H_{n-2}(S^{n-1})}_{0} \to \end{aligned}$$

Form this we conclude that $H_n(\mathbb{P}^n) \cong \mathbb{Z}$ and $H_{n-1}(\mathbb{P}^n) = 0$.

Next consider the case when $0 < k < n-1$ is odd. Then the sequence becomes

$$\begin{aligned} \ldots \to \underbrace{H_k(S^{n-1})}_{0} &\to \underbrace{H_k(\mathbb{P}^{n-1})}_{\mathbb{Z}/2} \to H_k(\mathbb{P}^n) \\ &\to \underbrace{H_{k-1}(S^{n-1})}_{0} \to \underbrace{H_{k-1}(\mathbb{P}^{n-1})}_{0} \to H_{k-1}(\mathbb{P}^n) \to \underbrace{H_{k-2}(S^{n-1})}_{0} \end{aligned}$$

which implies that $H_k(\mathbb{P}^n) \cong \mathbb{Z}/2$ and $H_{k-1}(\mathbb{P}^n) = 0$. The case of even n runs similarly. □

* * *

5.8 Relative homology

Definition 5.8.1. A **pair of spaces** is a pair (X, A), where X is a topological space and $A \neq \emptyset$ a subset of X. Let (X, A) be a pair of spaces. Define

$$C_n(X, A) := C_n(X)/C_n(A).$$

Because of $\partial C_n(A) \subset C_{n-1}(A)$ the boundary operator induces a map

$$\partial : C_n(X, A) \to C_{n-1}(X, A).$$

One has $\partial^2 = 0$, since this is already true in $C_n(X)$. The **relative homology** is defined by

$$H_n(X, A) := \ker \partial|_{C_n(X,A)} \Big/ \partial(C_{n+1}\,(X, A)).$$

We use the following conventions

- $\alpha \in C_n(X, A)$ with $\partial\alpha = 0$ is called a **relative cycle**.
- $\alpha \in C_n(X, A)$ with $\alpha = \partial\beta$ for some β is called a **relative boundary**.

An element $\alpha \in C_n(X, A) = C_n(X)/C_n(A)$ is a relative cycle, iff there is a representative α in $C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.

Likewise, α is a relative boundary iff there is $\beta \in C_{n+1}(X)$ with $\partial\beta - \alpha \in C_n(A)$.

Theorem 5.8.2. *For a pair (X, A) there is an exact sequence*

$$\cdots \to H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\delta} H_{k-1}(A) \to \dots$$

$$\cdots \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0,$$

where $i : A \hookrightarrow X$ is the inclusion and j is the chain map $C_k(X) \to C_k(X)/C_k(A) = C_k(X, A)$.

Proof. The maps i and j induce an exact sequence of chain complexes:

$$0 \longrightarrow C_k(A) \xrightarrow{i} C_k(X) \xrightarrow{j} C_k(X, A) \longrightarrow 0$$

So the claim follows from Theorem 5.5.3. □

Example 5.8.3. Consider the pair $(X, A) = (\Delta_n, \partial\Delta_n)$. Then

$$H_k(\Delta_n, \partial\Delta_n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

In the case $k = n$, the identity map on Δ_n yields a generator. This is clear from the Theorem, where we note that for $k = n$ the map j_* is induced by the identity map.

Definition 5.8.4. Let (X, A) and (Y, B) pairs of spaces. A **pair map** $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subset B$. Two pair maps $f, g : (X, A) \to (Y, B)$ are called **pair-homotopic**, or simply **homotopic**, if there is a homotopy h_s, where each h_s is a pair map.

A pair map $f : (X, A) \to (Y, B)$ induces a group homomorphism

$$f_* : H_k(X, A) \ \to \ H_k(Y, B)$$

by $f_*[\sigma] = [f \circ \sigma]$.

Proposition 5.8.5. *Let $f, g : (X, A) \to (Y, B)$ homotopic through pair maps. Then*

$$f_* = g_*.$$

Proof. The prism operator P maps $C_n(A)$ to $C_{n+1}(B)$, so it induces a relative prism operator $P : C_n(X,A) \to C_{n+1}(Y,B)$. The transition to the quotient preserves the equation $\partial P + P\partial = g_* - f_*$, hence $g_* - f_*$ is the zero map. □

Lemma 5.8.6. (a) *If X is path-connected, one has for any $x_0 \in X$,*

$$H_k(X,x_0) \cong H_k(X), \quad k \geq 1$$
$$H_0(X,x_0) = 0.$$

(b) *For a deformation retract $A \subset X$ one has*

$$H_k(X,A) = 0.$$

Proof. (a) The long exact sequence of Theorem 5.8.2 yields an exact sequence

$$H_k(x_0) \to H_k(X) \to H_k(X,x_0) \to H_{k-1}(x_0).$$

For $k \geq 2$ the two outer groups are zero, so the middle homomorphism is an isomorphism. For $k = 1$ one gets an exact sequence

$$0 \to H_1(X) \xrightarrow{a} H_1(X,x_0) \xrightarrow{b} H_0(x_0) \xrightarrow{c} H_0(X) \xrightarrow{d} H_0(X,x_0) \to 0.$$

The map c is an isomorphism, so b is zero and a is an isomorphism. Further, d is zero and $H_0(X,x_0) = 0$.

(b) Let $h : I \times X \to X$ be a deformation to A. Let $f : X \to X$, $f(x) = h(1,x)$. Then the pair map $f : (X,A) \to (X,A)$ is homotopic to the identity. Therefore $f_* = \mathrm{Id}$. On the other hand, $f(X) \subset A$ and so f_* is zero. It follows $\mathrm{Id} = 0$ on $H_p(X,A)$, hence $H_p(X,A) = 0$. □

Lemma 5.8.7. *Let X be a topological space and $B \subset A \subset X$. Then there is an* *exact sequence*

$$\begin{aligned}\cdots \to H_k(A,B) \to H_k(X,B) \to H_k(X,A)\\ \to H_{k-1}(A,B) \to \cdots \to H_0(X,A) \to 0.\end{aligned}$$

Proof. This sequence results from Theorem 5.5.3 applied to the exact sequence

$$0 \to C_k(A,B) \to C_k(X,B) \to C_k(X,A) \to 0. \qquad \square$$

Theorem 5.8.8 (Excision)**.** *Let X be a space and $Z \subset A \subset X$ subsets, where Z is closed and contained in the open interior $\mathring{A}$ of A. Then the inclusion $(X \smallsetminus Z, A \smallsetminus Z) \hookrightarrow (X,A)$ induces isomorphisms*

$$H_k(X \smallsetminus Z, A \smallsetminus Z) \xrightarrow{\cong} H_k(X,A), \quad k \geq 0.$$

Proof. Let $\psi : C_n(X \smallsetminus Z, A \smallsetminus Z) \to C_n(X,A)$ be the map induced by inclusion. Let $U = X \smallsetminus Z$ and $V = \mathring{A}$, then $U \cup V = X$. For every given class $[\alpha] \in H_n(X)$ there exists a power u^r of the barycentric division map, such that $u^r(\alpha) \in C_n(U) + C_n(V)$. The barycentric division map u maps $C_n(A)$ to itself. The inclusion gives a chain map

$$\phi_n : \Big(C_n(U) + C_n(V)\Big)\Big/\Big(C_n(A) \cap \Big(C_n(U) + C_n(V)\Big)\Big) \to C_n(X)/C_n(A) = C_n(X,A).$$

We claim that ϕ induces an isomorphism in homology. Call the complex on the left A_n and the one on the right B_n. So $\phi_n : A_n \to B_n$. We identify A_n with its image in B_n and view A_n as a subcomplex of B_n, where ϕ is the inclusion. For *injectivity* let $[\alpha] \in H_n(A_\bullet)$ be a homology class with $\phi_*[\alpha] = 0$. By Lemma 5.6.7 we have $[u^r\alpha] = [\alpha]$ it suffices to show that there exists $r \in \mathbb{N}$ such that $u^r\alpha = \partial\beta$ for some $\beta \in A_{n+1}$. Since

$\phi(\alpha)$ is trivial in homology, there exists $\eta \in B_{n+1}$ such that $\partial\eta = \alpha$. There exists $r \in \mathbb{N}$ such that $\beta = u^r\eta$ lies in A_n and then $\partial\beta = \partial u^r\eta = u^r\partial\eta = u^r\alpha$ as claimed.

For *surjectivity* let $[\tau]$ be a class in $H_n(B_\bullet)$. Then there exists $r \in \mathbb{N}$ such that $u^r\tau \in A_n$ and so $[\phi(u^r\tau)] = [u^r\tau] = [\tau]$.

So the inclusion $C_n(U) + C_n(V)$ into $C_n(X)$ induces an isomorphy in the homology of the complexes $C_n(X, A)$ and

$$\Big(C_n(U) + C_n(V)\Big)\Big/\Big(C_n(A) \cap \Big(C_n(U) + C_n(V)\Big)\Big) = C_n(U)\Big/\Big(C_n(A) \cap C_n(U)\Big).$$

By a definition we have

$$\begin{aligned} C_n(U)\Big/\Big(C_n(U) \cap C_n(A)\Big) &= C_n(X \smallsetminus Z)\Big/\Big(C_n(X \smallsetminus Z) \cap C_n(A)\Big) \\ &= C_n(X \smallsetminus Z, A \smallsetminus Z). \end{aligned}$$

Therefore, the inclusion maps induce an isomorphism between the relative homology $H_n(X, A)$ and the homology of the complex

$$C_n(X \smallsetminus Z, A \smallsetminus Z),$$

i.e., the map $\psi_* : H_n(X \smallsetminus Z, A \smallsetminus Z) \to H_n(X, A)$ is an isomorphism. □

* * *

5.9 The long exact sequence of a pair

Definition 5.9.1. A subset $\emptyset \neq A \subset X$ is called **regularly closed**, if A is closed in X and there is an open neighbourhood U of A, such that A is a deformation retract of U.

Examples 5.9.2. (a) An embedded smooth sub-manifold M in $X = \mathbb{R}^N$

is regularly closed, since there is a neighbourhood, spanned by normal vector fields. This is called a **tubular neighbourhood**.

(b) The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{0\right\}$ is closed in $\mathbb{R}$, but not regularly closed.

Lemma 5.9.3. *Let $A \subset X$ be regularly closed. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms*

$$q_* : H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A).$$

Proof. Let U be an open neighbourhood of A, such that A is a deformation retract of U. We have a commutative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "a"] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "e"] & H_n(X \setminus A, U \setminus A) \arrow[l, "b"'] \arrow[d, "f"] \\
H_n(X/A, A/A) \arrow[r, "c"] & H_n(X/A, U/A) & H_n(X/A \setminus A/A, U/A \setminus A/A) \arrow[l, "d"']
\end{tikzcd}$$

By the Excision theorem of relative homology, Theorem 5.8.8, the map b is an isomorphism.

The map a is an isomorphism, since then in the exact sequence of Lemma 5.8.7 for the triple $A \subset U \subset X$ all the groups $H_n(U, A)$ are zero by Lemma 5.8.6. By the same reason, the maps b, c and d are isomorphisms.

Since q is an isomorphism on the complement of A, the map f is an isomorphism. The commutativity of the diagram implies then that e and finally q_* are isomorphisms. The Lemma is proven. □

Theorem 5.9.4. *Let X be path-connected and $\emptyset \neq A \subset X$ regularly closed.*

Then there is an exact sequence:

$$\begin{aligned} \ldots \to H_k(A) &\xrightarrow{i_*} H_k(X) \xrightarrow{p_*} H_k(X/A) \\ &\xrightarrow{\delta} H_{k-1}(A) \xrightarrow{i_*} H_{k-1}(X) \xrightarrow{p_*} H_{k-1}(X/A) \ldots \end{aligned}$$

$$\cdots \to H_1(X/A) \xrightarrow{\delta} H_0(A) \to \underbrace{H_0(X)}_{\mathbb{Z}} \to 0.$$

Here i is the inclusion $A \hookrightarrow X$ and p the projection $X \to X/A$.

Proof. The long exact sequence of Theorem 5.8.2 looks like this one with $H_k(X, A)$ instead of $H_k(X/A)$. By the last Lemma 5.9.3 we have $H_k(X, A) \xrightarrow{\cong} H_k(X/A, A/A)$ and Lemma 5.8.6 (a) says that $H_k(X/A, A/A) \cong H_k(X/A)$ for $k \geq 1$ and $H_0(X, A) = 0$. □

Corollary 5.9.5. *The exact sequence of the theorem is functorial in the following way. Let $f : (X, A) \to (Y, B)$ be a pair map where A and B are regularly closed in X and Y respectively. Then the ensuing diagram*

$$\begin{tikzcd} H_k(A) \arrow[r] \arrow[d, "f_*"] & H_k(X) \arrow[r] \arrow[d, "f_*"] & H_k(X/A) \arrow[r] \arrow[d, "f_*"] & H_{k-1}(A) \arrow[d, "f_*"] \\ H_k(B) \arrow[r] & H_k(Y) \arrow[r] & H_k(Y/B) \arrow[r] & H_{k-1}(B) \end{tikzcd}$$

commutes.

Proof. The map f induces maps between the chain groups and thus the functoriality of the snake lemma induces this corollary. □

Corollary 5.9.6. *Suppose that the regularly closed subset $A \subset X$ is* *contractible. Then the projection map $j : X \to X/A$ induces isomorphisms*

$$H_k(X) \cong H_k(X/A)$$

for every $k \geq 0$.

Proof. For $k = 0$ we have that $H_0(X)$ is a direct sum of copies of $\mathbb{Z}$, the number of which equals the number of path-components of X, see Proposition 5.2.5. As A is path-connected, this number does not change under j. For $k \geq 1$ we have $H_{k+1}(A) = 0 = H_k(A)$ which implies the claim by the long exact sequence. □

* * *

5.10 Equivalence of simplicial and singular homology

Let X be a simplicial complex with a linear order $<$ on X_0. Let $\Delta = [v_0, \dots, v_n]$ be a simplex of X. For Δ define a singular Simplex $\sigma_\Delta : [v_0, \dots, v_n] \to X$ given by the inclusion of Δ in X. Let $C_n^{\mathrm{simp}}(X)$ be the free abelian group generated by the n-simplices of X. The map $\Delta \mapsto \sigma_\Delta$ induces a group homomorphism

$$\psi : C_n^{\mathrm{simp}}(X) \to C_n(X).$$

Lemma 5.10.1. *One has $\partial\psi = \psi\partial$.*

Proof.

$$\begin{aligned}\partial\psi([v_0,\dots,v_n]) &= \partial\sigma_{[v_0,\dots,v_n]}\\ &= \sum_{i=0}^{n}(-1)^i\ \sigma|_{[v_0,\dots\hat{v}_i\dots,v_n]}\\ &= \sum_{i=0}^{n}(-1)^i\ \sigma_{[v_0,\dots\hat{v}_i\dots,v_n]}\\ &= \psi\partial([v_0,\dots,v_n]).\end{aligned}$$

□

So ψ induces a map $\psi_*: H_n^{\text{simp}}(X) \to H_n(X)$.

Theorem 5.10.2. *ψ_* is an isomorphism $H_k^{\text{simp}}(X) \cong H_k(X)$.*

Proof. For $k = 0$ the claim is clear. So let $k \geq 1$. For a sub-complex $A \subset X$ let

$$C_k^{\text{simp}}(X, A) = C_k^{\text{simp}}(X)/C_k^{\text{simp}}(A).$$

As in the singular case, one defines the relative homology groups $H_k^{\text{simp}}(X, A)$. The exact sequence

$$0 \to C_k^{\text{simp}}(A) \to C_k^{\text{simp}}(X) \to C_k^{\text{simp}}(X, A) \to 0$$

induces a long exact sequence by Theorem 5.5.3

$$\begin{aligned}\dots \to H_{k+1}^{\text{simp}}(X, A) \to H_k^{\text{simp}}(A) \to H_k^{\text{simp}}(X)&\\ \to H_k^{\text{simp}}(X, A) \to H_{k-1}^{\text{simp}}(A) \to \dots&\end{aligned}$$

We replace X by the n-skeleton X_n and we set $A = X_{n-1}$. We get a commutative diagram with exact rows:

$$\begin{tikzcd}
H_{k+1}^{\mathrm{simp}}(X_n, X_{n-1}) \arrow[r] \arrow[d, "\alpha"] & H_k^{\mathrm{simp}}(X_{n-1}) \arrow[r] \arrow[d, "\beta"] & H_k^{\mathrm{simp}}(X_n) \arrow[r] \arrow[d, "\gamma"] & H_k^{\mathrm{simp}}(X_n, X_{n-1}) \arrow[r] \arrow[d, "\delta"] & H_{k-1}^{\mathrm{simp}}(X_{n-1}) \arrow[d, "\varepsilon"] \\
H_{k+1}(X_n, X_{n-1}) \arrow[r] & H_k(X_{n-1}) \arrow[r] & H_k(X_n) \arrow[r] & H_k(X_n, X_{n-1}) \arrow[r] & H_{k-1}(X_{n-1})
\end{tikzcd}$$

We show that α and δ are isomorphisms. One has $C_k^{\mathrm{simp}}(X_n, X_{n-1}) = 0$ for $k \neq n$ and $C_n^{\mathrm{simp}}(X_n, X_{n-1}) \cong C_n^{\mathrm{simp}}(X)$ is the free abelian group generated by the n-simplices. Then it follows

$$H_k^{\mathrm{simp}}(X_n, X_{n-1}) \cong \begin{cases} C_k^{\mathrm{simp}}(X) & k = n \\ 0 & k \neq n. \end{cases}$$

For the singular homology groups consider the map

$$\phi : \bigsqcup_\alpha (\Delta_\alpha^n, \partial\Delta_\alpha^n) \ \to \ (X_n, X_{n-1})$$

given by the inclusion map of the n-simplices Δ_α^n. The map ϕ induces a homeomorphism

$$\bigsqcup_\alpha \Delta_\alpha^n \Big/ \bigsqcup_\alpha \partial\Delta_\alpha^n \ \cong \ X_n/X_{n-1}.$$

Since X_{n-1} is regularly closed in X_n, we can apply excision twice to get

$$\begin{aligned}
H_k(X_n, X_{n-1}) &\cong H_k(X_n/X_{n-1}, X_{n-1}/X_{n-1}) \\
&\cong H_k\left(\bigsqcup_\alpha \Delta_\alpha^n \Big/ \bigsqcup_\alpha \partial\Delta_\alpha^n, \bigsqcup_\alpha \partial\Delta_\alpha^n \Big/ \bigsqcup_\alpha \partial\Delta_\alpha^n\right) \\
&\cong H_k\left(\bigsqcup_\alpha \Delta_\alpha^n, \bigsqcup_\alpha \partial\Delta_\alpha^n\right) \\
&\cong \bigoplus_\alpha H_k(\Delta_\alpha^n, \partial\Delta_\alpha^n).
\end{aligned}$$

For $k \neq n$, this is zero and for $k = n$, it is the free abelian group generated by the inclusions $\Delta_\alpha^n \hookrightarrow X$ of the n-simplices, see Example 5.8.3. Hence

α and δ are isomorphisms.

By induction on n we can assume that β and ε are isomorphisms, too. This is clear for $n = 0$, since then X_{n-1} is the empty set. From there on, the induction works.

Lemma 5.10.3 (Five-Lemma)**.** *Let there be given a commutative diagram of abelian groups with exact rows:*

$$\begin{tikzcd} A \arrow[r, "d_A"] \arrow[d, "\alpha"] & B \arrow[r, "d_B"] \arrow[d, "\beta"] & C \arrow[r, "d_C"] \arrow[d, "\gamma"] & D \arrow[r, "d_E"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r, "d'_A"'] & B' \arrow[r, "d'_B"'] & C' \arrow[r, "d'_C"'] & D' \arrow[r, "d'_D"'] & E' \end{tikzcd}$$

If α, β, δ and ε are isomorphisms, then γ is an isomorphism, too.

Proof. Let $c \in C$ with $\gamma(c) = 0$. Then $\delta(d_C(c)) = 0$ and since δ is injective, we have $d_C(c) = 0$. By the exactness of the top row, there is $b \in B$ with $c = d_B(b)$, hence $0 = \gamma(c) = d'_B(\beta(b))$. By the exactness of the lower row, there is $a' \in A'$ such that $\beta(b) = d'_A(a')$. As α is surjective, there is $a \in A$ with $a' = \alpha(a)$. Then $\beta(d_A(a)) = \beta(b)$ and since β is injective, we get $b = d_A(a)$ and so $c = d_B(b) = d_B(d_A(a)) = 0$ by exactness of the first row.

The surjectivity of γ is left to the reader. □

The five-lemma now implies that $\psi_* : H_k^{\mathrm{simp}}(X_n) \to H_k(X_n)$ is an isomorphism for every n. It remains to show this assertion for X instead of X_n. For this let $[\alpha] \in H_k(X)$ be a homology class with representative $\alpha \in C_k(X)$. Then α is a linear combination of singular simplices, each of which has a compact image. So any simplex only meets finally many cells in X, it follows that α already lies in $C_k(X_n)$ for some $n \in \mathbb{N}$. As ψ_* is surjective X_n, there is $\beta \in C_k^{\mathrm{simp}}(X_n) \subset C_k^{\mathrm{simp}}(X)$ with $\psi\beta - \alpha \in \partial(C_k(X_n)) \subset \partial(C_k(X))$. It follows $\psi_*[\beta] = [\alpha]$, hence ψ_* is surjective.

For injectivity note that for $n \geq k+1$ the inclusion $X_n \hookrightarrow X$ induces an isomorphism $H_k^{\mathrm{simp}}(X_n) \cong H_k^{\mathrm{simp}}(X)$. Let $[\alpha] \in H_k^{\mathrm{simp}}(X)$ lie in the kernel of ψ_*. Then $[\alpha]$ already lies in $H_k^{\mathrm{simp}}(X_n)$ for some $n \geq k+1$ and so $[\alpha] = 0$ holds in $H_k^{\mathrm{simp}}(X_n) \cong H_k^{\mathrm{simp}}(X)$. The theorem is proven. □

* * *

5.11 Applications

Theorem 5.11.1. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. If U and V are homeomorphic then we have $m = n$.*

Proof. Let $h : U \to V$ be a homeomorphism. If $n = 1$, then one can assume that U is an interval. Then for $x \in U$ the set $U \smallsetminus \left\{x\right\}$ Has two connected components. The same holds true for $V \smallsetminus \left\{h(x)\right\}$, so that also $m = 1$ must hold since otherwise a connected open set remains connected if one removes one point.

So let $m, n \geq 2$ and let $x \in U$. Let $W \subset U$ be a chart neighborhood of x and let $Z = U \smallsetminus W$. By the theorem on excision for relative homology we have

$$H_k\left(U, U \smallsetminus \left\{x\right\}\right) \cong H_k(U\smallsetminus Z, (U\smallsetminus\{x\})\smallsetminus Z) = H_k(W, W\smallsetminus\{x\}) \ \cong\ H_k\left(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \left\{x\right\}\right).$$

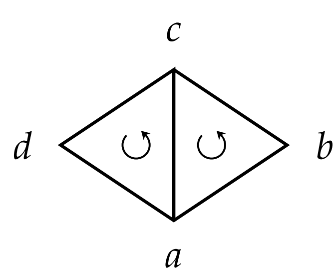
Since $\mathbb{R}^n$ is contractible, the long exact sequence of the pair $\left(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \left\{x\right\}\right)$ yields isomorphisms for $k \geq 2$

$$H_k\left(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \left\{x\right\}\right) \ \cong H_{k-1}\left(\mathbb{R}^n \smallsetminus \left\{x\right\}\right) \cong H_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = n, \\ 0 & n \neq k \geq 2. \end{cases}$$

In the middle step we used that $\mathbb{R}^n \setminus \{1\}$ retracts to the sphere S^{n-1}. The same holds fo V and so $m = n$. $\square$

Definition 5.11.2. We call a d-dimensional simplicial complex X is called a **manifold complex**, if the X us the closure of all d-dimensional cells and for every $(d-1)$-dimensional face f of a cell α there is exactly one cell β with $f = \alpha \cap \beta$. We then call β a **neighbour** of α.

Let α, β be two d-cells. Then we have $\partial(\alpha + \beta) = \sum_z k_z z$, where the sum runs over all $d-1$ cells and k_z is in $\{-2, -1, 0, 1, 2\}$. We call α and β **equally oriented** if $k_z = 0$, where z is is the common face of α and β.



In the picture: if for instance $a < b < c < d$, Then the two cells are equally oriented. If however, say, $a < b < d < c$, then they are not. The complex X is called **orientable** if there exists an ordering on the vertices, such that any two neighoured cells are equally oriented.

A triangulation of a smooth manifold is orientable iff the manifold is. (Take that as a definition of an orientable manifold if you are not familiar with the analytic notion.)

Theorem 5.11.3. *Let M be a connected compact differentiable manifold. Then for every $n \geq 0$ the group $H_n(M)$ is finitely generated. One has $H_0(M) \cong \mathbb{Z}$ and for $d = \dim M$,*

$$H_d(M) \cong \begin{cases} \mathbb{Z} & M \text{ orientable}, \\ 0 & \text{otherwise}. \end{cases}$$

In the first case, there is a generator of the form $\sigma = \sum_{j=1}^{n} k_j \sigma_j$, where the σ_j are positively oriented simplices and all k_j are positive. We call σ a **positive generator**.

Proof. Since M is path-connected, we already know that $H_0(M) \cong \mathbb{Z}$.

A compact differentiable manifold possesses a finite triangulation. The corresponding simplicial homology is finally generated and so is the singular homology.

Let M be orientable. We fix a compatible ordering of the vertices. Then let $A = \sum_\alpha \alpha \in C_d(M)$, Where the sum runs over all d-cells in M. Therefore, $\partial A = 0$ and we claim that A generates the group $H_d(M)$. So let $B \in \ker \partial|_{C_d(M)}$. Then $B = \sum_z m_z z$, where the sum runs over all d-cells. Consider two neighboured d-cells α and β. Then they have exactly one $d-1$-cell c in common and the other way round to any $d-1$-cell there are exactly two d-cells containing it. Therefore the equation $\partial \sum_z m_z z = 0$ implies that $m_z = m_w$, if z and w are neighboured. Since M is connected it follows that all m_z are equal. We can choose them equal to 1 to get a special generator.

For the result on the non-orientable case we show that if $H_d(M) \neq 0$, then M is orientable. So let $B = \sum_z m_z z$ be in the kernel of $\partial|_{C_d(M)}$. Since neighbour cells have to cancel in the common $d-1$-cell, it follows that $m_z = \pm m_w$ if z and w are neighboured. Let $\varepsilon_z = \text{sign}(m_z)$, then the family $(\varepsilon_z z)_z$ defines an orientation on M. □

Definition 5.11.4. Let $f : M \to N$ be a continuous map between compact orientable manifolds of the same dimension n. Then f induces a map $f_* : \mathbb{Z} \cong H_n(M) \to H_n(N) \cong \mathbb{Z}$ and by choosing positive generators, we get a group homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$, which then must be of the form $f_*(k) = df$ for some $d \in \mathbb{Z}$, called the **mapping degree** $\deg(f)$ of f.

Example 5.11.5. If f is an d-fold covering of an orientable manifold, then $\deg(f) = d$.

Proof. f_* maps the positive generator α to $n\beta$ for the positive generator on N as each cell in N has n preimages in M. □

Proposition 5.11.6. (a) *If f is homotopic to g, then* $\deg(f) = \deg(g)$.

(b) *One has* $\deg(f \circ g) = \deg(f)\deg(g)$.

(c) *If $f : S^n \to S^n$ is a reflection at a hyperplane, then we have* $\deg(f) = -1$.

(d) *The antipode map $S^n \to S^n$, $x \mapsto -x$ has degree* $(-1)^{n+1}$.

(e) *If the map $f : S^n \to S^n$ has no fixed point, then* $\deg(f) = (-1)^{n+1}$.

Proof. (a) follows from $f_* = g_*$ if f and g are homotopic.

(b) follows from $(fg)_* = f_*g_*$.

(c) The hyperplane divides S^n into two hemi-spheres A and B, each being the cone over S^{n-1} We give S^{n-1} a simplicial structure and on each hemisphere, we take the one given above. Take $\omega \in C_{n-1}(S^{n-1})$ such that $\partial\omega = 0$ and ω induces a non-trivial element in $H^{n-1}(S^{n-1})$. Let $C_A\omega$ and $C_B\omega$ be the respective cone constructions on the two hemispheres A and B and let $\eta = C_A\omega - \mathbb{C}_B\omega$, then by Lemma 5.12.2 we get $\partial\eta = 0$, so η defines a homology class in $H_n^{\mathrm{simp}}(S^n)$. It is non-trivial in $C_n^{\mathrm{simp}}(S^n)$ and therefore, as $C_{n+1}^{\mathrm{simp}}(S^n) = 0$, it defines a non-zero homology class. Now the reflection f swaps A and B and thus maps $C_A\omega$ to $C_B\omega$ and thus f induces multiplication by (-1) on $H_n(S^n)$.

(d) The antipode is a product of $n + 1$ reflections.

(e) Let $f : S^n \to S^n$ have no fixed point. Then the line through $f(x)$ and $-x$ does not contain zero. In particular, $(1-t)f(x) - tx$ is not zero for $0 \le t \le 1$. Hence

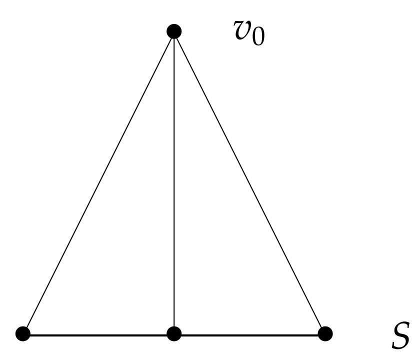
$$h(t,x) = \frac{(1-t)f(x) - tx}{\left\|(1-t)f(x) - tx\right\|}$$

defines a homotopy from f to the antipode. □

* * *

5.12 Hairy ball theorem

Definition 5.12.1. Let S be a finite simplicial complex of dimension n. The cone CS has a natural simplicial structure as follows. Add a new vertex v_0 to the vertex set and for each simplex $\{v_1, \dots, v_k\}$ add the simplex $\{v_0, \dots, v_k\}$. Finally, extend the ordering of the vertices in such a way that $v_0 < v$ for every vertex v of S. For a given simplex $\Delta = [v_1, \dots, v_{n+1}]$ in S let $C\Delta = [v_0, \dots, v_{n+1}]$ and extend the map $\Delta \mapsto C\Delta$ linearly to the chain group $C_\bullet^{\mathrm{simp}}(S)$



Lemma 5.12.2. *Let $\omega \in C_k^{\mathrm{simp}}(S)$. If $\partial\omega(S) = 0$, then*

$$\partial C\omega = \omega.$$

Proof. Let $\omega = \sum_{\nu=1}^{m} c_\nu [v_{\nu,1}, \ldots, v_{\nu,k}]$. Writing $v_{\nu,0} = v_0$ we have

$$\begin{aligned}
\partial C\omega &= \sum_{\nu=1}^{m} c_\nu \sum_{j=0}^{n} (-1)^j [v_{\nu,0}, \ldots \widehat{v_{\nu,j}} \ldots v_{\nu,n}] \\
&= \omega + \sum_{\nu=1}^{m} c_\nu \sum_{j=1}^{n} (-1)^j [v_{\nu,0}, \ldots \widehat{v_{\nu,j}} \ldots v_{\nu,n}] \\
&= \omega + C\partial\omega = \omega
\end{aligned}$$

□

Definition 5.12.3. A **vector field** on S^n is a continuous map $f : S^n \to \mathbb{R}^{n+1}$ such that $f(x)$ is perpendicular to x, i.e., we have

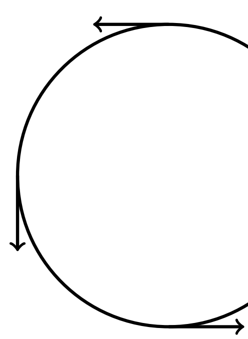
$$\langle f(x), x \rangle = 0$$

for every $x \in S^n$.

Theorem 5.12.4 (Hairy Ball Theorem)**.** *If n is even, then every vector field on S^n has a zero.*

"Every hairy ball has a whirl."

Here is an example of a zero-less field in the case $n = 1$.



Proof. Let $f : S^n \to \mathbb{R}^{n+1}$ be a vector field without zero. Replace $f(x)$ by $\frac{f(x)}{\|f(x)\|}$ to get $f(x) \in S^n$. Since $f(x) \perp x$, the line from $f(x)$ to x does not pass through zero, hence

$$h(t,x) = \frac{(1-t)f(x)+tx}{\left\|(1-t)f(x)+tx\right\|}$$

is a homotopy from $f : S^n \to S^n$ to the identity. This implies that $\deg(f) = 1$. On the other hand, f has no fixed point, so $1 = \deg(f) = (-1)^{n+1}$ and therefore n is odd. □

* * *

5.13 Homology and fundamental group

For a group G let $[G,G]$ be the **commutator group**, i.e., the subgroup of G generated by all commutators

$$[a,b] = aba^{-1}b^{-1}$$

with $a,b \in G$. Since $c[a,b]c^{-1} = [cac^{-1}, cbc^{-1}]$ the group $[G,G]$ is a normal subgroup of G.

Proposition 5.13.1. *The quotient $G_{\mathrm{ab}} = G/[G,G]$ the maximal abelian quotient of G. More precisely, the group G_{ab} is abelian and for every homomorphism $\phi : G \to A$ to an abelian group A there exists exactly one homomorphism $\phi_{\mathrm{ab}} : G_{\mathrm{ab}} \to A$ with $\phi = \phi_{\mathrm{ab}} \circ p$, where $p : G \to G_{\mathrm{ab}} = G/[G,G]$ is the natural projection.*

Proof. For $a,b \in G$ we have

$$[p(a),p(b)] = p([a,b]) = 1.$$

So G_{ab} is abelian.

Let $\phi : G \to A$ be a homomorphism to an abelian group. As A is abelian, one has for $a, b \in G$:

$$\phi([a,b]) = [\phi(a), \phi(b)] = 1.$$

So ϕ factors in a unique way through G_{ab}. □

Let $\gamma : I \to X$ be a closed path with endpoint x_0. Then one can view γ as a singular 1-simplex. This is a k, since $\partial\gamma = \gamma(1) - \gamma(0) = 0$. So γ defines a homology class in $H_1(X)$.

Theorem 5.13.2 (Hurewicz)**.** *Considering closed paths as simplices yields a group homomorphism $\psi : \Gamma = \pi_1(X, x_0) \to H_1(X)$. The kernel of ψ is the commutator subgroup* $[\Gamma, \Gamma]$.

If X is path-connected, then ψ is surjective, hence gives an isomorphism

$$\pi_1(X)_{\mathrm{ab}} \;\cong\; H_1(X).$$

Proof. In Lemma 5.7.5 we have shown that ψ is a group homomorphism. As $H_1(X)$ is abelian, it follows $[\Gamma, \Gamma] \subset \ker\psi$. One has to show that $[\Gamma, \Gamma] \supset \ker\psi$. Let $[\gamma]$ be in the kernel of ψ. Then γ is the boundary of some 2-chain $\sum_i n_i\sigma_i$. By writing simplices several times if necessary, we can assume $n_i = \pm 1$. Let $\tau_{i0}, \tau_{i1}, \tau_{i2}$ be the three boundary simplices of σ_i. Then one has $\partial\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$. and so

$$\gamma = \partial \sum_i n_i\sigma_i = \sum_i \sum_{j=0}^{2} (-1)^j n_i \tau_{ij}.$$

As γ is just one simplex, this means that the τ_{ij} can be ordered in pairs which cancel each other plus one τ_{ij}, which equals γ. Consider an edge

$\tau = \tau_{ij}$, which has x_0 for an endpoint, but not both ends equal to x_0. One can deform this to a constant path x_0 and this deformation extends to the 2-simplices σ_i, in which τ occurs in the boundary. One repeats this procedure until all edges τ_{ij} have both ends in x_0, hence can be viewed as elements of Γ. Note that γ was not changed in this procedure. The equation $\gamma = \sum_{i,j}(-1)^j n_i \tau_{ij}$ in $C_1(X)$ implies that in $\pi_1(X)$, γ is a product of the $\tau_{i,j}^{\pm 1}$ in some order (by a homotopy given via the simplices σ_i). Since the $\tau_{i,j}$ cancel pairwise in $C_1(X)$, modulo $[\Gamma, \Gamma]$ we can move each $\tau_{i,j}$ in $\pi_1(X)$ next to its inverse, so it cancels. This goes for all $\tau_{i,j}$, hence γ is trivial in $\pi_1(X)_{\mathrm{ab}}$.

Finally let X be path-connected. We need to show that ψ is surjective. Let $\sum_i n_i \tau_i$ be a 1-cycle, representing an element h of $H_1(X)$. Again we can assume $n_i = \pm 1$. Since $\check{\tau}_i = -\tau_i$ in $H_1(X)$, one can even assume $n_i = 1$. If some τ_i is not a closed path, then, since $\partial(\sum_i \tau_i) = 0$ there must exist some j, such that $\tau_i.\tau_j$ is defined. We collect these two and iterate this until all paths are closed. As X is path-connected, there is a path γ_i, that runs from x_0 to the end point of τ_i. Because of $\tau_i \sim \gamma_i.\tau_i.\check{\gamma}_i$, we can assume, that all paths start and end in x_0. By composing, these can be turned into one single path τ. It follows that $h = \psi(\tau)$ and therefore ψ is surjective. $\square$