Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 13

hand in by: 2.2.2026 12:00

Four points for each problem.

1. Let (X, A) be a pair of spaces. Show:
   (a) $H_0(Z, A) = 0 \Leftrightarrow A$ meets every path-component of Z.
   (b) $H_1(Z, A) = 0 \Leftrightarrow$ the map $H_1(A) \to H_1(Z)$ induced by inclusion is surjective and and every path-component of Z contains at most one path-component of A.

2. Let $x \neq y$ be points of S^2 and let $A = \{x, y\}$. Compute $H_p(S^2/A)$ for every $p \geq 0$.

3. Let $A \in \mathrm{GL}_{n+1}(\mathbb{R})$ and let $f_A : S^n \to S^n$ be given by
$$f_A(x) = \frac{1}{\|Ax\|} Ax.$$
   Show that $\deg(f_A)$ is ± 1 depending on whether the determinant of A is positive or negative.
   (Use multiplicativity of the degree to reduce to the case of elementary matrices. Then use homotopy invariance to reduce to the case of diagonal matrices.)

4. A polynomial map $f : \mathbb{C} \to \mathbb{C}$ can always be extended to a continuous map on $\widehat{\mathbb{C}} \cong S^2$. (If f is constant, extend it by the same constant, otherwise $f(z)$ tends to infinity as $z \to \infty$.)
   Show that the mapping degree equals its degree as a polynomial.

Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 12

hand in by: 26.1.2026 12:00

Four points for each problem.

1. Let $T = S^1 \times S^1$ the torus. Fix a point $x_0 \in S^1$ and let $A \subset T$ be the set
$$A = \{x_0\} \times S^1 \ \cup\ S^1 \times \{x_0\}.$$
Compute the homology groups of $X = T/A$.

2. Let X, Y be path-connected. We say that a continuous map $f : X \to Y$ *induces the trivial map on homology*, if f_* is zero on $H_p(X)$ for every $p \geq 1$.
Prove that the Hopf map $\eta : S^3 \to S^2$ induces the trivial homomorphism in homology, but a nontrivial in homotopy.

3. Show that every map $f : S^n \to S^n$, $n \in \mathbb{N}$ can be homotoped to have a fixed point.

4. Show that the spaces $X = S^2 \times \mathbb{P}^3(\mathbb{R})$ and $Y = S^3 \times \mathbb{P}^2(\mathbb{R})$ have isomorphic homotopy groups, but nonisomorphic homology groups.

Algebraic Topology **Sheet 11**

Winter 2025/26

Professor Deitmar, Giacomo Gavelli hand in by: 19.1.2026 12:00

Four points for each problem.

1. Let X and Y be path-connected spaces. Fix $(x_0, y_0) \in X \times Y$ and let $\eta = \eta_{y_0} : X \to X \times Y$, $x \mapsto (x, y_0)$. Likewise, let $\tau : Y \to X \times Y$, $y \mapsto (x_0, y)$. Show that the map
$$\begin{aligned} f : H_1(X) \oplus H_1(Y) &\to H_1(X \times Y), \\ \alpha \oplus \beta &\mapsto \eta_*(\alpha) + \tau_*(\beta) \end{aligned}$$
is an isomorphism.

2. Consider a chain complex $\mathcal{C}$ given by
$$0 \to C_1 \xrightarrow{d} C_0 \to 0$$
with $C_1 = \mathbb{Z}^m = C_0$ for some $m \in \mathbb{N}$.hen d is given by some integer matrix in $\mathrm{M}_n(\mathbb{Z})$. We assume that this matrix has full rank m. This implies that the homology of the complex $H_1 = 0$ and $H_0 = C_0/\operatorname{im}(d)$ is a finite group.
Let $d^* : \mathbb{C}^m \to \mathbb{C}^m$ be the adjoint of d, i.e., it is given by the adjoint matrix. Let $\Delta = dd^* : \mathbb{C}^m \to \mathbb{C}^m$.
Show that
$$\det(\Delta) = |H_0|^2.$$
(Use the Elementary Divisor Theorem for integer matrices.)

3. Let X be path-connected and let ΣX be the suspension. Show that
$$H_{p+1}(\Sigma X) \cong H_p(X)$$
holds for every $p \geq 1$.

4. Let $x \in \mathbb{R}^2/\mathbb{Z}^2$. Compute $H_p\big(\mathbb{R}^2/\mathbb{Z}^2 - \{x\}\big)$ for every $p \geq 0$.
(You may use that the space can be retracted to a bouquet of two circles.)

Algebraic Topology | **Sheet 10**

Winter 2025/26

Professor Deitmar, Giacomo Gavelli | hand in by: 12.1.2026 12:00

Four points for each problem.

1. Let $n \in \mathbb{N}$ be even. Show that for every continuous map $f : S^n \to S^n$ there exists $x \in S^n$ with $f(x) = \pm x$. (Hint: Use the Hairy Ball Theorem.)

2. Show that chain homotopy is an equivalence relation.

3. Let
$$\dots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \dots$$
be a chain complex. Put $D_p = C_p \oplus C_{p+1}$ and define $d_p : D_p \to D_{p-1}$ by the matrix $\left(\begin{smallmatrix} \partial & \\ (-1)^p & \partial \end{smallmatrix}\right)$. Show that $(D_\bullet, d_\bullet)$ is a complex and that it's homology is zero.

4. A chain complex $\mathcal{C} = (C_k, d_k)_{k\in\mathbb{Z}}$ is called *contractible*, if the identity map $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$ is nullhomotopic. A complex is called *acyclic*, if $H_p(\mathcal{C}) = 0$ holds for every $p \in \mathbb{Z}$. It is clear that every contractible complex is acyclic. Show that the complex
$$\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2 \to 0 \to 0 \to \dots$$
is acyclic, but not contractible, where p is the natural projection.

Algebraic Topology | **Sheet 9**
Winter 2025/26
Professor Deitmar, Giacomo Gavelli | hand in by: 15.12.12:00

Four points for each problem.

1. (Euler characteristic) Let $(E_k, \partial_k)_{k\in\mathbb{Z}}$ be a chain complex, where the E_k are finite-dimensional $\mathbb{Q}$-vector spaces and only finitely many of them are $\neq 0$. Also, the boundary operators ∂_k are supposed to be $\mathbb{Q}$-linear. Let H_k be the homology of the complex. Show that
$$\sum_{k\in\mathbb{Z}}(-1)^k \dim E_k = \sum_{k\in\mathbb{Z}}(-1)^k \dim H_k.$$

2. (Simplicial cone) Let $X = (V, \mathcal{S})$ be a simplicial complex. Let $V' = V \cup \{x_0\}$ for a new vertex x_0 and define $\mathcal{S}' \subset \mathcal{P}(V')$ by
$$E \in \mathcal{S}' \quad \Leftrightarrow \quad E \cap V \in \mathcal{S} \text{ or } E = \{x_0\}.$$
Show that $X' = (V', \mathcal{S}')$ is a simplicial complex and show that its homology groups satisfy
$$H_n(X') = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. For an abelian group A and a natural number k we write A/k for the group A/kA. Determine whether there exists a short exact sequence $0 \to \mathbb{Z}/4 \to \left(\mathbb{Z}/8 \times \mathbb{Z}/2\right) \to \mathbb{Z}/4 \to 0$.

4. (a) Let $\mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^n$ be $\mathbb{Z}$-linear maps such that $BA = 0$. Suppose that $k \leq m$ and the matrix of A can be extended to a matrix in $\mathrm{GL}_m(\mathbb{Z})$. Show that the abelian group $\ker(B)/\operatorname{im}(A)$ is torsion-free.
   (b) Let $X = I \times I$ and let $A = \left(\{0\} \times I\right) \cup \left(\{1\} \times I\right)$ Equip $Y = X/A$ with a simplicial structure and compute its simplicial homology.

Algebraic Topology **Sheet 8**
Winter 2025/26
Professor Deitmar, Giacomo Gavelli hand in by: 8.12.12:00

Four points for each problem.

1. Let G be a group acting on a set S. An element $g \in G$ is said to *act freely* on S if $gs \neq s$ holds for all $s \in S$.
   Let $\pi : E \to X$ be a finite galois covering. Then the group $D(\pi)$ of deck transformations acts on the set of isomorphy classes of intermediate coverings $E \xrightarrow{\alpha} F \xrightarrow{\beta} X$ by $d : \alpha \to \alpha \circ d^{-1}$. Show that for any orbit S of at least 2 elements there exists a deck transformation d acting freely on S.

2. (a) Show that if A is contractible, then $\pi_n(X, A, a_0) \cong \pi_n(X, a_0)$.
   (b) Show that if X is contractible, then $\pi_{n+1}(X, A, a_0) \cong \pi_n(A, a_0)$.

3. (a) Let $f, g : X \to Y$ be homotopic. Show that for $f_* \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ and likewise for g, there is a commutative diagram

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d, equal] & \pi_n(Y, f(x_0)) \arrow[d, "\cong"] \\ \pi_n(X, x_0) \arrow[r, "g_*"'] & \pi_n(Y, g(x_0)) \end{tikzcd}$$

   (b) Show that a homotopy equivalence $f : X \to Y$ on path-connected spaces induces isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for all $n \in \mathbb{N}$.

4. Show that for a cell complex X, a vertex $x_0 \in X_0$ and any $n \in \mathbb{N}$ one has

$$\pi_n(X, x_0) \cong \lim_{k \to \infty} \pi_n(X_k, x_0).$$

Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 7

hand in by: 1.12.12:00

Four points for each problem.

1. Let X and Y be spaces that admit universal covers $\tilde{X}, \tilde{Y}$. Let $f : X \to Y$ be a map and let $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ be the group homomorphism given by $f_*\gamma = f \circ \gamma$.

   (a) Pick pre-images $\tilde{p} \in \tilde{X}$ and $\tilde{q} \in \tilde{Y}$ of p and $q = f(p)$ Further let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the unique lift sending $\tilde{p}$ to $\tilde{q}$. Use the base-points $\tilde{p}$ and $\tilde{q}$ to identify the fundamental groups with the groups of deck transformations. Then show that for any deck transformation d on $\tilde{X}$ we have
   $$\tilde{f}(d(z)) = (f_*d)(\tilde{f}(z))$$
   for every $z \in \tilde{X}$.

   (b) Now suppose that $\tilde{Y}$ is contractible and f_* is the trivial group homomorphism. Show that f is nullhomotopic.

2. Define the *Moebius Band* as $I \times [-1, 1]/\sim$, where $\sim$ is the equivalence relation generated by
   $$(0, t) \sim (1, -t), \quad t \in [-1, 1].$$
   Construct a 2-dimensional cell complex which retracts to $I \times S^1$ and at the same time retracts to the Moebius Band.

3. Let $p : \tilde{X} \to X$ be the universal cover of a path-connected space X. Show that under the isomorphism $\pi_n(X) \cong \pi_n(\tilde{X})$, which holds for $n \geq 2$, the action of $\pi_1(X)$ on $\pi_n(X)$ corresponds to the action of $\pi_1(X)$ on $\pi_n(\tilde{X})$ induced by the action of $\pi_1(X)$ on $\tilde{X}$ as deck transformations.

   More precisely, let $e \in \tilde{X}$ and let $x_0 = p(e)$. For $\gamma \in \pi_1(X, x_0)$ let d_γ be the deck transformation attached to γ, i.e., the unique deck transformation with
   $$d_\gamma(e) = \tilde{\gamma}_e(1).$$
   Now let γ act on $\pi_n(\tilde{X}, e)$ by
   $$D_\gamma[f] = \left[(d_\gamma \circ f)^{\sigma^{-1}}\right],$$
   where σ is any path in $\tilde{X}$ connecting e to $d_\gamma(e)$ and g^σ for a spheroid g is defined as in the proof of Proposition 4.1.6. Then show that
   $$p_*D_\gamma[f] = \left(p_*[f]\right)^{\gamma^{-1}}.$$

4. Show that if $A \subset X$ is a deformation retract and $a_0 \in A$, then the inclusion $A \to X$ induces Isomorphisms $\pi_n(A, a_0) \to \pi_n(X, a_0)$.

Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 6

hand in by: 24.11.12:00

Four points for each problem.

1. Let $X = \mathbb{T}^2$ and let $E \to X$ be a finite connected covering. Show that E is homeomorphic to X.

2. Let $\pi : E \to X$ be a covering of connected and locally simply connected spaces. A diagram of coverings

$$\begin{tikzcd} E \arrow[dd, "\pi"'] \arrow[dr, "\alpha"] & \\ & F \arrow[dl, "\tau"] \\ X & \end{tikzcd}$$

is called an *intermediate covering*. Two intermediate coverings

$$\begin{tikzcd} E \arrow[dd, "\pi"'] \arrow[dr, "\alpha"] & \\ & F \arrow[dl, "\tau"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[dd, "\pi"'] \arrow[dr, "\alpha_1"] & \\ & F_1 \arrow[dl, "\tau_1"] \\ X & \end{tikzcd}$$

are called *isomorphic*, if there exists a homeomorphism $f : F \to F_1$ such that the diagram

$$\begin{tikzcd} & E \arrow[dl, "\alpha"'] \arrow[dr, "\alpha_1"] & \\ F \arrow[rr, "f"] \arrow[dr, "\tau"'] & & F_1 \arrow[dl, "\tau_1"] \\ & X & \end{tikzcd}$$

commutes

Show that if π is galois, then the map $\phi : H \mapsto H\backslash E$ is a bijection between the set of subgroups H of $G = \mathrm{D}(\pi)$ and the set of intermediate coverings up to isomorphy.

3. Let X be a path connected and locally simply connected space with finite fundamental group. Show that every continuous map $f : X \to S^1$ is nullhomotopic.

4. (a) Let X be a cell complex and γ a path in X with endpoints in X_1. Show that γ is homotopic with fixed ends to a path in X_1.
   (b) Show that a cell complex X is path-connected iff its 1-skeleton X_1 is and that in this case the homomorphism $\pi_1(X_1, x_0) \to \pi_1(X, x_0)$ is surjective for any $x_0 \in X_1$.
   (c) Show that any cell complex structure on the space $X = \mathbb{R}^2/\mathbb{Z}^2$ has at least 4 cells.

Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 5

hand in by: 17.11.12:00

Four points for each problem.

1. Let X be the following subset of $\mathbb{C}$:
$$X = \left(\mathbb{R} + i\mathbb{Q}\right) \cup i\mathbb{R}.$$
Show that the set $\left\{z\right\} \in X$ is a deformation retract iff $z \in i\mathbb{R}$.

2. Show that a graph can only be simply connected, if it is contractible.

3. (a) Show that for a bouquet of circles B the universal covering is a contractible graph and the fundamental group is a free group.
   (b) Show that for every group Γ there exists a connected covering $\pi : E \to X$ of graphs, such that Γ is isomorphic to the group of deck transformations $\mathrm{D}(\pi)$.

4. Let X be a path-connected space and $x_0 \in X$ a point. Let $\Gamma = \pi_1(X, x_0)$ be the fundamental group and let $S \subset \Gamma$ be a generating set. For each $[\gamma] \in S$ choose a representative $\gamma : [0,1] \to X$. Since $\gamma(0) = \gamma(1)$, one can γ consider as a continuous map $S^1 \to X$. Use this map, to attach a 2-cell to X. Let Y be the space that ensues from X by attaching all these 2-cells. Show that Y is simply connected.

   Hence, by attaching 2-cells, one can kill fundamental groups.

Algebraic Topology **Sheet 4**

Winter 2025/26

Professor Deitmar, Giacomo Gavelli hand in by: 10.11.12:00

Four points for each problem.

1. Show that for a path-connected space X the following are equivalent:

   (a) Every continuous map $S^1 \to X$ is freely homotopic to a constant map.

   (b) Every continuous map $S^1 \to X$ can be extended to a continuous map $D^2 \to X$, where $D^2 = \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\}$.

   (c) X is simply connected.

2. Let X be connected and locally simply connected and let $\pi : E \to X$ be a connected covering.

   Let $\mathrm{D}(\pi)$ denote the group of deck transformations. We say that the covering is a *Galois covering*, if the induced continuous map
   $$\phi : \mathrm{D}(\pi)\backslash E \to X$$
   is bijective (i.e., a homeomorphism).

   Then one has $E \cong \Sigma\backslash\tilde{X}$ for some group of deck transformations $\Sigma \subset \mathrm{D}(p)$ of the universal covering $p : \tilde{X} \to X$. We say that the covering π is *normal*, if the group Σ is a normal subgroup of $\mathrm{D}(p)$.

   Show that π is galois if and only if it is normal.

   Show that in this case one has a canonical isomorphism $\mathrm{D}(\pi) \cong \mathrm{D}(p)/\Sigma$. The following display illustrates this situation where we have written $\Gamma = \mathrm{D}(p)$.

$$\begin{tikzcd} \tilde{X} \arrow[dd, "p"'] \arrow[dr, "\phi"] & \\ & E \arrow[dl, "\pi"] \\ X & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \tilde{X} \arrow[dd, dash, "\Gamma"'] \arrow[dr, dash, "\Sigma"] & \\ & E \arrow[dl, dash, "\Gamma/\Sigma"] \\ X & \end{tikzcd}$$

3. (a) Let L_1, L_1 two disjoint lines in $\mathbb{R}^3$ and let $X = \mathbb{R}^3 \smallsetminus (L_1 \cup L_2)$. Compute $\pi_1(X)$.

   (b) Let $Z = \mathbb{R}^3 \smallsetminus S^1$, where S^1 stands for a circle embedded into $\mathbb{R}^3$. Compute $\pi_1(Y)$.

4. Let $S_1, \dots, S_n$ be disjoint copies of the circle S^1. In each S_j fix a point $a_j \in S_j$ and let
   $$X_n = \left(S_1 \cup S_2 \cup \cdots \cup S_n\right)\Big/\left\{a_1, \dots, a_n\right\}.$$
   This space is called a *bouquet of circles*. The collapsed set $\{a_1, \dots, a_n\}$ forms a single point b in X_n, called the *base point*. Show that the fundamental group of a bouquet of n circles is a free group in n generators.

Algebraic Topology
Winter 2025/26
Professor Deitmar, Giacomo Gavelli

Sheet 3

hand in by: 3.11.12:00

Four points for each problem.

1. Show that the fundamental group of the Hawaian Earring is uncountable.

2. For any two spaces Y, Z let $[Y, Z]$ denote the set of free homotopy classes of continuous maps $f : Y \to Z$.

   Let X be a path-connected space. Show that the canonical map $\phi : \pi_1(X, x_0) \to [S^1, X]$, $[\gamma] \mapsto [e^{2\pi it} \mapsto \gamma(t)]$ induces a bijection between the set of conjugacy classes of $\pi_1(X, x_0)$ and $[S^1, X]$.

3. A topological space X is called *contractible*, if the identity map is homotopic to a constant map, i.e., if there exists a point $x_0 \in X$ and a continuous map $f : [0, 1] \times X \to X$ with

   $$\begin{aligned} f(0, x) &= x, \\ f(1, x) &= x_0 \end{aligned}$$

   for every $x \in X$.

   Show that every contractible space is simply connected.
   (Maybe it is possible to apply the last problem?)

4. Let X, Y, Z be connected and locally simply-connected spaces and let the commutative diagram of continuous maps be given:

   $$\begin{tikzcd} Y \arrow[dd, "p"'] \arrow[dr, "r"] & \\ & Z \arrow[dl, "q"] \\ X & \end{tikzcd}$$

   Suppose that p is a covering. Which of the following is true?

   (a) If r is a covering, then so is q.

   (b) If q is a covering, then so is r.

Algebraic Topology **Sheet 2**
Winter 2025/26
Professor Deitmar, Giacomo Gavelli hand in by: 27.10.12:00

Four points for each problem.

1. Show that the space $\mathbb{D}/\mathbb{T}$ is homeomorphic to S^2, where $\mathbb{D}$ is the closed unit disk in $\mathbb{C}$.

2. Let X be a topological space. The *suspension* of X is the space
$$\Sigma X := \Big([-1,1]\times X\Big)\Big/\Big[(-1,x)\sim(-1,y),\ \ (1,x)\sim(1,y),\ x,y\in X\Big].$$
Show that $\Sigma S^n \cong S^{n+1}$ holds for every $n\in\mathbb{N}_0$, where $\cong$ means homeomorphy.

3. Let $(X_j)_{j\in J}$ be a family of topological spaces.
   (a) Show that there is a smallest topology on $X=\prod_{j\in J}X_j$ such that all projections $p_j : X\to X_j$ are continuous. This is called the *product topology*.
   (b) Show that a set is open in the product topology iff it is a union of sets of the form
   $$\prod_{j\in E}U_j\ \times\ \prod_{j\notin E}X_j,$$
   where $E\subset J$ is a finite subset and for every $j\in E$, the set U_j is an open subset of X_j.
   (c) Let $y\in X$. Show that the set
   $$D=\Big\{x\in X : x_j=y_j \text{ for all but finitely many } j\Big\}$$
   is dense in X.

4. (a) Let $\alpha : Z\to X$ and $\beta : Z\to Y$ be continuous maps. Show that there exists a space P and continuous maps $p : X\to P$ and $q : Y\to P$ making the diagram
   $$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "p"] \\ Y \arrow[r, "q"'] & P \end{tikzcd}$$
   commutative in a universal way, i.e., such that for every other commutative diagram
   $$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "r"] \\ Y \arrow[r, "s"'] & T \end{tikzcd} \qquad (*)$$
   of continuous maps there exists a unique continuous map $\eta : P\to T$ such that the diagram
   $$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "r"] \arrow[ddr, "p"] & \\ Y \arrow[r, "s"] \arrow[drr, "q"'] & T & \\ & & P \arrow[ul, dotted] \end{tikzcd} \qquad (**)$$
   commutes, where the dotted arrow is η.
   (Hint: define P to be the disjoint union of X and Y modulo the equivalence generated by $\alpha(z)\sim\beta(z)$, $z\in Z$.)
   (b) Show that the universal property of P determines P up to homeomorphy. More precisely show that if P' is another space with maps p', q' as above and the property that for every commutative diagram $(*)$ there exists a unique $\eta' : P'\to T$ making the diagram $(**)$ commutative (with P, p, q, η replaced by P', p', q', η'), then P is homeomorphic to P'.

   P is called the *pushout* and is written as $P = X\sqcup_Z Y$, if the maps α and β are understood.

Algebraic Topology **Sheet 1**

Winter 2025/26

Professor Deitmar, Giacomo Gavelli hand in by: 20.10.12:00

For points for each exercise.

1. Show that a topological space X is a Hausdorff space if and only if the diagonal
$$\Delta = \Big\{(x,x) : x \in X\Big\} \subset X \times X$$
is closed in $X \times X$.

2. Let X be a set. For a subset $A \subset X$, write $A^c = X \smallsetminus A$. Show that
$$\mathcal{U} = \Big\{U \subset X : U^c \text{ is finite}\Big\} \cup \Big\{\emptyset\Big\}$$
is a topology, called the *cofinite topology*. Show that this topology is hausdorff if and only if X is finite. Show further, that every subset of X is compact.

3. Let $X = \mathbb{R} \times \mathbb{R}$ with the product topology, where the first factor carries the usual topology of $\mathbb{R}$, whereas the second factor is equipped with the discrete topology. Let
$$Y_1 = \Big\{(a,a) \in X : a \in \mathbb{R}\Big\} \qquad\qquad Y_2 = \Big\{(a,-a) \in X : a \in \mathbb{R}\Big\}$$
The real vector space X is a direct sum of the subspaces Y_1 and Y_2, so X equals $Y_1 \oplus Y_2 \cong Y_1 \times Y_2$. Determine the subspace topology on Y_1, Y_2. Is the topology on X equal to the product topology of $Y_1 \times Y_2$?

4. Let $f : X \to Y$ be a continuous bijection, where X is compact, and Y is a Hausdorff space. Show that f is a homeomorphism.