Lie groups

Contents

1 Lie groups

1.1 Manifolds

Definition 1.1.1. A topological space X is **locally euclidean** of dimension $n \in \mathbb{N}$, if for every $x \in X$ there exists an open neighbourhood U of x, which is homeomorphic to an open subset of $\mathbb{R}^n$.

In that case, a homeomorphism $\phi : U \to W \subset \mathbb{R}^n$, where U is a neighbourhood of x, is called a **chart**.

Definition 1.1.2. A topological space X is called a **manifold** of dimension n if it is locally euclidian of dimension n and

(a) a Hausdorff space, as well as

(b) **separable**, i.e., it contains a countable dense subset.

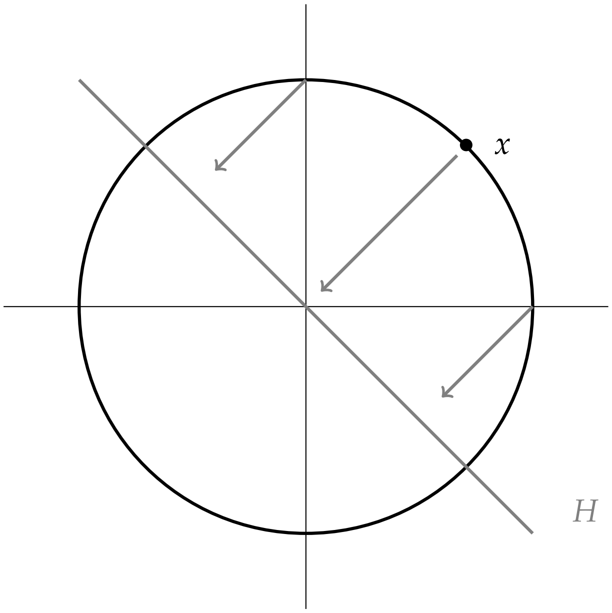
Examples 1.1.3.

(a) The space $\mathbb{R}^n$ is an n-dimensional manifold.

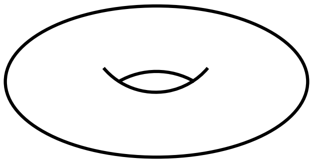
(b) The n**-sphere**

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \right\}$$

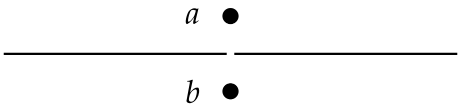
is a manifold of dimension n, where $\|x\| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. Charts are constructed as follows: For $x \in S^n$ let $p : \mathbb{R}^{n+1} \to H$ be the orthogonal projection onto $H = \{x\}^\perp$. The set H is homeomorphic to $\mathbb{R}^n$ and p induces a homeomorphism of a neighbourhood of x to an open subset of H.



(c) The **torus** $T_2 \subset \mathbb{R}^3$ is homeomorphic to $(S^1) \times (S^1)$. One can also define it as the set of all $x \in \mathbb{R}^3$, which have distance 1 to the circle $2S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$.



(d) The Hausdorff property does not follow from the other axioms, as the following example shows: Let $X = \mathbb{R}^\times \sqcup \{a, b\}$, where a, b are two new points.



We define a topology on X: A set $U \subset X$ is open if and only if

(a) $U \cap \mathbb{R}^\times$ is open in $\mathbb{R}^\times$ and

(b) if a or b lies in U, then there exists $\varepsilon > 0$, such that $(-\varepsilon, \varepsilon) \setminus \{0\}$ is contained in U.

The points a and b cannot be separated by open neighbourhoods. The space X is second countable and locally euclidean, but not Hausdorff, hence not a manifold.

(e) The separability doesn't follw from the other axioms, either. A counterexample is the **long line**. To construct it, let Ω be the class of ordinal numbers. For every ordinal number α we construct a linearly ordered set $L(\alpha)$ by the inductive rules

$$\begin{aligned} L(0) &= (0,1), \\ L(\alpha+1) &= L(\alpha) \sqcup [0,1), \\ L(\lambda) &= \bigcup_{\alpha<\lambda} L(\alpha). \end{aligned}$$

In the second step we have written $[0,1)$ for a new copy of the interval which we put to the right of the ordered set. Further, the symbol λ denotes an arbitrary liwith ordinal. Let λ_0 be the smallest uncountable ordinal. We equip $L = L(\eta)$ with the topology generated by all open intervals

$$(a,b) = \Big\{x \in L : a < x < b\Big\}, \quad a,b \in L.$$

Then L is locally euclidean, but not separable.

Definition 1.1.4. Let M be a manifold. Let (U,ϕ) and (V,ψ) be charts. Then $\eta := \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a homeomorphism between some open sets in $\mathbb{R}^n$.

$$\begin{tikzcd} & & U \cap V \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & \\ \mathbb{R}^n \supset & \phi(U \cap V) \arrow[rr, dotted, "\eta"'] & & \psi(U \cap V) & \subset \mathbb{R}^n \end{tikzcd}$$

Every such map is called an **coordinate change** map.

Definition 1.1.5. Let M be a manifold. An **atlas** is a set of charts $\mathcal{A}$, such that $\bigcup_{(U,\phi)\in\mathcal{A}} U = M$.
An atlas $\mathcal{A}$ is called **smooth**, if all coordinate change maps are C^∞ maps.

Examples 1.1.6.

(a) Smooth atlases are given in Example 1.1.3

(b) An example of a non-smooth altas is easy to give: Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism, which is not smooth. Then ψ and the identity map make an atlas which is not smooth.

Definition 1.1.7. Let $\mathcal{A}$ be a smooth atlas on M. A given chart (U,ϕ) of M, not necessarily in $\mathcal{A}$, is called **compatoble** with $\mathcal{A}$, if all coordinate changes $\phi \circ \psi^{-1}$, $\psi \in \mathcal{A}$ are smooth. This means that $\mathcal{A} \cup \{\phi\}$ is a smooth atlas as well.

A smooth atlas $\mathcal{A}$ is called a **differentiable structure**, if it is maximal, i.e., if every compatible chart already belongs to $\mathcal{A}$.

Lemma 1.1.8. *For every smooth atlas $\mathcal{A}$ there is exactly one differentiable structure containing $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a smooth atlas on M and let $\mathcal{M}$ be the set of all charts, which are compatible to $\mathcal{A}$. We claim that $\mathcal{M}$ is a maximal smooth atlas. Since $\mathcal{A}$ is smooth, one gets $\mathcal{A} \subset \mathcal{M}$ and therefore is $\mathcal{M}$ is an atlas. To show smoothness, consider two charts (U, ϕ) and (V, ψ) in $\mathcal{M}$. We have to show, that $\psi \circ \phi^{-1}$ is mooth on $\phi(U \cap V)$. Let $x \in U \cap V$ and let (W, η) be a chart around x in the atlas $\mathcal{A}$. On $\phi(U \cap V \cap W)$ one has

$$\psi \circ \phi^{-1} = \left(\psi \circ \eta^{-1}\right) \circ \left(\eta \circ \phi^{-1}\right).$$

This is a komposition of smooth maps, hence smooth. THis means that $\psi \circ \phi^{-1}$ is smooth in a neighbourhood of $\phi(x)$, hence it is a smooth map.

This atlas is maximally smooth, since aver chart, which is compatible with $\mathcal{M}$, is compatioble with $\mathcal{A}$ and so belongs to $\mathcal{M}$. Finally, $\mathcal{M}$ is uniquely determined, as it contains every smooth atlas, which contains $\mathcal{A}$. □

Definition 1.1.9. (a) A **differentiable manifold** is a pair $(M, \mathcal{A})$, consisting of a manifold M and a differentiable structure $\mathcal{A}$ on M. The charts of $\mathcal{A}$ are called the **smooth charts** of M.

(b) A function $f : M \to \mathbb{C}$ is called **smooth**, if for every smooth chart ϕ the function $f \circ \phi^{-1}$ is smooth (where defined). We write $C^\infty(M)$ for the set of all smooth functions $f : M \to \mathbb{C}$.

(c) A map map $f : M \to N$ between smooth manifolds is called a **smooth map**, if $\psi \circ f \circ \phi^{-1}$ is smooth for every choice of smooth charts ϕ and ψ of M and N respectively.

Definition 1.1.10. Let M be a smooth manifold of dimension n. A **smooth submanifold** of dimension $k \le n$ is a subset $S \subset M$ with a family of smooth charts $(U_i, \phi_i)_{i \in I}$, such that $S \subset \bigcup_{i \in I} U_i$ and for every $i \in I$ one has

$$S \cap U_i = \phi_i^{-1}\left(\mathbb{R}^k \times \{0\}\right).$$

These charts induce a differentiable structure on S, such that the inclusion $S \hookrightarrow M$ is a smooth map.

* * *

1.2 Fundamental groups

We shall first pick up some notation about paths. Details can be found in the lecture on Copmplex Analysis.

Definition 1.2.1. A **path** in a topological space X is a continuous map $\gamma\colon [0,1] \to X$. One can compose paths, so $\gamma.\tau$ shall denote the path, which runs through γ first and then through η. Further, $\check{\gamma}$ is the reverse path to γ.

Definition 1.2.2. Let X, Y be topological spaces. Two continuous maps $f, g\colon X \to Y$ are called **freely homotopic**, if there is a continuous map $h\colon I \times X \to Y$, such that for every $x \in X$ one has $h(0,x) = f(x)$ and $h(1,x) = g(x)$.

This means that two maps are freely homotopic if one can be deformed continuously into the other. Every map h as above is called **homotopy** from f to g.

For two paths, γ and η, one can also fix the endpoints through the homotopy to get a **homotopy with fixed ends**.

(a) Homotopy with fixed ends is an equivalence relation on the set of paths in X. We write this relation as "$\simeq$".

(b) If $\gamma \simeq \gamma'$ and $\eta \simeq \eta'$ and further $\gamma(1) = \eta(0)$, then we have $\gamma.\eta \simeq \gamma'.\eta'$.

(c) If γ, η, τ are paths in X with $\gamma(1) = \eta(0)$ and $\eta(1) = \tau(0)$, then one has

$$(\gamma.\eta).\tau \ \simeq \ \gamma.(\eta.\tau).$$

(d) If c is a constant path $c(t) = p$, then one has

$$\eta.c \simeq \eta \quad \text{und} \quad c.\tau \simeq \tau$$

for all paths η and τ with $\eta(1) = p = \tau(0)$.

(e) If γ is an arbitrary path in X, then $\gamma.\check{\gamma}$ is **nullhomotopic**, i.e., it is homotopic with fixed ends to a constant path.

Definition 1.2.3. If x_0 is a point in X and $G(x_0)$ is the set of all closed paths in X with end-point x_0. Further let

$$\pi_1(X, x_0) = G(x_0)/\simeq$$

be the set of homotopy-classes (with fixed ends) in the endpoint x_0. The remarks above imply that the composition

$$[\gamma][\eta] \coloneqq [\gamma.\eta]$$

on $\pi_1(X, x_0)$ is well-defined and turns $\pi_1(X, x_0)$ into a group. The neutral element is the class of the constant path and the inverse of a class $[\gamma]$ is $[\check{\gamma}]$. This group is called the **fundamental group** of X.

Definition 1.2.4. A space X is called **path-connected**, if any two points can be connected by a path, i.e., for any two $x, y \in X$ there exists a path γ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 1.2.5. *A path-connected space is* **connected**, *i.e., If $X = U \sqcup V$ for two open sets U, V, then either $U = \emptyset$ or $V = \emptyset$.*

Proof. Analysis 2. □

Lemma 1.2.6. *Let X be a path-connected space. Then for any two $x_0, x_1 \in X$, the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Proof. Let γ be a path from x_0 to x_1. Then the map $\phi_\gamma \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$, given by

$$\phi_\gamma(\eta) = \check{\gamma}.\eta.\gamma$$

is an isomorphism of groups. The inverse map is $\phi_{\check{\gamma}}$. □

Notation. If the base-point x_0 is fixed, or if X is path-connected, we shall often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Note that in the path-connected case, the group does not depend upon the choice of a base point. So, as long as assertions about the group structure are concerned, like $\pi_1(X) \cong \mathbb{Z}$, the base-point doesn't play a role anyway, so it can very well be left out of the notation.

Example 1.2.7. Consider the space $X = \mathbb{T} = \left\{ z \in \mathbb{C} : |z| = 1 \right\} \cong S^1$. Then we have

$$\pi_1(X) \ \cong\ \mathbb{Z}.$$

For $z = 1$ an isomorphism $\mathbb{Z} \to \pi_1(X, 1)$ is given by

$$k \ \mapsto \ [\gamma_k],$$

where $\gamma_k(t) = e^{2\pi i k t}$.

We will only sketch a proof, as this assertion follows easily from coming sections. Consider the map $\pi\colon \mathbb{R} \to \mathbb{T}; t \mapsto e^{2\pi i t}$. A continuous map $\gamma\colon I \to \mathbb{T}$ with $\gamma(0) = \gamma(1) = 1$ can, in a unique way, be lifted to a continuous map $\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$, such that $\gamma = \pi \circ \tilde{\gamma}$. The map $\gamma \mapsto \tilde{\gamma}(1)$ is an inverse to $k \mapsto [\gamma_k]$.

Definition 1.2.8. A space $X \neq \emptyset$ is called **simply-connected**, if

- X is path connected and
- the fundamental group $\pi_1(X)$ is trivial.

Definition 1.2.9. A subset $S \subset \mathbb{R}^n$ is called **star shaped** if there exists a point $s_0 \in S$ such that for any point $s \in S$ the line segment

$$[s_0, s] = \left\{(1-t)s_0 + ts : t \in I\right\}$$

is contained in S. Any such s_0 is called a **central point** of S. This in particular means that S is path connected.

Lemma 1.2.10. *Let $n \in \mathbb{N}$. Any star shaped subset of $\mathbb{R}^n$ is simply-connected. If $n \geq 2$, then $S^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$ is simply-connected.*

Proof. Let $S \subset \mathbb{R}^n$ be star shaped. Translating S we can assume that $s_0 = 0$ is a central point. Let $\gamma : I \to S$ be continuous with $\gamma(0) = \gamma(1) = 0$. Then $h(s,t) = (1-s)\gamma(t)$ is a homotopy with fixed ends from γ to the constant path 0.

For S^n, let N be the north pole $N = (1, 0, \ldots, 0)^t$ in S^n. WLOG we assume $p \neq N$. Every closed path γ with endpoint p is homotopic to a path avoiding N: To see this, let B be a small open ball around N, such that $p \notin \overline{B}$. Then $\gamma^{-1}(B)$ is an open subset of the interval $(0, 1)$. Hence it is a disjoint union of its connected components, which are open intervals. Let $J = (a, b)$ be one such component. This means that $\gamma(t)$ enters B at $t = a$ and leaves it at $t = b$. The closure $\overline{B}$ is simply connected, hence $\gamma|_{[a,b]}$ can be homotoped with fixed ends to a path avoiding N. This construction can be done simultaneously with all connected components of $\gamma^{-1}(B)$ to end up with a path homotopic to γ, avoiding N. So we can replace γ with that graph and assume that γ avoids N.

The **stereographic projection** maps $S^n \smallsetminus \{N\}$ homeomorphically to $\mathbb{R}^n$. In the latter, γ is homotopic to a constant path, so the same holds in S^n. □

Lemma 1.2.11. *Let X be simply-connected and let $p, q \in X$. Then any two paths γ and τ from p to q are homotopic with fixed ends.*

Proof. One has

$$\gamma \simeq \gamma.\check{\tau}.\tau \simeq p.\tau \simeq \tau. \qquad \square$$

1.3 Coverings

Definition 1.3.1. A **covering** of a manifold M is a continuous map $f : N \to M$ such that every point $p \in M$ possesses an open neighbourhood U, such that

$$f^{-1}(U) = \bigsqcup_{i \in I} V_i,$$

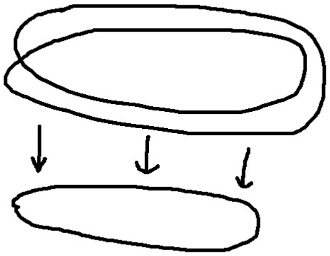
where f mapst each V_i homeomorphically onto U and the topologys on $f^{-1}(U)$ is the disjoint union topology.

Such a neighbourhood U is called a **trivialising neighbourhood**.

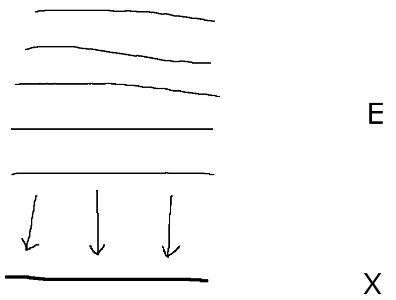
Examples 1.3.2.

(a) Let M be a manifold and I a non-empty set. The projection $p : I \times M \to M$ is a covering. Every such covering will be called a **trivial covering**.

(b) An example of a non-trivial covering of the circle S^1 is given in the drawing:



Locally, a covering always looks like this:



Definition 1.3.3. Let $\pi : N \to M$ be a covering. A **deck transformation** is a homeomorphism $d : N \to N$, such that the diagram

$$\begin{tikzcd} N \arrow[rr, "d"] \arrow[dr, "\pi"'] & & N \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. The deck transformations form a group $\mathrm{Deck}(\pi)$.

Example 1.3.4. The map

$$\begin{aligned} p : \mathbb{R} &\to S^1, \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

is a covering of the set $\mathbb{T} = \left\{ z \in \mathbb{C} : |z| = 1 \right\}$. The deck transformations are the maps

$$d_k : t \mapsto t + k, \qquad k \in \mathbb{Z}$$

Definition 1.3.5. A **homomorphism of coverings** from $E \to X$ to $F \to X$ is a continuous map $\psi : E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

commutes. If no confusion occurs, we call such a map a continuous map **over** X, or an X**-map**. An **isomorphism** is a homomorphism $\phi : E \to F$ which is bijective, such that the inverse map is a homomorphism, too.

Definition 1.3.6 (Lift)**.** Let $p : E \to X$ be a covering and let $f : S \to X$ be a continuous map. A continuous map $\tilde{f} : S \to E$ such that $f = p \circ \tilde{f}$ is called a **lift** of f to E. A lift does not necessarily exist.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
S \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Lemma 1.3.7 (Lifting of paths). *Let $p : E \to M$ be a covering. Let $\gamma : I \to M$ be a continuous map and let $x = \gamma(0)$. Then to every $e \in p^{-1}(x)$ there is exactly one lift $\tilde{\gamma}_e : I \to E$ of γ, such that $\tilde{\gamma}_e(0) = e$.*

The map $e \mapsto \tilde{\gamma}_e(1)$ is a bijection from the fibre $p^{-1}(x)$ to the fibre $p^{-1}(x')$, where $x' = \gamma(1)$.

If γ and τ are paths in M with $\gamma \simeq \tau$, then one has $\tilde{\gamma}_e \simeq \tilde{\tau}_e$.

Proof. Let $e \in p^{-1}(x)$ and let $U \subset M$ be a trivialising neighbourhood of x, i.e., $p^{-1}(U) \cong U \times D$. Then there exists a neighbourhood U_e of e such that $p|_{U_e}$ is a homeomorphism from U_e to U. Let $t_0 > 0$ such that $\gamma([0, t_0)) \subset U$. Then on the interval $[0, t_0)$ the path γ has a unique lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = e$. Let $t_1 > 0$ be the supremum of all $t_0 > 0$ such that $\gamma|_{[0,t_0)}$ has a unique lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = e$. Let V be a trivialising neighbourhood of the point $\gamma(t_1)$. In this neighbourhood the lift can be extended in a unique way, if $t_1 < 1$, so we conclude that $t_1 = 1$ and that γ has a unique lift on the entire interval I.

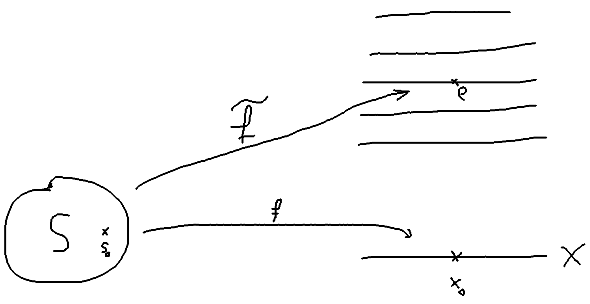
The map $e \mapsto \tilde{\gamma}_e(1)$ is bijective, since the same map for the reverse path $\check{\gamma}$ is an inverse.

Let $\gamma \simeq \tau$ in M and let $h : I \times I \to M$ be a homotopy with fixed ends. As above one sees that also the map h has a unique lift to a continuous map $\tilde{h} : I \times I \to E$ with $p \circ \tilde{h} = h$ and $\tilde{h}(0, 0) = e$. Then $\tilde{h}$ is the desired homotopy. □

Theorem 1.3.8. (a) *Let $p : E \to M$ be a covering and let $f : S \to M$ a continuous map, where S is simply-connected. Let $s_0 \in S$ and let $x_0 = f(s_0) \in M$. Choose a point $e \in E$, with $p(e) = x_0$. Then there exists exactly one lift $\tilde f$ of f to E with $\tilde f(s_0) = e$. If f is a covering, then the map $\tilde f$ is a homomorphism of coverings.*

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
S \arrow[ur, dotted, "\tilde f"] \arrow[r, "f"'] & M
\end{tikzcd}$$

(b) *If M has a simply-connected covering, then it is uniquely determined up to isomorphy. We call it the* **universal covering** *and write it as $\tilde M \to M$.*



Proof. (a) Let $s \in S$ and fix a path γ_s in S from s_0 to s. Define

$$\tilde f(s) = (\widetilde{f \circ \gamma_s})_e(1).$$

That means, we first map γ_s to M, then lift it to E and then we evaluate at 1. Since p is a local homeomorphism, it is easy to see that $\tilde f$ is continuous. The lifting property follows by construction. The uniqueness is clear as a given lift from S to E must map the path γ_s to the unique lift of $f \circ \gamma_s$.

(b) Let $p_E : E \to M$ and $p_F : F \to M$ be two simply-connected coverings, let $e \in E$ and $f \in F$. By part (a), there are uniquely determined M-homomorphisms $\psi : E \to F$ and $\phi : F \to E$ with $\psi(e) = f$ and $\phi(f) = e$. Then $\phi \circ \psi$ is the uniquely determined continuous M-map $E \to E$ mapping e to e. Therefore $\phi \circ \psi = \mathrm{Id}$. Similarly we get $\psi \circ \phi = \mathrm{Id}$. □

Corollary 1.3.9. *If M is simply-connected, then every covering is trivial.*

Proposition 1.3.10. *Let $p : \tilde{M} \to M$ be a covering, where $\tilde{M}$ is simply connected. Pick a base-point $x_0 \in M$ and a preimage $\tilde{x}_0 \in \tilde{M}$. Take $[\gamma] \in \pi_1(M, x_0)$ and let $\tilde{\gamma}$ be the unique lift with $\tilde{\gamma}(0) = \tilde{x}_0$. Then the endpoint $\tilde{\gamma}(1)$ only depends on γ up to homotopy and there exists exactly one deck transformation d_γ with $d_\gamma = \tilde{\gamma}(\tilde{x}_0)$. The map $\gamma \mapsto d_\gamma$ is an isomorphism.*

In particluar, M is simply connected iff it has no nontrivial coverings.

Proof. You find a proof in the Algebraic Topology Skript. □

Theorem 1.3.11. *Every connected manifold M admits a universal covering $p : \tilde{M} \to M$. If $\Gamma = \mathrm{Deck}(p)$ is the deck group, then p induces a homeomorphism*

$$\tilde{M}/\Gamma \;\cong\; M.$$

The covering p is uniquely determined up to isomorphy.
For every covering $f : N \to M$ with connected N is isomorphic to a quotient of $\tilde{M}$ in the sense that $N \cong \tilde{M}/\Sigma$ for a subgroup $\Sigma \subset \Gamma$, making the diagram

$$\begin{tikzcd}
\tilde{M}/\Sigma \arrow[r, "\cong"] \arrow[d] & N \arrow[d, "f"] \\
\tilde{M}/\Gamma \arrow[r, "\cong"] & M
\end{tikzcd}$$

commutative. The intermediate coverings $\tilde{M} \to N \to M$ are in bijection with the subgroups of Γ (Galois Theory of coverings)

Proof. Algebraic topology. □

* * *

1.4 Lie groups

Definition 1.4.1. A **Lie group** is a group G, which at the same time is a smooth manifold, such that multiplication and inversion

$$\begin{aligned} G\times G &\to G, & G &\to G,\\ (x,y) &\mapsto xy & x &\mapsto x^{-1} \end{aligned}$$

are smooth maps.

Remark 1.4.2. It suffices to assume that multiplication is smooth, since the Implicit Function Theorem says that then inversion is smooth, too. We only formulate it this way, because of the context of topological groups.

Remark 1.4.3. The following resulte have been shown in the 1950ies by Montgomery and Zippin:

(a) If G is a group, which is a topological manifold, such that multiplication and inversion are continuous, then there exists exactly one differentiable structure on G, which makes G a Lie group.

(b) Every closed subgroup of a Lie group is a smooth submanifold and thus itself a Lie group.

Examples 1.4.4. (a) $(\mathbb{R},+)$, $(\mathbb{R}^\times,\times)$ are Lie groups as well as $(\mathbb{C},+)$ and $(\mathbb{C}^\times,\times)$.

(b) $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}\cong\mathbb{R}/\mathbb{Z}$, the **circle group** is a Lie group of dimension one.

(c) $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$ and the same with $\mathbb{C}$ in place of $\mathbb{R}$.
This is not completely obvious. First observe that $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})\cong\mathbb{R}^{n^2}$, as it is the set of all matrices with non-zero determinant and the determinant map is continuous. The group $\mathrm{GL}_n(\mathbb{R})$ get a differentiable structure from $\mathbb{R}^{n^2}$.

The set $\mathrm{SL}_n(\mathbb{R})$ is a smooth manifold by the Imimplicite Function Theorem. Multiplication is given by sums of products of the entries, hence smooth.

Definition 1.4.5. Recall that a **regular** measure on a locally compact space X is a Borel measure μ such that

(a) $\mu(K)<\infty$ if K is compact,

(b) $\mu(A) = \inf\left\{\mu(U) : U \supset A,\ \text{open}\right\}$ holds for every measurable A,

(c) $\mu(U) = \sup\left\{\mu(K) : K \subset U, \text{compact}\right\}$ for every open U.

If μ is a regular measure, then the map $I_\mu : C_c(X) \to \mathbb{C}$, $f \mapsto \int_X f\, d\mu$ is a positive linear functional. The map $\mu \mapsto I_\mu$ is a bijection between the set of all regular measures and the set of all positive linear functionals on $C_c(G)$.

Proof. Analysis. □

Finally, we cite from Harmonic Analysis:

Theorem 1.4.6. *Every Lie group G possesses a regular Radon measure μ which is invariant under left translates, i.e., $\mu(xA) = \mu(A)$. It is uniquely determined up to scaling. It is called a* **Haar measure***.*
Every non-empty open set has Haar measure > 0, every compact set has Haar measure $< \infty$.

* * *

1.5 Vector fields

Definition 1.5.1. Let M be a smooth manifold and let $p \in M$ be a point. A **Point derivation** in p is a linear map $D = D_p : C^\infty(M) \to \mathbb{R}$, satisfying the **Leibniz rule** at p:

$$D(fg) = D(f)g(p) + f(p)D(f).$$

The **tangent space** in p is the real vector space T_pM of all point derivations at p.

Lemma 1.5.2. *Every point derivation kills constants. This means that if $f = c$ is a constant function, then $X(f) = 0$ holds for every $X \in T_pM$.*

Proof. By linearity one has $X(f) = X(c \cdot 1) = cX(1)$. By the Leibniz rule we also have $X(1) = X(1 \cdot 1) = X(1) + X(1) = 2X(1)$, so $X(1) = 0$. □

Example 1.5.3. Let $\gamma : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$ a smooth curve in M. Let $p = \gamma(0)$. For $f \in C^\infty(M)$ set

$$D_\gamma(f) = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)).$$

The classical Leibniz rule for derivatives shows that D_γ lies in the tangent space T_pM.

Definition 1.5.4. Let $\phi : U \to \mathbb{R}^n$ be a smooth chart and $p \in U$ with $\phi(p) = 0$. The coordinates x_j of $\mathbb{R}^n$ give coordinates on U, so called **local coordinates**. If $x(p) = 0$, we call them **local coordinates around** p. around the point p. One writes $x : U \to \mathbb{R}^n$ instead of ϕ and $x_j(m)$ for the j-th coordinate of $\phi(m)$. For $f \in C^\infty(M)$ let

$$D_j f = \frac{\partial}{\partial x_j}\left(f \circ \phi^{-1}\right)(0).$$

Then $D_j \in T_pM$. One writes $\frac{\partial}{\partial x_j}\Big|_p$ for this derivation.

Proposition 1.5.5. *Let M be a smooth manifold.*

(a) *For $X \in T_pM$ and $f \in C^\infty(M)$ the value $X(f)$ only depends on $f|_U$ for any neighbourhood U of p.*

(b) *Let $x = (x_1, \dots, x_n) : U \to \mathbb{R}^n$ be a local coordinates around p. Then $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p$ is a basis of the real vector space T_pM.*

(c) *(Change of coordinates) If x_i and y_j are local coordinates, then, where both are defined, one has*

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^n \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial x_j}.$$

Proof. (a) Let $f, f_1 \in C^\infty(M)$ and let U be an open neighbourhood of p such that $f|_U = f_1|_U$. We have to show that $X(f) = X(f_1)$.
For this let $g = f - f_1$, so $g|_U \equiv 0$. Then there is an open neighbourhood V of p with $V \subset \overline{V} \subset U$ and there is a function $h \in C^\infty(M)$ with $h|_V \equiv 0$, $h|_{M \smallsetminus U} \equiv 1$. (This is the smooth form of Urysohn's lemma. You get it from the continuous form by convolving with a smooth function.) We then have $g = hg$ and

$$X(g) = X(hg) = X(h)\underbrace{g(p)}_{=0} + \underbrace{h(p)}_{=0} X(g) = 0.$$

(b) For $\mu \in \mathbb{R}^n$ let $X_\mu = \mu_1 \frac{\partial}{\partial x_1}\Big|_p + \dots + \mu_n \frac{\partial}{\partial x_n}\Big|_p \in T_pM$. We show that the map $\mu \mapsto X_\mu$ is injective. Let x_j be the j-th coordinate map. Then

$$X_\mu(x_j) = \mu_1 \frac{\partial x_j}{\partial x_1} + \dots + \mu_n \frac{\partial x_j}{\partial x_n} = \mu_j.$$

Hence the map is injective. To see surjectivity as well, for given $X \in T_pM$ let $\mu_j = X(x_j)$ for $j = 1, \dots, n$. Set $Y = X - X_\mu$. We show $Y = 0$.

Observe that $Y(x_j) = 0$ for $j = 1, \dots, n$. Let $h \in C^\infty(M)$ such that in a neighbourhood of p one has $h(x) = x_j g(x)$ for a $g \in C^\infty(M)$. Then

$$Y(h) = Y\big(x_j g\big) = \underbrace{x_j(p)}_{=0}\, Y(g) + \underbrace{Y(x_j)}_{=0}\, g(p) = 0.$$

By the theory of Taylor series, every $f \in C^\infty(M)$ can be written as

$$f(x) = c + \sum_{j=1}^{n} x_j g_j(x),$$

with x in a neighbourhood of p and $g_j \in C^\infty(M)$. One has $Y(f) = Y(c) = 0$.

(c) Let f be a smooth function on M and let $\phi = f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}$. Then $\frac{\partial f}{\partial x_j} = D_j\phi$ is the j-th coordinate derivative of ϕ. Let $\psi = f \circ y^{-1} : \mathbb{R}^n \to \mathbb{R}$. Then

$$\phi = f \circ x^{-1} = f \circ y^{-1} \circ y \circ x^{-1} = \psi \circ \alpha$$

with $\alpha = y \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. By the chain rule we have

$$D\phi(x) = D\psi\,(\alpha(x))\, D\alpha(x).$$

On the other had we have $D\alpha(x) = \left(\frac{\partial y_i}{\partial x_j}\right)_{i,j}$, which implies the claim. □

Definition 1.5.6. Let $F : M \to N$ be a smooth map between smooth manifolds. Let $p \in M$ and $q = F(p) \in N$. Then F gives a map

$$\begin{aligned} F^* &: C^\infty(N) \to C^\infty(M), \\ f &\mapsto f \circ F. \end{aligned}$$

Define

$$\begin{aligned} DF(p) : T_pM &\to T_qN, \\ X &\mapsto X \circ F^*. \end{aligned}$$

Then one has

$$DF(p)X\ (f) = X(f \circ F)$$

for $f \in C^\infty(N)$. This map is called the **differential** of F at p.

Example 1.5.7. For $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ one can use the standard coordinates to identify $T_pM = T_p\mathbb{R}^m$ with $\mathbb{R}^m$ and the same for T_q. Then the linear map $DF(p)$ is represented by the Jacobi matrix, which appears in Analysis 2.

Definition 1.5.8. A **vector field** is a map

$$X : M \to \bigsqcup_{p\in M} T_pM$$

with the property that $X_p = X(p) \in T_pM$ for every $p \in M$ and such that for every $f \in C^\infty(M)$ the map

$$p \mapsto X_p(f)$$

is continuous.

The vector field X is called **smooth vector field**, if $p \mapsto X_p(f)$ is smooth for every $f \in C^\infty(M)$.

Example 1.5.9. Let $M = \mathbb{R}^n$. For continuous functions $a_1, \dots, a_n : \mathbb{R}^n \to \mathbb{R}$ the map

$$X_p = a_1(p)\frac{\partial}{\partial p_1} + \dots + a_n(p)\frac{\partial}{\partial p_n}$$

is a vector field. This field is smooth iff all a_j are smooth. Every vector field on $\mathbb{R}^n$ is of this form.

Definition 1.5.10. Let X be a smooth vector field and for $f \in C^\infty(M)$ define the function Xf by

$$Xf(p) = X_p(f).$$

In local coordinates this means

$$\begin{aligned} X_p &= a_1(p)\frac{\partial}{\partial p_1} + \dots + a_n(p)\frac{\partial}{\partial p_n} \\ \Rightarrow Xf(p) &= a_1(p)\frac{\partial f}{\partial p_1}(p) + \dots + a_n(p)\frac{\partial f}{\partial p_n}(p). \end{aligned}$$

Therefore $Xf \in C^\infty(M)$, so X defines a linear map $X : C^\infty(M) \to C^\infty(M)$.

Proposition 1.5.11. *Let X be a smooth vector field. The induced map on $C^\infty(M)$ is a* **Derivation** *of the algebra $C^\infty(M)$, i.e.,*

(a) $X : C^\infty(M) \to C^\infty(M)$ *is linear and*

(a) *satisfies the* **Leibniz rule***:*

$$X(fg) = X(f)g + fX(g).$$

In this way we get a bijection

$$\Big\{ \textit{smooth vector fields on } M \Big\} \quad \leftrightarrow \quad \Big\{ \textit{derivations on } C^\infty(M) \Big\}.$$

Proof. This follows from Proposition 1.5.5. □

* * *

1.6 Integral curves

Definition 1.6.1. Let X be a smooth vector field. An **integral curve** for X in the point p is a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and

$$\gamma'(t) = X_{\gamma(t)}$$

for every $t \in (-\varepsilon, \varepsilon)$.

In other words this means:
For every $f \in C^\infty(M)$ and every $t \in (-\varepsilon, \varepsilon)$ one has

$$X_{\gamma(t)}(f) = \frac{d}{dt} f\left(\gamma(t)\right).$$

Lemma 1.6.2. *Let X be a smooth vector field. For every point $p \in M$ there is a unique integral curve for X in p.*

The uniqueness means that any two integral curves agree where both are defined.

Proof. In local coordinates around p we can write $X = \sum_{j=1}^n c_j(x) \frac{\partial}{\partial x_j}$. Using these coordinates, an integral curve is a curve $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ in $\mathbb{R}^n$ such that

$$\gamma_j'(t) = c_j\left(\gamma_1(t), \dots, \gamma_n(t)\right).$$

The Theorem of Picard-Lindeloef of the theory of ordinary differential equations yields the existence and uniqueness of an integral curve. In this theorem one needs local Lipschitz continuity, which is granted by the smoothness of the vector field. □

* * *

1.7 Lie algebras

Definition 1.7.1. A **Lie algebra** over $\mathbb{R}$ is a vector space $\mathfrak{g}$ with a bilinear map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

(a) $[X, X] = 0$ und

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

for all $X, Y, Z \in \mathfrak{g}$.

Remark 1.7.2. (a) Note that one does not insist in associativity.

(b) In every Lie algebra $\mathfrak{g}$ one has

$$[X, Y] = -[Y, X]$$

for all $X, Y \in \mathfrak{g}$. This follows from

$$0 = [X + Y, X + Y] = \underbrace{[X, X]}_{=0} + \underbrace{[Y, Y]}_{=0} + [X, Y] + [Y, X].$$

Examples 1.7.3. (a) Every vector space becomes a Lie algebra by setting $[X, Y] = 0$ for all X, Y. One calls this an **abelian Lie algebra**.

(b) The vector space $\mathfrak{gl}_n(\mathbb{R})$ of all real $n \times n$ matrices is a Lie algebra with the **commutator bracket**:

$$[X, Y] = XY - YX.$$

(Verify the Jacobi identiyy!)

(c) A 2-dimensional $\mathbb{R}$-vector space with basis (e, f) becomes a Lie algebra by

$$[e, e] = 0 = [f, f], \qquad [e, f] = -[f, e] = f.$$

(d) The space $\mathbb{R}^3$ is a Lie algebra with the vecyor product.

(e) Let $\mathcal{A}$ be any associative algebra, then it becomes a Lie algebra with

$$[X, Y] = XY - YX.$$

Definition 1.7.4. Let $\mathfrak{g}$ be a Lie algebra. For $X, Y \in \mathfrak{g}$ we write $\mathrm{ad}(X)Y = [X, Y]$. Then $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is a linear map. The Jacobi identity gives

$$\mathrm{ad}\,([X, Y]) = \Big[\mathrm{ad}(X), \mathrm{ad}(Y)\Big],$$

where on the right we mean the commutator bracket of the algebra $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. So ad is a Lie algebra homomorphism.

Definition 1.7.5. A **multi index** is an element $\alpha \in \mathbb{N}_0^n$. For $x \in \mathbb{R}^n$ one writes

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

For a multi index α let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $k \in \mathbb{N}_0$. A **differential operator** is a linear map $D : C^\infty(M) \to C^\infty(M)$, such that for all local coordinates $(x_1, \dots, x_n)$ one has

$$Df(x) = \sum_\alpha c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} f(x),$$

where the sum runs over a finite set of multi indices (depending on the coordinates) and every c_α is a smooth function.

If there is a $k \in \mathbb{N}$, such that $c_\alpha \neq 0 \Rightarrow |\lambda| \leq k$, then we say that the differential operator has order $\leq k$. The **order** of D is the smallest such k.

By the formula for change of coordinates in Proposition 1.5.5, the order does not depend on the chosen coordinates, but only on the operator and the point.

Proposition 1.7.6. *The set* $\mathrm{vect}(M)$ *of all smooth vector fields on a smooth manifold M is a Lie algebra with the Lie bracktet*

$$[X, Y] = XY - YX,$$

where smooth vector fields are considered as derivations, i.s., as linear maps $C^\infty(M) \to C^\infty(M)$.

Proof. A smooth vector field $\neq 0$ is the same as a differential operator of order 1, which kills constants. In other words, a vector field X is a differential operator, which in all local coordinates is of the form

$$X = \sum_{j=1}^n c_j(x) \frac{\partial}{\partial x_j}.$$

In order to show that the commutator $XY - YX$ is a vector field, we write in local coordinates

$$X = \sum_{i=1}^{n} c_j(x)\frac{\partial}{\partial x_i}, \qquad Y = \sum_{j=1}^{n} d_j(x)\frac{\partial}{\partial x_j}.$$

Then one has

$$\begin{aligned}
XY - YX &= \left(\sum_{i=1}^{n} c_i(x)\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} d_j(x)\frac{\partial}{\partial x_j}\right)\\
&\quad - \left(\sum_{j=1}^{n} d_j(x)\frac{\partial}{\partial x_j}\right)\left(\sum_{i=1}^{n} c_i(x)\frac{\partial}{\partial x_i}\right)\\
&= \sum_{i,j} c_i(x)\frac{\partial d_j}{\partial x_i}(x)\frac{\partial}{\partial x_j} + c_i(x)d_j(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\\
&\quad - \sum_{i,j} d_j(x)\frac{\partial c_i}{\partial x_j}(x)\frac{\partial}{\partial x_i} - c_i(x)d_j(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\\
&= \sum_{i,j} c_i(x)\frac{\partial d_j}{\partial x_i}(x)\frac{\partial}{\partial x_j} - \sum_{i,j} d_j(x)\frac{\partial c_i}{\partial x_j}(x)\frac{\partial}{\partial x_i}\\
&= \sum_{k=1}^{n}\left(\sum_{j=1}^{n} c_j(x)\frac{\partial d_k}{\partial x_j}(x) - d_j(x)\frac{\partial c_k}{\partial x_j}(x)\right)\frac{\partial}{\partial x_k}.
\end{aligned}$$

□

Let G be a Lie group. For $y \in G$ defa the **left translation action** $L_y : C^\infty(G) \to C^\infty(G)$ be defined by $L_y f(x) = f(y^{-1}x)$. A linear map $T : C^\infty(G) \to C^\infty(G)$ is called **left-invariant**, if

$$TL_y = L_y T$$

holds for every $y \in G$.

Example 1.7.7. For $z \in G$ let the **right translation action**$R_z : C^\infty(G) \to C^\infty(G)$ be defined by

$$R_z f(x) = f(xz).$$

Then R_z is left-invariant, since

$$R_z L_y f(x) = L_y f(xz) = f(y^{-1}xz) = R_z f(y^{-1}x) = L_y R_z f(x).$$

Proposition 1.7.8. *Let G be a Lie group.*

(a) *The set of all left-invariant vector fields* $\mathrm{vect}(G)^G$ *is stable under the commutator bracket,* *hence is a Lie sub-algebra of* vect(G). *This Lie algebra is called* **the Lie algebra** *of G and one writes it as*

$$\mathfrak{g} \quad := \quad \Big\{ \textit{left-invariant vector fields} \Big\}.$$

One uses the small script letter to denote the Lie algebra. For instance, if H is a Lie group we would write its Lie algebra as $\mathfrak{h}$.

(b) *The Lie algebra of a Lie group G is finite-dimensional. Its dimension equals the dimension of the manifold G.*

(c) *For a given vector field X on G the following are equivalent*

  (i) *X is left-invariant,*

  (ii) *for all $x, y \in G$ one has*

$$Dl_x(y)X_y = X_{xy},$$

where l_x is the map

$$\begin{aligned} l_x : G &\to G, \\ y &\mapsto xy. \end{aligned}$$

(d) *Evaluating a vector field at the point $e \in G$ yields a linear bijection*

$$\mathfrak{g} \xrightarrow{\cong} T_eG.$$

Proof. The sum and the composition of left-invariantoperators is left-invariant, hence the same holds true for the commutator bracket. Hence $\mathfrak{g}$ is a Lie algebra. Let X be a left-invariant vector field. For $y \in G$ let $l_y : G \to G, x \mapsto yx$. Then one has $L_y(f) = f \circ l_{y^{-1}}$ and so $X(f \circ l_y) = X(f) \circ l_y$. The differential $Dl_y(p) : T_pG \to T_{yp}G$ is given by composition with l_y and hence $Dl_y(X_p)f = X_p(f \circ l_y) = X(f) \circ l_y(p) = X_{yp}(f)$. For $p = e$ we get

$$X_y = Dl_y(X_e).$$

This means that the map $\mathfrak{g} \to T_eG$, which maps X to X_e, is injective. The same formula shows that for a given vector $v \in T_eG$ the prescription $X_y = Dl_y(v)$ defines a left-invariant vector field X, so the evaluation map is also surjective. □

Proposition 1.7.9. *A* **one parameter subgroup** *(1PSG) of G is a Lie group homomorphism $P : \mathbb{R} \to G$.*

For every $X \in \mathfrak{g}$ the integral curve through $e \in G$ defines a 1PSG. This induces a bijection

$$\mathfrak{g} \leftrightarrow \left\{ \text{1PSGs} \right\}.$$

Proof. Let $X \in \mathfrak{g}$ and let $\gamma : (-\varepsilon, \varepsilon) \to G$ be the integral curve through $e = \gamma(0)$. Let $s \in (-\varepsilon, \varepsilon)$. We claim that the two curves

$$\alpha(t) = \gamma(s)\gamma(t), \quad \beta(t) = \gamma(s + t)$$

both are integral curves for X at the point $\gamma(s)$. For this we use the invariance and the chain rule:

$$X_{\alpha(t)} = X_{\gamma(s)\gamma(t)} = Dl_{\gamma(s)} X_{\gamma(t)} = Dl_{\gamma(s)}\gamma'(t) = \left(l_{\gamma(s)}\gamma(t)\right)' = (\gamma(s)\gamma(t))' = \alpha'(t).$$

On the other hand we have

$$X_{\beta(t)} = X_{\gamma(s+t)} = \gamma'(s + t) = \frac{d}{dt}\gamma(s + t) = \beta'(t).$$

The uniqueness of integral curves implies $\alpha = \beta$, so $\gamma(s + t) = \gamma(s)\gamma(t)$. This means that γ also is defined in $s + t$ and by iteration we extend it to all of $\mathbb{R}$ and it yields a 1PSG.

Conversely let $P : \mathbb{R} \to G$ be a 1PSG. Then $X_e = P'(0)$ is an element of T_eG, hence defines a left-invariant vector field X and one has

$$X_{P(t)} = Dl_{P(t)}(e)(X_e) = Dl_{P(t)}(P'(0)) = P'(t),$$

since P is a group homomorphism. □

Definition 1.7.10. Let P_X be the 1PSG generated by $X \in \mathfrak{g}$. The evaluation at 1 defines a map, the so called **exponential map**:

$$\begin{aligned} \exp : \mathfrak{g} &\to G, \\ X &\mapsto P_X(1). \end{aligned}$$

One has

$$\exp(tX) = P_{tX}(1) = P_X(t).$$

Examples 1.7.11. (a) Let $G = \mathbb{T} \subset \mathbb{C}^\times$ be the circle group. For $r \in \mathbb{R}$ let X_r be the vector field

$$X_r f(e^{2\pi i x}) = \frac{d}{dt}\bigg|_{t=0} f\left(e^{2\pi i(x+tr)}\right).$$

We claim that $\gamma(t) = e^{2\pi irt}$ is the integral curve through the point $e = 1$. We compute

$$\gamma'(t) = 2\pi ir\, e^{2\pi irt} = X_{r,\gamma(t)}.$$

(b) Let $G = \mathrm{GL}_n(\mathbb{R})$. For a matrix $A \in \mathrm{M}_n(\mathbb{R})$ the exponential series

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

converges in $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. For $A, B \in \mathrm{M}_n(\mathbb{R})$ with $AB = BA$ one has

$$\exp(A + B) = \exp(A)\exp(B).$$

The matrix $\exp(A)$ is invertible, i.e., in $\mathrm{GL}_n(\mathbb{R})$ and the vector field

$$X_A f(x) = \frac{d}{dt}\bigg|_{t=0} f(x\exp(A))$$

ist left-invariant. The map $A \to X_A$ is a linear isomorphism $\mathrm{M}_n(\mathbb{R}) \xrightarrow{\cong} \mathfrak{g}$. The 1PSG generated by X_A is $\Big\{\exp(tA) : t \in \mathbb{R}\Big\}$.

Proof. The euclidean norm on $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ is given by

$$\|A\|^2 = \mathrm{tr}(A^t A) = \sum_{i,j=1}^{n} A_{ij}^2.$$

For $A, B \in \mathrm{M}_n(\mathbb{R})$ the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|AB\|^2 &= \sum_{i,j} (AB)_{ij}^2 \\ &= \sum_{i,j} \left(\sum_k A_{ik} B_{kj} \right)^2 \\ &\le \sum_{i,j} \left(\sum_k A_{ik}^2 \right) \left(\sum_l B_{lj}^2 \right) = \|A\|^2 \ \|B\|^2, \end{aligned}$$

hence $\|AB\| \le \|A\|\,\|B\|$ and in particular $\|A^n\| \le \|A\|^n$. Since the series $\exp(\|A\|) = \sum_{k=1}^{\infty} \frac{1}{k!} \|A\|^k$ converges, the exponential series converges absolutely. The equation $\exp(A + B) = \exp(A)\exp(B)$ is proven exactly as in the Analysis lecture. By $\exp(A)\exp(-A) = \exp(0) = 1_n$ the matrix $\exp(A)$ is invertible. Finally, the group

$\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$ and the map

$$X \mapsto \exp(X) = 1_n + X + X^2\, R(X)$$

hat differential Id. This differential is invertible and maps $\mathrm{M}_n(\mathbb{R})$ bijectively to the tangent space of $\mathrm{GL}_n(\mathbb{R})$ at e. □

Theorem 1.7.12. *Let G be a Lie group.*

(a) *If $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, then*

$$Xf(p) = \frac{d}{dt}\Big|_{t=0} f(p\exp(tX)).$$

(b) *The image of the exponential map* exp *contains a unit neighbourhood in G. If G is connected, then every unit neighbourhood generates the group G.*

(c) *Let $\phi : G \to H$ be a homomorphism of Lie groups. Define $\phi' : \mathfrak{g} \to \mathfrak{h}$ by $\phi = D\phi(e) : T_eG \to T_eH$. Then the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi'"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

commutes.

(d) *The map $\phi' : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, i.e., one has*

$$\phi'([X, Y]) = [\phi' X, \phi' Y].$$

(e) *As $\mathfrak{g}$ is a vector space, the tangent space $T_0\mathfrak{g}$ is naturally identified with $\mathfrak{g}$. Then the differential of* exp *gives a map $D\exp(0) : \mathfrak{g} \to T_eG = \mathfrak{g}$. his map is the identity, i.e.,*

$$D\exp(0) = \mathrm{Id}_{\mathfrak{g}}.$$

Proof. (a) As $t \mapsto \exp(tX)$ is the integral curve of X through e, the left-invariance implies that the curve $t \mapsto p\exp(tX)$ is the integral curve of X through p. This implies part (a).

(b) Every $0 \neq X \in \mathfrak{g}$ generates a non-trivial 1PSG, so the differential $D\exp(0)$ of the map exp in $0 \in \mathfrak{g}$ has trivial kernel, hence is invertible.

By the theorem on local inverses the image of exp contains an open unit neighbourhood $U \subset G$. Assume now that G is connected and let H be the subgroup generated by an open unit neighbourhood U. The group H is open, since for every point $p \in H$ the open neighboruhood pU also lies in H. Now G is the union of H and $\bigcup_{x \notin H} xH$, where the second sett is a union of open sets, hence open. The group G is connected, hence the second set must be empty, i.e., $G = H$.

(c) For $X \in \mathfrak{g}$ and $f \in C^\infty(H)$ one has

$$\begin{aligned} D\phi(X_{\exp(tX)})(f) &= X_{\exp(tX)}(f \circ \phi) \\ &= \frac{d}{dt} f(\phi(\exp(tX)), \end{aligned}$$

which implies that $t \mapsto \phi(\exp(tX))$ is the integral curve to $\phi'(X)$, or

$$\phi\left(\exp(tX)\right) = \exp\left(t\phi'(X)\right).$$

Putting $t = 1$, the commutativity of the diagram follows.

(d) Let $X, Y \in \mathfrak{g}$ and let $f \in C^\infty(H)$. We know, that $\phi(\exp(tX)) = \exp(t\phi' X)$. The map ϕ is a group homomorphism, hence

$$\begin{aligned} \phi'[X,Y]_e(f) &= [X,Y]_e(f \circ \phi) \\ &= XY(f \circ \phi) - YX(f \circ \phi)\Big|_e \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f(\phi(\exp(tX)\exp(sY))) \\ &\qquad\qquad - f(\phi(\exp(sY)\exp(tX))) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f(\phi(\exp(tX))\phi(\exp(sY))) \\ &\qquad\qquad - f(\phi(\exp(sY))\phi(\exp(tX))) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f(\exp(t\phi' X)\exp(s\phi' Y)) \\ &\qquad\qquad - f(\exp(s\phi' Y)\exp(t\phi' X)) \\ &= \phi' Y \phi' Y(f) - \phi' Y \phi' X(f)|_e \\ &= [\phi' X, \phi' Y]_e(f). \end{aligned}$$

(e) Let $X \in \mathfrak{g}$. For the corresponding element of $T_0\mathfrak{g}$ we write X'. For a function ϕ, which is smooth in $0 \in \mathfrak{g}$ it follows that

$$X'\phi = \frac{d}{dt}\bigg|_{t=0} \phi(tX).$$

For $f \in C^\infty(G)$ one has

$$\begin{aligned}(D\exp(0)X)\, f &= X'(f \circ \exp) \\ &= \frac{d}{dt}\bigg|_{t=0} f(\exp(tX) = Xf.\end{aligned}$$

□

1.8 The Baker-Campbell-Hausdorff formula

Lemma 1.8.1. *Let $X \in \mathfrak{g}$. In the vector space* $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ *the following holds*

(a) $\frac{d}{ds}\big|_{s=0} \mathrm{Ad}\,(\exp(sX)) = \mathrm{ad}(X)$,

(b) $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$.

Proof. (a) Let $X, Y \in \mathfrak{g}$. For $s \in \mathbb{R}$ let $x_s = \exp(sX)$. Then $\frac{d}{ds}\big|_{s=0} x_s Y x_s^{-1}$ is an element of $\mathfrak{g}$. Since the exponential function has differential Id at zero, we get for $f \in C^\infty(G)$, that

$$\begin{aligned}\left(\frac{d}{ds}\bigg|_{s=0} x_s Y x_s^{-1}\right)_e f &= \frac{d}{dt}\bigg|_{t=0} f\left(\exp\left(t\,\frac{d}{ds}\bigg|_{s=0} x_s Y x_s^{-1}\right)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f\left(\exp\left(t x_s Y x_s^{-1}\right)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f\left(x_s \exp(tY)\, x_s^{-1}\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f\left(\exp(sX)\exp(tY)\exp(-sX)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f\left(\exp(sX)\exp(tY)\right) \\ &\quad - \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f\left(\exp(tY)\exp(sX)\right) \\ &= [X, Y]_e f.\end{aligned}$$

(b) Let $\gamma(t) = \mathrm{Ad}(\exp(tX))$. In $\mathrm{End}(\mathfrak{g})$ one has $\gamma(s+t) = \gamma(s)\gamma(t)$. The same holds for

$\eta(t) = \exp(\operatorname{ad}(tX))$. By part (a),

$$\frac{d}{dt}\bigg|_{t=0} \gamma(t) = \operatorname{ad}(X) = \frac{d}{dt}\bigg|_{t=0} \eta(t).$$

Hence η and γ coincide, in particular $\gamma(1) = \eta(1)$, which is the claim. □

Definition 1.8.2. We shall use the following abbreviation: For $Y \in \mathfrak{g}$ and $g, h \in G$ we write

$$gYh = D\ell_g(Dr_{h^{-1}}(Y)) \in T_{gh}G.$$

It's the left translation action by g and right action by h, which makes the notation plausible. Note that these two actions commute with each other.

Proposition 1.8.3. *Let $X, Y \in \mathfrak{g}$. Then*

$$\frac{d}{dt}\bigg|_{t=0} e^{X+tY} = \left(\frac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}(Y)\right)_{\exp(X)}.$$

More generally, for any $\mathfrak{g}$-valued smooth function $Z(t)$ one has

$$\frac{d}{dt}e^{Z(t)} = e^{Z(t)}\left(\frac{1 - e^{Z(t)}}{\operatorname{ad}(Z(t))}\right)\frac{d}{dt}Z(t).$$

Proof. The general statement follws from the special one by applying the chain rule. We show the specialised version. Let

$$\Delta(X, Y) = \frac{d}{dt}\bigg|_{t=0} e^{X+tY}.$$

Then $\Delta(X, Y)$ is linear in Y. For $m \in \mathbb{N}$ one has

$$\begin{aligned} e^{X+tY} &= \left(\exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\right)^m \\ &= \exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\cdots\exp\left(\frac{X}{m} + t\frac{Y}{m}\right) \end{aligned}$$

This equals $F(t, t, \dots, t)$ and for the function F the derivative $\frac{d}{dt}\big|_{t=0} F(t, \dots, t)$ is a sum of derivatives in the single slots, so, using the above notation,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} e^{X+tY} &= \sum_{k=1}^{m} \exp\left(\frac{X}{m}\right)^{k-1} \left(\frac{d}{dt}\Big|_{t=0} \exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\right) \exp\left(\frac{X}{m}\right)^{m-k} \\
&= \exp\left(\frac{m-1}{m}X\right) \sum_{k=1}^{m} \exp\left(\frac{X}{m}\right)^{k-m} \left(\frac{d}{dt}\Big|_{t=0} \exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\right) \exp\left(\frac{X}{m}\right)^{m-k} \\
&= \exp\left(\frac{m-1}{m}X\right) \sum_{k=1}^{m} \exp\left(\frac{X}{m}\right)^{k-m} \Delta\left(\frac{X}{m}, \frac{Y}{m}\right) \exp\left(\frac{X}{m}\right)^{m-k} \\
&= \exp\left(\frac{m-1}{m}X\right) \frac{1}{m} \sum_{k=1}^{m} \exp\left(\frac{\operatorname{ad}(X)}{m}\right)^{k-m} \Delta\left(\frac{X}{m}, Y\right).
\end{aligned}$$

For $m \to \infty$ the first term tends to $\exp(X)$, the last to $\Delta(0, Y) = Y$. It remains to show

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \exp\left(\frac{-\operatorname{ad}(X)}{m}\right)^{k} = \frac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.$$

We have

$$\frac{1}{m} \sum_{k=0}^{m-1} \exp\left(\frac{-\operatorname{ad}(X)}{m}\right)^{k} = \frac{1}{m} \frac{1 - \exp(-\operatorname{ad}(X))}{1 - \exp(-\operatorname{ad}(X)/m)}$$

and this converges to $\frac{1-\exp(-\operatorname{ad}(X))}{\operatorname{ad}(X)}$ as $m \to \infty$. □

Definition 1.8.4. Let G, H be Lie groups. A **local homomorphism** from G to H is a pair (U, ϕ), where U is a unit neighbourhood in G and $\phi : U \to H$ is a smooth map with

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in U$ with $xy \in U$. We consider two local homomorphisms $(U, \phi), (V, \psi)$ as equal, if ϕ and ψ agree on $U \cap V$.

A **local isomorphism** is a local homomorphism with a local inverse.

Theorem 1.8.5 (Baker-Campbell-Hausdorff or BCH)**.** *For all sufficiently small* $0 \neq X, Y \in \mathfrak{g}$ *one has*

$$\exp(X)\exp(Y) = \exp\left(X + \int_0^1 \frac{\log\left(e^{\operatorname{ad}(X)} e^{t\operatorname{ad}(Y)}\right)}{1 - e^{-t\operatorname{ad}(Y)} e^{-\operatorname{ad}(X)}}(Y)\, dt \right).$$

In the theorem we take a quotient in $\operatorname{End}(\mathfrak{g})$. This makes sense, as enumerator and denominator commute. It is part of the statement, that the denominator is invertible.

Corollary 1.8.6. *If* $X, Y \in \mathfrak{g}$ *have zero Lie bracket, i.e.,* $[X, Y] = 0$, *then* e^X *and* e^Y *commute and one has*

$$\exp(X + Y) = \exp(X)\ \exp(Y).$$

Proof. The first assertion follows from the second, which, in tuirn, follows from BCH because of $\operatorname{ad}(X)Y = 0$. □

Corollary 1.8.7. *Let* G, H *be Lie groups and let* $\rho : \mathfrak{g} \to \mathfrak{h}$ *a Lie algebra homomorphism. For* $X \in G$ *define*

$$\phi\Big(\exp(X)\Big) = \exp\left(\rho(X)\right).$$

Then one has

$$\phi\Big(\exp(X)\exp(Y)\Big) = \phi\Big(\exp(X)\Big)\,\phi\Big(\exp(Y)\Big),$$

this means that ϕ *is a local homomorphism.*

Proof of the corollary. Let $g(z) = \frac{\log z}{1-\frac{1}{z}}$. This function is holomorphic in the open disk $B_1(1)$ and so has a convergent power series expansion

$$g(z) = \sum_{k=0}^{\infty} a_k\,(z-1)^k.$$

The BCH-formula then says

$$\exp(X)\exp(Y) = \exp\left(X + \int_0^1 \sum_{k=0}^{\infty} a_k \left(e^{\operatorname{ad}(X)} e^{\operatorname{ad}(Y)} - 1\right)^k (Y)\, dt \right).$$

As ρ is a Lie algebra homomorphism, one has $\rho([X, Y]) = [\rho(X), \rho(Y)]$ or

$\rho(\operatorname{ad}(X)Y) = \operatorname{ad}(\rho(X))\rho(Y)$. Iteration yields $\rho(\operatorname{ad}(X)^k Y) = \operatorname{ad}(\rho(X))^k\rho(Y)$ and finally

$$\rho\Big(P(\operatorname{ad}(X))Y\Big) = P\Big(\operatorname{ad}(\rho(X))\Big)\rho(Y)$$

for every convergent power series $P(z)$. As ρ is a linear map on finite dimensional spaces, it is continuous and so

$$\begin{aligned}
\phi\left(\exp(X)\exp(Y)\right) &= \exp\left(\rho(X) + \rho\left(\int_0^1 \sum_{k=0}^{\infty} a_k \left(e^{\operatorname{ad}(X)}e^{\operatorname{ad}(Y)} - 1\right)^k (Y)\, dt\right)\right)\\
&= \exp\left(\rho(X) + \int_0^1 \sum_{k=0}^{\infty} a_k\, \rho\left(\left(e^{\operatorname{ad}(X)}e^{\operatorname{ad}(Y)} - 1\right)^k (Y)\right)\, dt\right)\\
&= \exp\left(\rho(X) + \int_0^1 \sum_{k=0}^{\infty} a_k\, \left(e^{\operatorname{ad}(\rho(X))}e^{\operatorname{ad}(\rho(Y))} - 1\right)^k (\rho(Y))\, dt\right)\\
&= \exp(\rho(X))\ \exp(\rho(Y))\\
&= \phi(\exp(X))\ \phi(\exp(Y)),
\end{aligned}$$

where we have applied BCH once on G and once on H. □

Proof of BCH. Let $Z(t) = \log\left(e^X e^{tY}\right)$. For X and Y sufficiently small, $Z(t)$ is defined for $0 \le t \le 1$. One has $e^{Z(t)} = e^X\, e^{tY}$, so that

$$e^{-Z(t)}\frac{d}{dt}e^{Z(t)} = (e^X e^{tY})^{-1} e^X e^{tY} Y = Y.$$

On the other hand, by Proposition 1.8.3 one has

$$e^{-Z(t)}\frac{d}{dt}e^{Z(t)} = \frac{1 - e^{-\operatorname{ad}(Z(t))}}{\operatorname{ad}(Z(t))}\ \frac{d}{dt}Z(t).$$

Hence

$$\frac{1 - e^{-\operatorname{ad}(Z(t))}}{\operatorname{ad}(Z(t))}\ \frac{d}{dt}Z(t) = Y.$$

If X and Y are small enough, then

$$\frac{1 - e^{-\operatorname{ad}(Z(t))}}{\operatorname{ad}(Z(t))} = 1 - \sum_{k=2}^{\infty} \frac{1}{(n+1)!}\left(\operatorname{ad}(Z(t))\right)^n$$

is invertible and therefore

$$\frac{d}{dt}Z(t) = \frac{\mathrm{ad}(Z(t))}{1-e^{-\mathrm{ad}(Z(t))}}Y. \tag{$*$}$$

One has $e^{Z(t)} = e^X e^{tY}$, hence

$$\mathrm{Ad}\left(e^{Z(t)}\right) = \mathrm{Ad}\left(e^X\right)\mathrm{Ad}\left(e^{tY}\right).$$

By Lemma 1.8.1 it follows

$$e^{\mathrm{ad}(Z(t))} = e^{\mathrm{ad}(X)}e^{t\,\mathrm{ad}(Y)},$$

or

$$\mathrm{ad}(Z(t)) = \log\left(e^{\mathrm{ad}(X)}e^{t\,\mathrm{ad}(Y)}\right).$$

We plug this into ($*$) to get

$$\frac{d}{dt}Z(t) = \frac{\log\left(e^{\mathrm{ad}(X)}e^{t\,\mathrm{ad}(Y)}\right)}{1-e^{-t\,\mathrm{ad}(Y)}e^{-\mathrm{ad}(X)}}Y = g\left(e^{\mathrm{ad}(X)}e^{t\,\mathrm{ad}(Y)}\right)Y.$$

Since $Z(0) = X$, it follows

$$e^X e^Y = Z(1) = X + \int_0^1 g\left(e^{\mathrm{ad}(X)}e^{t\,\mathrm{ad}(Y)}\right)Y\,dt.$$

This finishes the proof of BCH. □

* * *

1.9 Coverings of Lie groups

Theorem 1.9.1. *Let G be a connected Lie group. Let $p : H \to G$ be a connected covering. Choose a point $\tilde{e}$ in the fibre $p^{-1}(e)$. On the manifold H there is exactly one Lie group structure, which has the point $\tilde{e}$ as neutral element and which turns the projection $p : \tilde{G} \to G$ into a Lie group homomorphism.*

Proof. As H is covered by the universal covering $\tilde{G}$, it suffices to assume $H = \tilde{G}$. One constructs the universal covering $\tilde{G}$ as follows: Define $\tilde{G}$ as the set of all homotopy classes (with fixed ends) of paths $\gamma : [0,1] \to G$ with $\gamma(0) = e$. Then the map

$$\begin{aligned} p : \tilde{G} &\to G, \\ [\gamma] &\mapsto \gamma(1) \end{aligned}$$

is the universal covering. As multiplication is continuous, the pointwise product of two paths is a path, and pointwise product of a homotopy is a homotopy. Hence the product

$$[\gamma][\eta] \mapsto [\gamma\eta]$$

on $\tilde{G}$ is well-defined. This multiplication turns $\tilde{G}$ into a topological group.

Existence and uniqueness of a differentiable structure is part of the Montgomery-Zippin result. □

Proposition 1.9.2. *Let G be a connected Lie group. The kernel $\Gamma \subset \tilde{G}$ of the universal covering $p : \tilde{G} \to G$ acts on $\tilde{G}$ by left-translation, which induces an isomorphism $\Gamma \cong \mathrm{Deck}(p)$. The group Γ is a discrete subgroup and lies in the center of $\tilde{G}$.*

Proof. As p is a local homeomorphism, its kernel is discrete. Hence Γ is a discrete subgroup of $\tilde{G}$. Left-translation makes Γ a subgroup of $\mathrm{Deck}(p)$. Since $\mathrm{Deck}(p)$ leaves the kernel $p^{-1}(e)$ stable, we get $\Gamma = \mathrm{Deck}(p)$.

Let $H \subset \tilde{G}$ be the centraliser, so $H = \left\{h \in \tilde{G} : h\gamma h^{-1} = \gamma \ \forall_{\gamma\in\Gamma}\right\}$. For $\gamma \in \Gamma$ consider the map

$$\begin{aligned} \mu : G &\to G, \\ g &\mapsto g\gamma g^{-1}. \end{aligned}$$

As μ is continuous, the image $\mu(G)$ is connected. As Γ is normal, the image lies in the discrete group Γ, which means that the image is a single point $\mu(G) = \mu(e) = \{\gamma\}$. Hence $\gamma = \mu(x) = x\gamma x^{-1}$ for every $x \in G$ and γ is central. □

Example 1.9.3. Let

$$G = \mathrm{SL}_2(\mathbb{R}), \qquad A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G,$$

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset G, \qquad K = \mathrm{SO}(2) \subset G.$$

Then the map $A \times N \times K \to G$, $(a, n, k) \mapsto ank$ is a diffeomorphism. (Exercise) This is known as the **Iwasawa decomposition**. It implies that the fundamental group of G is $\mathbb{Z}$. Hence for every $n \in \mathbb{N}$ there is exactly one covering G_n of G of degree n. The groups G_n for $n \geq 2$ are called the **metaplectic groups.**

Theorem 1.9.4. *Let G, H be Lie groups, where G is connected. Every local homomorphism f from G to H lifts to a unique Lie group homomorphism $\tilde{f} : \tilde{G} \to H$. We get a bijection*

$$\mathrm{Hom}_{\mathrm{loc}}(G, H) \quad \leftrightarrow \quad \mathrm{Hom}(\tilde{G}, H).$$

Proof. Let (f, U) be a local homomorphism, where U is chosen connected and trivialising. Let $\tilde{U}$ be the unit-neighbourhood in $\tilde{G}$, which is mapped homeomorphically to U. As $\tilde{G}$ is connected, the group $\tilde{G}$ is generated by the uni-neighbourhood $\tilde{U}$. Since $p : \tilde{G} \to G$ is a group homomorphism, $(\tilde{U}, f \circ p)$ is a local homomorphism f_1 from $\tilde{G}$ to H.

The group $\tilde{G}$ is connected, so this group is generated by the unit-neighbourhood $\tilde{U}$ and hence for a given $x \in \tilde{G}$ there are elements $u_1, \dots, u_r \in \tilde{U}$, such that $x = u_1 \cdots u_r$. We set $\tilde{f}(x) = f_1(u_1) \cdots f_1(u_r)$. We have to show that $f(x)$ does not depend on the choice of $u_1, \dots, u_r$. Let $x = v_1 \cdots v_s$ be another choice. Choose paths η_j in $\tilde{U}$, connecting the unit with u_j. Then $\gamma_j = u_1 \cdots u_{j-1}\, \eta_j(t)$ is a path connecting $u_1 \cdots u_{j-1}$ with $u_1 \cdots u_j$. Let γ be the composition of the paths $\gamma_1, \dots, \gamma_r$. Then γ connects the unit with x. The image of γ lies in $\bigcup_{j=1}^{r} u_1 \cdots u_{j-1} \tilde{U}$. In the same way one obtains a second path τ, connecting the unit with x lying in $\bigcup_{i=1}^{s} v_1 \dots, v_{i-1} \tilde{U}$. The group $\tilde{G}$ is simply connected, so there exists a homotopy $h : [0, 1]^2 \to \tilde{G}$ with fixed ends from γ to τ. Covering every intermediate path $\gamma_s(t) = h(s, t)$ with translates of $\tilde{U}$, one gets, by compactess, finitely many paths $\gamma = \alpha_0, \dots, \alpha_m = \tau$ such that α_{j+1} is contained in the given covering of α_j. Moving from translation point to translation point one sees that α_{j+1} defines the same extension of f as α_j. In conclusion, f is a well-defined group homomorphism. If ϕ_1, ϕ_2 are two lifts of f, then one has $\phi_1|_{\tilde{U}} = \phi_2|_{\tilde{U}}$ and since $\tilde{U}$ generates the group $\tilde{G}$, they are equal. So the map $\mathrm{Hom}_{\mathrm{loc}}(G, H) \to \mathrm{Hom}(\tilde{G}, \tilde{H})$ is defoned and is obviously injective.

A given $f \in \mathrm{Hom}(\tilde{G}, H)$, induces by restriction a local homomorphism, which projects to a local homomorphism from G to H. Surjectivity follows. □

Proposition 1.9.5. *For $y \in G$ let* $\mathrm{int}(y) : G \to G$, $x \mapsto yxy^{-1}$ *be the conjugation map. Then* $\mathrm{int}(y)$ *is a smooth map, sending e to e. Let* $\mathrm{Ad}(y) : T_eG \to T_eG$ *be its differential at the point e. Identifying $T_eG = \mathfrak{g}$ one gets*

(a) $y\exp(X)y^{-1} = \exp\left(\operatorname{Ad}(y)X\right),$

(b) $\operatorname{Ad}(y)\left([X,Y]\right) = \left[\operatorname{Ad}(y)X, \operatorname{Ad}(y)Y\right],$

(c) $\operatorname{Ad}\left(\exp(X)\right) = \exp\left(\operatorname{ad}(X)\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\operatorname{ad}(X)^k,$

for all $y \in G$ and $X, Y \in \mathfrak{g}$. The group homomorphism $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$ *is called the* **adjoint representation**.

Proof. (a) For $f \in C^\infty(G)$ one has

$$\begin{aligned}(\operatorname{Ad}(y)X)_e f &= X_e(f\circ \operatorname{int}(y))\\ &= \frac{d}{dt}\Big|_{t=0} f\circ\operatorname{int}(y)(\exp(tX))\\ &= \frac{d}{dt}\Big|_{t=0} f\left(y\exp(tX)y^{-1}\right)\end{aligned}$$

(b) Using

$$\begin{aligned}\operatorname{Ad}(y)\left([X,Y]\right)_e\ f &= [X,Y]_e\left(f\circ\operatorname{int}(y)\right)\\ &= XY\left(f\circ\operatorname{int}(y)\right)(e) - YX\left(f\circ\operatorname{int}(y)\right)(e).\end{aligned}$$

We write $f_y = f\circ\operatorname{int}(y)$ and compute

$$\begin{aligned}XY\left(f\circ\operatorname{int}(y)\right)(e) = XY\left(f_y\right)(e) &= \frac{d}{dt}\Big|_{t=0}(Yf_y)(\exp(tX))\\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f_y\left(\exp(tX)\exp(sY)\right)\\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f\left(y\exp(tX)y^{-1}y\exp(sY)y^{-1}\right)\\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f\left(\exp(t\operatorname{Ad}(y)X)\exp(s\operatorname{Ad}(y)Y)\right)\\ &= (\operatorname{Ad}(y)X)\operatorname{Ad}(y)Y)f(e).\end{aligned}$$

(c) Let $\phi(t) = \operatorname{Ad}\left(\exp(tX)\right)$ and $\psi(t) = \exp\left(\operatorname{ad}(tX)\right)$. These are curves in $\operatorname{GL}(\mathfrak{g})$, for which $\phi(t+s) = \phi(s)\phi(t)$ halds and the same for ψ. Let $A = \frac{d}{dt}\big|_{t=0}\phi(t) \in \operatorname{End}(\mathfrak{g})$ and $B = \frac{d}{dt}\big|_{t=0}\psi(t)$. Then ϕ and ψ are the 1PSG generated by A, and B respectively. By

Lemma 1.8.1 one has

$$\begin{aligned} A &= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}\left(\exp(sX)\right) = \mathrm{ad}(X) \\ &= \left.\frac{d}{ds}\right|_{s=0} \exp\left(\mathrm{ad}(tX)\right) = B. \end{aligned}$$

We conclude $\phi = \psi$. □

Definition 1.9.6. We use the notation

$$yXy^{-1} := \mathrm{Ad}(y)X.$$

Theorem 1.9.7. *If ϕ is a local homomorphism, then $D\phi(e) : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*
One has

$$\phi\left(\exp(X)\right) = \exp(D\phi(e)X).$$

We get a bijection

$$\mathrm{Hom}_{\mathrm{loc}}(G, H) \quad \leftrightarrow \quad \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}).$$

In particular, two Lie groups are locally isomorpic if and only if the Lie algebras are isomorphic.

Proof. The assertion $\phi\left(\exp(X)\right) = \exp\left(D\phi(e)X\right)$ is clear by definition of the differential. Let $\phi, \psi : G \to H$ satisfy $D\phi(e) = D\psi(e)$. Then they agree on every 1PSG, hence are equal, so the map is injective.

Conversely, let $\rho : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then
$\phi(\exp(X)) = \exp(\rho(X))$ is a local homomorphism by Corollary 1.8.7. The map ρ is teh differential of ϕ. □

Proposition 1.9.8. *The group* SU(2) *is simply connected. Differentiation yields a bijection*

$$\mathrm{Hom}\left(\mathrm{SU}(2), \mathrm{GL}_n(\mathbb{C})\right) \quad \leftrightarrow \quad \mathrm{Hom}\left(\mathfrak{su}(2), \mathfrak{gl}_n(\mathbb{C})\right).$$

Proof. We write

$$\begin{aligned} \mathrm{SU}(2) &= \left\{A \in \mathrm{M}_2(\mathbb{C}) | A^*A = I,\ \det(A) = 1\right\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}. \end{aligned}$$

The last representation shows that SU(2) is the set of vectors of length 1 in the space $\mathbb{C}^2 = \mathbb{R}^4$, hence the 3-sphere S^3. The latter is simply-connected as it has Dimension > 1 and when you remove one point, you get $\mathbb{R}^3$ after stereographic projection. The latter is contractible, hence simply connected. The rest follows from Theorem 1.9.7 and Theorem 1.9.4. □

* * *

1.10 The universal envelope

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorem 1.10.1. *There is an asociative algebra* $U(\mathfrak{g})$ *with unit and a linear injective map* $\phi : \mathfrak{g} \to U(\mathfrak{g})$ *such that*

(a) $\phi([X, Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$ *for all* $X, Y \in \mathfrak{g}$,

(a) *the algebra* $U(\mathfrak{g})$ *is generated by the image* $\phi(\mathfrak{g})$,

(a) *the algebra* $U(\mathfrak{g})$ *has the universal property, that for every associative algebra* $\mathcal{B}$ *and every Lie-homomorphism* $\eta : \mathfrak{g} \to \mathcal{B}$ *there is exactly one algebra homomorphism* $\alpha : U(\mathfrak{g}) \to \mathcal{B}$, *which extends* η, *i.e., which makes the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[dr, "\eta"'] & U(\mathfrak{g}) \arrow[d, dotted, "\exists!\alpha"] \\ & \mathcal{B} \end{tikzcd}$$

commutative.

These properties determine the algebra $U(\mathfrak{g})$ *up to isomorphy. It is called the* **universal enveloping algebra** *of* $\mathfrak{g}$.

Remark 1.10.2. The universal property can also be formulated by saying that there is a bijection

$$\mathrm{Hom}_{\mathrm{alg}}(U(\mathfrak{g}),\mathcal{B}) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g},\mathcal{B}),$$

which is functorial in $\mathfrak{g}$ and $\mathcal{B}$. One says that the functor $\mathcal{Lie} \to \mathcal{Alg}$, sending $\mathfrak{g}$ to $U(\mathfrak{g})$ is **left adjoint** to the forgetful functor $\mathcal{Alg} \to \mathcal{Lie}$. Here $\mathcal{Alg}$ and $\mathcal{Lie}$ are the categories of algebras and Lie algebras respectively.

Proof of the theorem. For a vector space V over $\mathbb{K}$ let

$$TV = \mathbb{K} \oplus V \oplus \Big(V \otimes V\Big) \oplus \Big(V \otimes V \otimes V\Big) \oplus \dots$$

be the **tensorial algebra**. This is an associative algebra over $\mathbb{K}$ with the product

$$(v,w) \mapsto v \otimes w.$$

The natural map $\psi : V \to TV$ has the following universal property: If $\mu : V \to \mathcal{A}$ is a linear map, where $\mathcal{A}$ is an associative algebra, then there is exactly one algebra homomorphism $\theta : TV \to \mathcal{A}$ extending μ. For the proof one defines $\theta(v_1 \otimes \dots \otimes v_k) = \mu(v_1)\cdots\mu(v_k)$ and one easily sees that the property is satisfied.

Let $T\mathfrak{g}$ be the tensorial algebra of $\mathfrak{g}$ and let $I \subset T\mathfrak{g}$ be the two sided ideal generated by all elements of the form

$$[X,Y] - XY + YX,$$

where X and Y run through all elements of $\mathfrak{g}$. Then $U(\mathfrak{g}) := T\mathfrak{g}/I$ is an algebra and the natural embedding $\mathfrak{g} \to T\mathfrak{g}$ yields the injective linear map $\phi : \mathfrak{g} \to U(\mathfrak{g})$. The algebra $U(\mathfrak{g})$ is generated by the image of ϕ and the condition

$$\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$$

has been forced in, so that ϕ is a Lie algebra homomorphism.

Now for the universal property. Let $\eta : \mathfrak{g} \to \mathcal{B}$ be a Lie homomorphism to an associative algebra $\mathcal{B}$. The universal property of the tensorial algebra yields an algebra homomorphism $t : T\mathfrak{g} \to \mathcal{B}$ extending η. As η is a Lie homomorphism, we get $t(I) = 0$, so t factors uniquely through $U(\mathfrak{g})$. □

Theorem 1.10.3 (Poincaré-Birkhoff-Witt). *Let $\mathfrak{g}$ be a Lie algebra and $X_1, \dots, X_n$ a basis of the vactor space $\mathfrak{g}$. Then the elements of the form*

$$X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n},$$

$k \in \mathbb{N}_0^n$, are a basis of the vector space $U(\mathfrak{g})$.

Definition 1.10.4. A **filtration** on an associative algebra A is a sequence of subspaces $F_0 \subset F_1 \subset F_2 \subset \dots$ such that

$$A = \bigcup_k F_k \qquad \text{and} \qquad F_k F_l \subset F_{k+1}.$$

A **gradation** is a family of subspaces $G_k \subset A$ such that

$$A = \bigoplus_k G_k \qquad \text{and} \qquad G_k G_l \subset G_{k+1}.$$

Lemma 1.10.5. *Let (F_k) be a filtration on the algebra A. Set $F_{-1} = 0$. Then*

$$GR(A) = \bigoplus_{k=0}^{\infty} F_k / F_{k-1}$$

*is a graded algebra. We write $GR(A)_k = F_k/F_{k-1}$ and call it the k-**th graded piece**. For each k, let $[v_{k,1}], \dots, [v_{k,n_k}]$ be a basis of $GR(A)_k$, with $v_{k,j} \in A$. Then $(v_{k,j})_{\substack{k \geq 0 \\ 0 \leq j \leq n_k}}$ is a basis of A.*

Proof. We show by induction on $K \in \mathbb{N}_0$, that $(v_{k,j})_{k \leq K}$ is a basis of F_K. For $K = 0$ there is nothing to prove. So let $v \in F_{K+1}$. Then there are uniquely determined coefficients $\lambda_{K+1,j}$such that with $v' = \sum_{j=0}^{n_{K+1}} \lambda_{K+1,j} v_{K+1,j}$ one has $v - v' \in F_k$. By induction hypothesis, there are uniquely determined coefficients $\mu_{k,j}$ such that

$$v - v' = \sum_{k=0}^{K} \sum_{j=1}^{n_k} \lambda_{k,j} v_{k,j}.$$

The existence and uniqueness of the $\lambda_{k,j}$ proves the claim. □

Proof of the theorem. We want to apply the lemma with the gradation of $U(\mathfrak{g})$ coming from the degree. In the relation $XY - YX - [X, Y] = 0$, as the last term is of lower degree, it follows that the associated graded algebra $GR(U(\mathfrak{g}))$ is isomorphic to the graded algebra one obtains from the trivial Lie bracket $[X, Y] = 0$. This, however, is the algebra of polynomials in any basis of $\mathfrak{g}$ and thus has a basis of the described type. $\square$

Definition 1.10.6. Let G be a Lie group. We write $\mathbb{D}(G)$ for the (associative) algebra of all left-invariant differential operators on G, i.e., of all differentialoperators $D : C^\infty(G) \to C^\infty(G)$ with

$$L_y D = D L_y$$

for every $y \in G$.

Definition 1.10.7. For a smooth map $\phi : M \to N$ between manifolds let

$$\begin{aligned} \phi^* : C^\infty(N) &\to C^\infty(M), \\ f &\mapsto f \circ \phi. \end{aligned}$$

Assume that ϕ is an isomorphism. For a differentialoperator D on M let $\phi_* D$ be defined by

$$\phi_* D = (\phi^{-1})^* \circ D \circ \phi^*.$$

Lemma 1.10.8. (a) *Let $\phi : M \to N$ be an isomorphism of smooth manifolds. If D is a differential operator or order k, then $\phi_* D$ again is a differentialoperator of order k.*

(b) *Every left-invariant differential operator on a Lie group G has finite order.*

Proof. (a) As ϕ defines new local charts, part (a) follows from the independence of the order of the choice of charts.

(b) Let $D \in \mathbb{D}(G)$. Let U be an open unit neighbourhood, on which D has order k and let $x \in G$¿ Then xU is a neighbourhood of x, on which D has, by part (a), again order k. $\square$

Definition 1.10.9. A **filtration** of a vector space V is a sequence of subspaces (V_j) with $V_j \subset V_{j+1}$ and $V = \bigcup_{j=1}^\infty V_j$.

A **gradation** is a sequence of subspaces G_i such that $V = \bigoplus_i G_i$.

A given filtration on V induces a graded space $GR(V) = \bigoplus_i V_i / V_{i-1}$.

Definition 1.10.10. A linear map $\phi : V \to W$ between fitered vector spaces is called a **filtered map**, if $\phi(V_k) \subset W_k$ for every $k \in \mathbb{N}_0$. Such a map then induces a linear map $\overline{\phi} : \mathrm{Gr}(V) \to \mathrm{Gr}(W)$.

Lemma 1.10.11. *Let $\phi : V \to W$ be filtered. If $\overline{\phi}$ is bijective, then ϕ is bijective.*

Proof. Let $x \in \ker(\phi) \setminus 0$ and let $k \in \mathbb{N}_0$ minimal with $x \in V_k$. Then $\phi(x) + W_{k-1} \neq 0$, so $\phi(x) \neq 0$.

For surjectivity we show by induction on $k \geq -1$, that $\phi(V_k) = W_k$. For $k = -1$ there is nothing to show. Let $\phi : V_k \to W_k$ be surjective and let $y \in W_{k+1}$. Then there exists $x \in V_{k+1}$, such that $x + V_k$ maps to $y + W_k$, hence $y - \phi(x) \in W_k$. By induction hypothesis there is $z \in V_k$, such that $\phi(z) = y - \phi(x)$, so $y = \phi(z + x)$. □

Definition 1.10.12. As every vector field X is a differential operator, we get an inclusion $\mathfrak{g} \hookrightarrow D(G)$. The universal property of the enveloping algebra yields an algebra homomorphism

$$\theta : U(\mathfrak{g}) \to D(G).$$

Theorem 1.10.13. *The algebra homomorphism θ is an isomorphism, i.e., the algebra $D(G)$ of all left-invariant differential operators is isomorphic to the universal enveloping algebra.*

Proof. Fix a coordinate system x_j around the point $e \in G$. For $D \in \mathbb{D}(G)$ write $D = \sum_\alpha c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$. Let $\eta(D) = \sum_\alpha c_\alpha(e) x^\alpha \in \mathbb{R}[x_1, x_2, \dots, x_n]$. We show that η is a linear filtered bijection.

Firstly, η maps the unit to the unit, so it is bijective on $U_0(\mathfrak{g}) \to \mathbb{R}$. Let $X_j = \frac{\partial}{\partial x_j} \in T_e G \cong \mathfrak{g}$. these X_j span $\mathfrak{g}$. One has

$$X_{j_1} X_{j_2} \cdots X_{j_k} = \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j+k}} + \text{ terms of lower order.}$$

The Poincaré-Birkhoff-Witt Theorem implies, that η mapsthe space $\mathrm{Gr}(U(\mathfrak{g}))$ bijectively to $\mathrm{Gr}(\mathbb{R}[x_j])$ and therefore η is a bijection.

By PBW one can define a linear bijection $\gamma : U(\mathfrak{g}) \to \mathbb{R}[x_1, \dots, x_n]$ by

$$\gamma(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

It is easy to verify that $\theta = \eta^{-1} \circ \gamma$ ist. □

Example 1.10.14. Consider the Lie algebra $\mathfrak{g} = s\ell_2(\mathbb{R})$ of all 2×2-matrices of trace zero. We choose a standard basis of $s\ell_2(\mathbb{R})$ consisting of

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A computation shows:

$$\begin{aligned} [H,E] &= 2E, \\ [H,F] &= -2F, \\ [E,F] &= H. \end{aligned} \tag{$*$}$$

Let

$$\Omega = H^2 + 2EF + 2FE \ \in \ U(\mathfrak{g}).$$

The element Ω is called the **Casimir element**, or **Casimir operator** in $U(\mathfrak{g})$. It lies in the center of the algebra $U(\mathfrak{g})$.

Proof. We need to show that Ω commutes with H, E, F, as they generate the algebra $U(\mathfrak{g})$. Using the fact, that in any algebra one has

$$[XY,Z] = XYZ - ZXY = XYZ - XZY + XZY - ZXY = X[Y,Z] + [X,Z]Y,$$

the claim is established in a straightforward manner. □

* * *

1.11 Abelian groups

Lemma 1.11.1. *Let V be a finite-dimensional $\mathbb{R}$-vector space and let $\Lambda \subset V$ be a discrete subgroup. Then there exists a basis $v_1, \dots, v_n$ of V and a unique $0 \le k \le n$, such that*

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k.$$

Proof. Replacing V with the span of Λ, we can assume that Λ spans the space V.

For a given V-basis $v_1, \dots, v_n \in \Lambda$ let

$$\mathcal{F} = \left\{ \lambda_1 v_1 + \cdots + \lambda_n v_n : 0 \le \lambda_j < 1 \right\}.$$

If $\mathcal{F}$ contains another vector $v \in \Lambda$, then one can replace one of the v_j by v and shrink $\mathcal{F}$ in this way. As Λ is discrete, this process terminates and then one has found a basis. □

Definition 1.11.2. A **torus** is a Lie group isomorphic to $\mathbb{T}^n$ for an $n \in \mathbb{N}_0$.

Lemma 1.11.3. *For every torus T there exists $\tau \in T$, such that the group $\tau^{\mathbb{Z}}$ generated by τ is dense in T.*

Such an element τ is called a **topological generator** *of T. The set of all topological generators is dense in T.*

Proof. As $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, we can assume $T = (\mathbb{R}/\mathbb{Z})^n$ and write this group additively. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ numbers, which are linearly independent over $\mathbb{Q}$ and let

$$\tau = (\alpha_1, \alpha_2, \dots, \alpha_n) + \mathbb{Z}^n \in T.$$

Let S be the closure of the group generated by τ. As S is compact, S/S^0 is finite, where S^0 is the connected component of the unit. Let $p : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the projection and let U be the connected component of the pre-image $p^{-1}(S)$. Then U is a connected Lie subgroup of $\mathbb{R}^n$, hence a linear subspace. The projection $p : U \to S^0$ is a local isomorphism, hence inducs an isomorphism of topological groups

$$p : U \Big/ \underbrace{(U \cap \mathbb{Z}^n)}_{\Lambda} \xrightarrow{\cong} (U + \mathbb{R}^n) \Big/ \mathbb{Z}^n = S^0.$$

Hence U/Λ is compact and so Λ contains a basis of U. In particular, U is generated by $U \cap \mathbb{Q}^n$. **Assume** that $U \neq \mathbb{R}^n$. Then there is a $0 \neq \mu \in \mathbb{Q}^n$, such that

$$\mu_1 u_1 + \cdots + \mu_n u_n = 0$$

for every $u \in U$. In particular, one has this for $u = \tau$, contradicting the linear indipendence of $\alpha_1, \dots, \alpha_n$!

The second assertion follows from the construction, since the set of vectors τ we have used in the construction, is dense. □

Theorem 1.11.4. *A connected abelian Lie group G is a product of a compact torus and a vector space, i.e.,*

$$G \cong \mathbb{T}^r \times \mathbb{R}^d.$$

For a connected abelian Lie group the exponential map $\exp : \mathfrak{g} \to G$ *is a surjective continuous group homomorphism, the kernel of which is a discrete subgroup.*

Proof. As G is abelian, multiplication $m : G \times G \to G$ is a group homomorphism. We claim that the Lie derivative $Lm : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ coincides with the addition $X \oplus Y \mapsto X + Y$. This follows, since Lm is linear and equals the identity on every summand. By Theorem 1.7.12 the diagram

$$\begin{tikzcd}
\mathfrak{g} \oplus \mathfrak{g} \arrow[r, "Lm"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

commutes, so $\exp(X + Y) = \exp(X)\exp(Y)$, which means that $\exp : \mathfrak{g} \to G$ is a group homomorphism. As G is connected, the image of $\exp : \mathfrak{g} \to G$ generates the group G. On the other hand, the image is already a subgroup, hence exp is surjective. This means that G, as a group, is isomorphic to $\mathfrak{g}/K$, where $K = \ker(\exp)$ is a closed subgroup of the finite-dimensional $\mathbb{R}$-vector space $\mathfrak{g}$. The dimension of G equlas the dimension of $\mathfrak{g}$, so Λ must be a discrete subgroup. The theorem therefore follows from Lemma 1.11.1. □

* * *

1.12 Matrix groups

Definition 1.12.1. A **matrix Lie group** or **matrix group** is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$.

Lemma 1.12.2. (a) *Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be a continuous map with the property, that that for all $s, t \in \mathbb{R}$ with $|s|, |t|, |s + t| < \varepsilon$ one has $\alpha(s) + \alpha(t) = \alpha(s + t)$. Then there is a $v \in \mathbb{R}^n$ with $\alpha(t) = tv$.*

(b) *Let $\gamma : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ be a continuous group homomorphism. Then there is a uniquely determined matrix $X \in \mathrm{M}_n(\mathbb{R})$, such that $\gamma(t) = \exp(tX)$. In particular, γ is a smooth* *map and one has*

$$\gamma'(t) = \frac{d}{dt}\exp(tX) = X\exp(tX).$$

Proof. (a) First w extend α to a continuous group homomorphism $\tilde{\alpha} : \mathbb{R} \to \mathbb{R}^n$. Let $t \in \mathbb{R}$. Then there is $n \in \mathbb{N}$ with $|t/n| < \varepsilon$. We set

$$\tilde{\alpha}(t) = n\alpha(t/n).$$

We need to show well-definedness. Fior thei let $m \in \mathbb{N}$ another number with $|t/m| < \varepsilon$. Then one has $n\alpha(t/n) = mn\alpha(t/mn) = m\alpha(t/m)$, so $\tilde{\alpha}$ is well-defined. For given $s, t \in \mathbb{R}$ let $n \in \mathbb{N}$ such that $(|s| + |t|)/n < \varepsilon$. Then

$$\tilde{\alpha}(s + t) = n\alpha(s/n + t/n) = n\alpha(s/n) + n\alpha(t/n) = \tilde{\alpha}(s) + \tilde{\alpha}(t).$$

For continuity: as $\tilde{\alpha}$ is a group homomorphism, it suffices to show continuity at the point $t = 0$. This, however, was assumed.

We can assume that α is defined on all of $\mathbb{R}$. Set $v = \alpha(1)$. For $k \in \mathbb{N}$ one has $\alpha(k) = \alpha(1) + \cdots + \alpha(1) = k\alpha(1) = kv$. The sam efollows for $k \in \mathbb{Z}$, since $\alpha(-t) = -\alpha(t)$. For $p/q \in \mathbb{Q}$ one has $q\alpha(p/q) = \alpha(p) = pv$, so $\alpha(p/q) = \frac{p}{q}v$. By continuity one getst $\alpha(t) = tv$ for all $t \in \mathbb{R}$.

(b) If $s, t \in \mathbb{R}$ are sufficiently small, then $\left\|1 - \gamma(s)\right\|, \left\|1 - \gamma(t)\right\|, \left\|1 - \gamma(st)\right\| < 1$ and so

$$\log\big(\gamma(s + t)\big) = \log\big(\gamma(s)\gamma(t)\big) = \log\big(\gamma(s)\big) + \log\big(\gamma(t)\big).$$

We infer that $t \mapsto \log(\gamma(t))$ extends to a continuous group homomorphism $\mathbb{R} \to \mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. By part (a) there exists a matrix $X \in \mathrm{M}_n(\mathbb{R})$ such that $\log(\gamma(t)) = tX$, so $\gamma(t) = \exp(tX)$. Finally for differentiability. For $h \in \mathbb{R}$, $h \neq 0$ one has

$$\begin{aligned}
\frac{1}{h}\Big(\exp((t + h)X) - \exp(tX)\Big) &= \frac{1}{h}\Big(\exp(tX)\exp(hX) - \exp(tX)\Big)\\
&= \exp(tX)\,\frac{1}{h}\Big(\exp(hX) - 1\Big)\\
&= \exp(tX)\,\frac{1}{h}\sum_{k=1}^{\infty}\frac{h^k}{k!}X^k\\
&= \exp(tX)\,\sum_{k=1}^{\infty}\frac{h^{k-1}}{k!}X^k.
\end{aligned}$$

For $h \to 0$ konvergiert dies gegen

$$\exp(tX)X = X\exp(tX). \qquad \square$$

Theorem 1.12.3. *Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a matrix group and let $\mathfrak{g}$ be its Lie algebra. For every $X \in \mathfrak{g}$ there is a unique matrix $\tilde{X} \in \mathrm{M}_n(\mathbb{R})$ such that*

$$\exp(t\tilde{X}) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\tilde{X}\right)^k$$

is the integral curve of X through the point e. One gets a bijection

$$\mathfrak{g} \xrightarrow{\cong} \tilde{\mathfrak{g}} = \left\{X \in \mathrm{M}_n(\mathbb{R}) : \exp(tX) \in G\ \forall_{t\in\mathbb{R}}\right\}.$$

This map is a Lie algebra homomorphism, i.e.,

$$\widetilde{[X,Y]} = [\tilde{X}, \tilde{Y}],$$

where on the right hand side we have the commutator bracket. In particular, $\tilde{\mathfrak{g}}$ is closed under the commutator bracket.

Proof. Follows from the lemma. $\square$

Examples 1.12.4. (a) The Lie algebra of the special linear group $\mathrm{SL}_n(\mathbb{R})$ is

$$sl_n(\mathbb{C}) = \left\{X \in \mathrm{M}_n(\mathbb{R}) : \mathrm{tr}(X) = 0\right\}.$$

(b) We consider complex matrices in $\mathrm{M}_n(\mathbb{C})$ as real matrices in $\mathrm{M}_{2n}(\mathbb{R})$. Then the unitary group $U(n) \subset \mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R})$ is a matrix group. The Lie algebra of the unitary group $\mathrm{U}(n)$ is the real vector space of skew-symmetric matrices:

$$u(n) = \left\{X \in \mathrm{M}_n(\mathbb{C}) : X^* = -X\right\},$$

where $X^* = \overline{X}^t$ is the adjoint matrix.

Proof. (a) Let X be in the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$. Then one has

$1 = \det(\exp(tX)) = \exp(t\operatorname{tr}(X))$ for every $t \in \mathbb{R}$. Taking the derivative in t yields

$$0 = \frac{d}{dt}\exp(t \operatorname{tr} X)\bigg|_{t=0} = \operatorname{tr}(X).$$

Conversely assume $\operatorname{tr}(X) = 0$. Then $\det(\exp(tX)) = \exp(\operatorname{tr}(tX)) = \exp(0) = 1$ for every $t \in \mathbb{R}$, so $X \in sl_n(\mathbb{R})$.

(b) Let X be in $u(n)$. Then one has

$$\exp(tX)^* = \overline{\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k\right)}^t = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(X^*)^k\right) = \exp(tX^*) = \exp(-tX) = \exp(tX)^{-1}.$$

Therefore $\exp(tX) \in U(n)$.

Conversely assume $\exp(tX) \in U(n)$ for every $t \in \mathbb{R}$. Then

$$X^* = \frac{d}{dt}\bigg|_{t=0} \exp(tX^*) = \frac{d}{dt}\bigg|_{t=0} \exp(tX)^* = \frac{d}{dt}\bigg|_{t=0} \exp(tX)^{-1} = \frac{d}{dt}\bigg|_{t=0} \exp(-tX) = -X. \qquad \square$$

* * *

2 Representations

2.1 Vector valued integrals

Definition 2.1.1. Let G be a Lie group. A function $f : G \to V$ into a Banach space V is called **integrable**, if the is a vector $v_0 \in V$, such that

$$\alpha(v_0) = \int_X \alpha(f(x))\, d\mu(x)$$

for every continuous linear functional $\alpha : V \to \mathbb{C}$.

Lemma 2.1.2. *If f is integrable, the vector v_0 is uniquely determined. We then write $v_0 = \int_G f\, d\mu$. The defining property can be written in the form*

$$\alpha\left(\int_G f\, d\mu\right) = \int_G \alpha(f)\, d\mu.$$

Proof. As continuous linear functiomnals separate points, v_0 is unique. $\square$

Theorem 2.1.3. *Let V be a Banach space, G a Lie group and μ a Haar measure on G. Let $f : G \to V$ be contiunuous with compact support, i.e., $f \in C_c(G, V)$. Then f is intgerable.*

The proof will fill the rest of the section.

Definition 2.1.4. A **step function** is a map $s : G \to V$ of the form

$$s = \sum_{j=1}^{n} \mathbf{1}_{A_j} v_j,$$

where $v_j \in V$ and the $A_j \subset G$ are measurable sets of finite Haar measure.

Lemma 2.1.5. *Every step function $s = \sum_{j=1}^{n} \mathbf{1}_{A_j} v_j$ is integrable. The integral is*

$$\int_G s\, d\mu = \sum_{j=1}^{n} \mu(A_j)\, v_j \in V.$$

Proof. For $\alpha \in V'$ one has

$$\alpha\left(\sum_{j=1}^{n} \mu(A_j)\, v_j\right) = \sum_{j=1}^{n} \mu(A_j)\, \alpha(v_j) = \int_G \alpha(s)\, d\mu.$$

Therefore $\sum_{j=1}^{n} \mu(A_j)\, v_j$ is the integral. □

Lemma 2.1.6. *The map $s \mapsto \int_G s\, d\mu$ is linear on the vector space of all step functions. For a step function s one has*

$$\left\| \int_G s\, d\mu \right\| \le \int_G \|s\|\, d\mu.$$

Proof. Linearity is clear. For $s = \sum_{i=1}^{n} \mathbf{1}_{A_i} v_i$ one has

$$\left\| \int_G s\, d\mu \right\| = \left\| \sum_i \mu(A_i) v_i \right\| \le \sum_i \mu(A_i)\, \|v_i\| = \int_G \|s\|\, d\mu.$$ □

Definition 2.1.7. A measurable function $f : X \to V$ is called **Bochner approximable**, if there is asequence (s_i) of step functions, such that

$$\int_G \left\| f - s_i \right\|\, d\mu \to 0.$$

In this case the sequence (s_i) is called an **approximating sequence**.

Lemma 2.1.8. *Every $f \in C_c(G, V)$ is Bochner-approximable.*

Proof. Let $K = \operatorname{supp}(f)$ and let $\varepsilon > 0$. For every $x \in K$ the set $U_x = f^{-1}\left(B_\varepsilon(f(x)\right)$ is an open neighbourhood of x and as K is compact, there are $x_1, \dots, x_n \in K$ with

$$K \subset U_{x_1} \cup \dots \cup U_{x_n}.$$

Let $A_1 = K \cap U_1, A_2 = K \cap U_2 \smallsetminus A_1, \dots, A_n = K \cap U_n \smallsetminus (A_1 \cup \dots \cup A_{n-1})$. The A_i are pairwise disjoint. Further let $v_i = f(x_i)$ for $i = 1, \dots, n$. The step function

$$s = \sum_{i=1}^{n} \mathbf{1}_{A_i} v_i$$

satisfies $\operatorname{supp}(s) = K$ and $\left\| f(x) - s(x) \right\| < \varepsilon$ for every $x \in K$. Therefore

$$\int_G \left\| f - s \right\| \, d\mu = \int_K \left\| f - s \right\| \, d\mu < \varepsilon \mu(K).$$

As $\varepsilon > 0$ is arbitrary, the claim follows. □

Proposition 2.1.9. *Let $f : X \to V$ be Bochner approximable. Then f is integrable. More precisely one has: For every approximating sequence $(s_j)_{j \in \mathbb{N}}$ the sequence of integrals $\left(\int_X s_j \, d\mu\right)_j$ converges. The limit v_0 is independend of the seuquence and equals the integral of f.*

Proof. According to Lemma 2.1.6 one has

$$\begin{aligned} \left\| \int_G s_i \, d\mu - \int_X s_j \, d\mu \right\| &= \left\| \int_G (s_i - s_j) \, d\mu \right\| \\ &\le \int_G \left\| s_i - s_j \right\| \, d\mu \\ &\le \int_G \left\| s_i - f \right\| + \left\| f - s_j \right\| \, d\mu \\ &= \int_G \left\| s_i - f \right\| \, d\mu + \int_G \left\| f - s_j \right\| \, d\mu. \end{aligned}$$

The two sequences at the end tend to zero, so $\int_G s_i \, d\mu$ is a Cauchy sequence, hence convergent. The limit is independent of the sequence, since, if t_i is a second approximating sequnece, then $r_{2i} = s_i$ and $r_{2i-1} = t_i$ again is an approximating sequence. Finally we show, that v_0 is the integral. For this let $\alpha \in V'$. Then one has

$$\begin{aligned}\left|\alpha(v_0) - \int_G \alpha(f)\,d\mu\right| &= \left|\lim_i \alpha\left(\int_G s_i\,d\mu\right) - \int_G \alpha(f)\,d\mu\right| \\ &= \left|\lim_i \int_G \alpha(s_i)\,d\mu - \int_G \alpha(f)\,d\mu\right| \\ &= \left|\lim_i \int_G \alpha(s_i) - \alpha(f)\,d\mu\right| \\ &= \lim_i \left|\int_G \alpha(s_i - f)\,d\mu\right| \\ &\le \lim_i \int_G |\alpha(s_i - f)|\,d\mu \\ &\le \lim_i \int_G \|\alpha\|_{\text{op}} \left\|s_i - f\right\| \,d\mu = 0.\end{aligned}$$

The Proposition is proven and so is Theorem 2.1.3. □

Corollary 2.1.10. *Let $f \in C_c(G, V)$. Then one has*

(a) $\left\|\int_G f\,d\mu\right\| \le \int_G \left\|f\right\|\,d\mu.$

(b) *For a continuous linear operator $T : V \to W$ between Banach spaces one has*

$$T\left(\int_G f\,d\mu\right) = \int_G T(f)\,d\mu.$$

The map $C_c(G, V) \to V$, $f \mapsto \int_G f\,d\mu$ is linear.

Proof. (a) As in the proof of Lemma 2.1.8, using Proposition 2.1.9, we find for given $\varepsilon > 0$ a step function s with $\mathrm{supp}(s) = K = \mathrm{supp}(f)$ and $\left\|s(x) - f(x)\right\| < \varepsilon$ for every

$x \in K$, as well as $\left\| \int_G f \, d\mu - \int_G s \, d\mu \right\| < \varepsilon$. It follws that

$$\begin{aligned}\left\| \int_G f \, d\mu \right\| &\le \left\| \int_G f \, d\mu - \int_G s \, d\mu \right\| + \left\| \int_G s \, d\mu \right\| \\ &< \varepsilon + \int_G \|s\| \, d\mu \\ &\le \varepsilon + \int_K \underbrace{\|s - f\|}_{<\varepsilon} \, d\mu + \int_G \|f\| \, d\mu \\ &\le \varepsilon + \varepsilon\mu(K) + \int_G \|f\| \, d\mu.\end{aligned}$$

For $\varepsilon \searrow 0$ the claim follows.

(b) Let $\alpha \in W'$. Then $\alpha \circ T \in V'$ and therefore

$$\alpha\left(T\left(\int_G f \, d\mu\right)\right) = \int_G \alpha\left(T(f)\right) \, d\mu.$$

The claim follows.

For linearity we show additivity, scalar multiplicativity follows similarly. For $f, g \in C_c(G, V)$ and $\alpha \in V'$ one has

$$\begin{aligned}\alpha\left(\int_G f + g \, d\mu\right) &= \int_G \alpha(f + g) \, d\mu \\ &= \int_G \alpha(f) + \alpha(g) \, d\mu = \int_G \alpha(f) \, d\mu + \int_G \alpha(g) \, d\mu \\ &= \alpha\left(\int_G f \, d\mu\right) + \alpha\left(\int_G g \, d\mu\right) \\ &= \alpha\left(\int_G f \, d\mu + \int_G g \, d\mu\right).\end{aligned}$$

As α is arbitrary, we get additivity. □

2.2 Representations

Definition 2.2.1. Let V be a Banach space. A **representation** of a Lie group G on V is a group homomorphism $\pi : G \to \mathrm{GL}(V)$, such that the map

$$\begin{aligned} G \times V &\to V, \\ (x, v) &\mapsto \pi(x)v \end{aligned}$$

is continuous.

Lemma 2.2.2. *Let (π, V) be a representation. For every $x \in G$ the linear operator $\pi(x)$ is continuous, so has finite operator norm $\|\pi(x)\|_{\mathrm{op}}$. The map $x \to \|\pi(x)\|_{\mathrm{op}}$ is locally bounded on G.*

Proof. The first Remark follows from the definition of a representation. For the second let $x_j \to x$ be a convergent sequence in G. Then $\pi(x_j)v \to \pi(x)v$ for an given v, hence there is a c_v, such that $\left\|\pi(x_j)v\right\| \le c_v$ for every j. By the principle of uniform boundedness, the sequence $\left\|\pi(x_j)\right\|_{\mathrm{op}}$ is bounded. □

Proposition 2.2.3. *Let (π, V) be a representation and let $f \in C_c(G)$. Then one defines a linear operator $\pi(f) : V \to V$ by*

$$\pi(f)v = \int_G f(x)\, \pi(x)v \, dx.$$

This operator is continuous. More precisely one has for $K = \mathrm{supp}(f)$, that

$$\left\|\pi(f)\right\|_{\mathrm{op}} \le \sup_{x \in K} \|\pi(x)\|_{\mathrm{op}} \int_K |f(x)| \, dx.$$

Proof. As $G \to V, x \mapsto f(x)\pi(x)v$ is continuous with compact support, the inmtegral exists. Since the vector valued integral is linear, we conclude that $\pi(f)$ is linear. Finally let $K = \mathrm{supp}(f)$ and $C = \sup_{x \in K} \|\pi(x)v\|$. Then the map $x \mapsto \|\pi(x)\|_{\mathrm{op}}$ is bounded on K. For $v \in V$ this implies

$$\begin{aligned} \left\|\pi(f)v\right\| &= \left\| \int_G f(x)\pi(x)v \, dx \right\| \\ &\le \int_G \left\| f(x)\pi(x)v \right\| \, dx \\ &\le \sup_{x \in K} \|\pi(x)\|_{\mathrm{op}} \int_G |f(x)| \, \|v\| \, dx. \end{aligned}$$

□

2.3 Smooth vectors

Definition 2.3.1 (Smooth maps to a Banach space)**.** Let V be a Banach space. A map $f : \mathbb{R}^n \to V$ is called **continuously partially differentiable**, if for every $j = 1, \dots, n$ the limit

$$D_j f(x) = \lim_{h \to 0} \frac{1}{h} \left[f(x + he_j) - f(x) \right]$$

exists in every point $x \in \mathbb{R}^n$ and depends continuously on x. The function is called two times continuously differentiable, if every partial derivative is continuously differentiable and so on.

The fanction is called **smooth**, if it is infinitely differentiable.

Definition 2.3.2. Let G be a Lie group and let (π, V_π) be a representation. A vector $v \in V$ is called a **smooth vector**, if the map $G \to V_\pi$, $x \mapsto \pi(x)v$ is smooth. The smooth vectors form a linear subspace V_π^∞.

Theorem 2.3.3. *Let (π, V_π) be a representation of a Lie group G. Then V_π^∞ is dense in V_π. If V_π is finite-dimensional, every vector is smooth.*

Proof. Let $f \in C_c^\infty(G)$. We claim that for every vector $v \in V_\pi$ one has

$$\pi(f)v \in V_\pi^\infty.$$

For this note

$$\begin{aligned} \pi(x)\pi(f)v &= \pi(x) \int_G f(y)\pi(y)v\,dy \\ &= \int_G f(y)\pi(xy)v\,dy = \int_G f(x^{-1}y)\pi(y)v. \end{aligned}$$

As $f \in C_c^\infty(G)$, this expression depends smoothly on x .

Let $\varepsilon > 0$ and let $v \in V_\pi$. Then there is an open unit neighbourhood U, such that for every $u \in U$ one has

$$\|\pi(u)v - v\| < \varepsilon.$$

Let $f \geq 0$ with $\int_G f(x)\,dx = 1$ and $\operatorname{supp}(f) \subset U$. Then it follows

$$\begin{aligned}\left\|\pi(f)v - v\right\| &= \left\|\int_U f(x)(\pi(x)v - v)\,dx\right\| \\ &\leq \int_G f(x) \underbrace{\|\pi(x)v - v\|}_{<\varepsilon}\,dx < \varepsilon.\end{aligned}$$

This means that the space V_π^∞ is dense in V_π.

If V_π is finite-dimensional, then V_π is the only dense subspace and so we get $V_\pi^\infty = V_\pi$. □

Example 2.3.4. Let G be a Lie group and let $\phi \in C_c^\infty(G) \subset L^2(G)$. Then ϕ is a smooth vector with respect to the left translation representatiopn of G on $L^2(G)$.

Definition 2.3.5. Let $\mathfrak{g}$ be a Lie algebra. A **representation** of $\mathfrak{g}$ on a vector space V is a lineae map $\pi : \mathfrak{g} \to \operatorname{End}(V)$ with the property

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

for all $X, Y \in \mathfrak{g}$. We call a representation π **irreducible**, if it does not contain any proper sub-representation, i.e., if the only $\pi(\mathfrak{g})$-stable subspaces are $\{0\}$ and V_π. Note that in the case of Lie algebras not topology is involved.

Lemma 2.3.6. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. If (π, V_π) is a representation of G, then*

$$\pi'(X)v = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX))\,v, \qquad v \in V$$

is a representation of $\mathfrak{g}$ on V_π^∞. We call π' the **derived representation**. *There will be no danger of confusion, so we simply write $\pi(X)$ instead of $\pi'(X)$.*

Proof. For linearity note that $\pi(sX)v = s\pi(X)v$ for every $s \in \mathbb{R}$ and every $v \in V_\pi^\infty$. Let $X, Y \in \mathfrak{g}$. Consider the smooth map $F : \mathbb{R} \times \mathbb{R} \to V_\pi$ with $F(s, t) = \pi(\exp(sX + tY))\,v$.

Then

$$\begin{aligned}\pi(X+Y)v &= \frac{d}{dt}\bigg|_{t=0} F(t,t)\\ &= \frac{d}{dt}\bigg|_{t=0} F(t,0) + \frac{d}{dt}\bigg|_{t=0} F(0,t)\\ &= \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tX)v + \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tY))\\ &= \pi(X)v + \pi(Y)v.\end{aligned}$$

Let $X, Y \in \mathfrak{g}$ as well as $s, t \in \mathbb{R}$ and write $x_s = \exp(sX)$. In Lemma 1.8.1 we saw that

$$\frac{d}{ds}\bigg|_{s=0} x_s Y x_s^{-1} = XY - YX = [X, Y].$$

If $v \in V_\pi^\infty$, then the map

$$\begin{aligned}&\mathfrak{g} \to V_\pi\\ &Y \mapsto \pi(Y)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tY))v\end{aligned}$$

is linear, hence it commutes with derivatives. One has

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} \pi(x_s Y x_s^{-1})v &= \pi\left(\frac{d}{ds}\bigg|_{s=0} x_s Y x_s^{-1}\right)v\\ &= \pi\left([X, Y]\right)v.\end{aligned}$$

On the other hand,

$$\begin{aligned}\pi(x_s Y x_s^{-1}) &= \frac{d}{dt}\bigg|_{t=0} \pi(\exp(t x_s Y x_s^{-1}))\\ &= \frac{d}{dt}\bigg|_{t=0} \pi(x_s \exp(tY) x_s^{-1})\\ &= \frac{d}{dt}\bigg|_{t=0} \pi(x_s)\pi\left(\exp(tY)\right)\pi(x_s^{-1})\\ &= \pi(x_s)\pi(Y)\pi(x_s^{-1}).\end{aligned}$$

Together we get

$$\begin{aligned}\pi([X,Y]) &= \frac{d}{ds}\Big|_{s=0} \pi(x_s)\pi(Y)\pi(x_s^{-1})\\ &= \left(\frac{d}{ds}\Big|_{s=0} \pi(x_s)\right)\pi(Y)\pi(x_0^{-1}) + \pi(x_0)\pi(Y)\left(\frac{d}{ds}\Big|_{s=0} \pi(x_s^{-1})\right)\\ &= \left(\frac{d}{ds}\Big|_{s=0} \pi(\exp(sX))\right)\pi(Y) + \pi(Y)\left(\frac{d}{ds}\Big|_{s=0} \pi(\exp(-sX))\right)\\ &= \pi(X)\pi(Y) - \pi(Y)\pi(X).\end{aligned}$$

□

Theorem 2.3.7. *Let G be connected and (π, V_π) a finite-dimensional representation of G. The following are equivalent:*

(a) *π is irreducible,*

(b) *π' is irreducible.*

If η is another representation of G, then the following are equivalent:

(i) *$\pi \cong \eta$,*

(ii) *$\pi' \cong \eta'$.*

Proof. (a)⇒(b): Let $U \subset V_\pi$ be a $\pi'(\mathfrak{g})$-stable subspace. Consider the Lie group $H = \mathrm{GL}(V)$ and its Lie algebra $\mathfrak{h}$. By definition we have $\pi' = D\pi : T_eG \to T_eH$. Let $X \in \mathfrak{g}$. Then $\pi(\exp(tX))$ is a 1PSG in H, hence is generated by one $Y \in \mathfrak{h}$. It follows $Y = \pi'(X)$ and therefore $\pi(\exp(tX)) = \exp(\pi'(tX)) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \pi(X)^k$ and so U is stable under every 1PSG in G and hence under G. The converse (b)⇒(a) is clear.

In the second part, the non-trivial direction is (ii)⇒(i) and this follows from $\pi(\exp(X)) = \exp(\pi(X))$. □

Theorem 2.3.8. *In this theorem let G be a compact group.*

(a) *Every finite-dimensional representation (ρ, V) can be made unitary, i.e., there is an inner product on V, under which ρ is unitary.*

(b) *Every unitary representation is an orthogonal sum of irreducibles and every irreducible is finite-dimensional.*

(c) *Every representation (π, W) has a finite-dimensional quotient $\neq 0$.*

(d) *In a given representation (ρ, V), the finite dimensioanl subrepresentations are dense. In other wirds, consider the subspace*

$$\tilde{V} = \sum_{E \subset V} E,$$

where the sum runs over all finite-dimensional subrepresentations. Then $\tilde{V}$ is dense in V.

Proof. (a) Let $(.,.)$ be any inner product and set

$$\langle v, w \rangle = \int_G (\rho(k)v, \rho(k)w)\, dk.$$

This inner product makes the representation unitary.

(b) This is part of the Theorem of Peter-Weyl. A proof can be found in Deitmar/Echterhoff.

(c) Let $0 \neq \alpha \in W'$ be a continuous linear functional and let $w \in W$. Consider the **matrix coefficient** $\psi_{\alpha,w} : G \to \mathbb{C}$,

$$\psi_{\alpha,w}(x) = \alpha\left(\pi(x)w\right).$$

The continuity of π implies that $\psi_{\alpha,w}$ is a continuous function on G, hence in $L^2(G)$. One gets $\psi_{\alpha,\pi(y)w}(x) = \psi_{\alpha,w}(xy)$, or $\psi_{\alpha,\pi(y)w} = R_y\psi_{\alpha,w}$. By this the map $\phi : w \mapsto \psi_{\alpha,w}$ is a G-homomorphism $\neq 0$ of π to $(R, L^2(G))$. As $L^2(G)$ is a direct sum of irreducibles, for any projection $P : L^2(G) \to E$ to an irreducible E we have $P(\psi_{\alpha,-}) \neq 0$, which means that $P \circ \phi$ is a G-homomorphism $\neq 0$ on a finite-dimensional representation.

(d) Let U be the closure of $\tilde{V}$. Then the induced representation on V/U has no finite-dimensioanle subrepresentations, or the dual representation on $(V/U)'$ has no finite-dimensional qutotient. By part (c) we have $(V/U) = 0$, so $V/U = 0$ or $V = U$. □

* * *

2.4 Complexification

Definition 2.4.1. If $\mathfrak{g}$ is a real Lie algebra, its **complexification**

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus \mathfrak{g} i$$

becomes a complex Lie algebra by extending the Lie bracket to a bilinear map on $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$.

Similarly, every representation of $\mathfrak{g}$ is a $\mathbb{C}$-linear representation of $\mathfrak{g}_{\mathbb{C}}$.

Remark 2.4.2. It may happen that completely differnet Lie groups have isomorphic complexifications of their Lie algebras, as the following example shows.

Example 2.4.3. Let $G = \mathrm{SL}_2(\mathbb{R})$, $H = \mathrm{SU}(2)$ and let $\mathfrak{g}$ and $\mathfrak{h}$ denote their Lie algebras. Then one has

$$\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{h}_{\mathbb{C}}$$

as complex Lie algebras.

Proof. One has

$$\begin{aligned} \mathfrak{g} &= \Big\{X \in \mathrm{M}_2(\mathbb{R} : \mathrm{tr}(X) = 0\Big\}, \\ \mathfrak{h} &= \Big\{Z \in \mathrm{M}_2(\mathbb{C}) : \mathrm{tr}(Z) = 0,\ Z^* = -Z\Big\}. \end{aligned}$$

The space $\mathfrak{g}$ has the following basis

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad E = \begin{pmatrix} & 1 \\ & \end{pmatrix} \quad F = \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

Here the so called sl(2)-relations:

$$[H,E] = 2E \qquad [H,F] = -2F \qquad [E,F] = H$$

hold. Since E, F, H is a basis, the sl(2)-relations fix the Lie algebra up to isomorphy.

The complexification $\mathfrak{h}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{h}$ has the following basis

$$H' = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad E' = \frac{1}{2}\begin{pmatrix} & 1 \\ -1 & , \end{pmatrix} \quad F' = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

A direct computation shows that H', E', F' also satisfy the sl(2)-relations and the claim follows. □

* * *

2.5 Finite-dimensional representations of SL(2)

We shall determine all finite-dimensional representations of the group

$$\mathrm{SL}_2(\mathbb{R}) = \Big\{A \in M_2(\mathbb{R}) : \det(A) = 1\Big\}.$$

The Lie algebra is the set $s\ell_2(\mathbb{R})$ of all 2×2-matrices of trace zero.

Recall Example 1.10.14 and accordingly let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Recall that

$$\begin{aligned} [H, E] &= 2E, \\ [H, F] &= -2F, \\ [E, F] &= H. \end{aligned} \tag{$*$}$$

Definition 2.5.1. A of a triple (H, E, F) of elements of a Lie algebra $\mathfrak{g}$ is called **sl2-triple**, if the relations $(*)$ hold. An sl2-triple gives an embedding $s\ell_2(\mathbb{R}) \hookrightarrow \mathfrak{g}$.

Let $\pi : s\ell_2(\mathbb{R}) \to \mathrm{End}(V)$ be a Lie algebra representation on a finite-dimensional $\mathbb{C}$-vector space and let $L = \pi(H)$, as well as $A = \pi(E)$ and $B = \pi(F)$. As π is a Lie algebra homomorphism, we get

$$[L, A] = 2A, \quad [L, B] = -2B, \quad [A, B] = L.$$

Lemma 2.5.2. *Let Ω be the Casimir element of $U(\mathfrak{g})$ and let*

$$C = \pi(\Omega) = L^2 + 2AB + 2BA$$

be the **Casimir operator**. *For geven $X \in s\ell_2(\mathbb{R})$ one has*

$$\pi(X)C = C\pi(X).$$

If π is irreducible, then there is $\mu \in \mathbb{C}$, such that $C = \mu\,\mathrm{Id}$.

Proof. As Ω is central in $U(\mathfrak{g})$ we get

$$\pi(X)C = \pi(X)\pi(\Omega) = \pi(X\Omega) = \pi(\Omega X) = \pi(\Omega)\pi(X) = C\pi(X).$$

For the second assertion let π be irreducible and let μ be an eigenvalue of C. Then the eigenspace $\mathrm{Eig}(\mu, C)$ is stable under $\pi(\mathfrak{g})$, hence is an invariant subspace and so $\mathrm{Eig}(\lambda, C) = V$. □

Lemma 2.5.3. (a) *Let $v \in V$ be in the eigenspace $V_\lambda = \mathrm{Eig}(L, \lambda)$ for the eigenvalue $\lambda \in \mathbb{C}$. Then Av lies in $\mathrm{Eig}(L, \lambda + 2)$ and $Bv \in \mathrm{Eig}(L, \lambda - 2)$. In other words we have*

$$A\Big(V_\lambda\Big) \subset V_{\lambda+2}, \qquad B\Big(V_\lambda\Big) \subset V_{\lambda-2}.$$

(b) *(π, V) is irreducible if and only if there is $k \in \mathbb{N}_0$, such that $V = \bigoplus_{j=0}^{k} \mathbb{C}v_j$, where $Lv_j = (2j - k)v_j$.*

(c) *Any two irreducible representations of the same dimension are isomorphic.*

Proof. (a) Let $w = Av$. Then one has

$$Lw = LAv = \underbrace{[L, A]}_{2A}v + ALv = 2Av + \lambda Av = (\lambda + 2)w.$$

The case B is proven in the same way.

(b) Let (π, V) be a finite-dimensional representation. We show that V contains a $\mathfrak{g}$-stable subspace U of claimed form. We can exchange V for an eigenspace of the Casimir, as the latter is stable.

As V is finite-dimensional, there is an eigenvector of L and by part (a) there is an eigenvector v_0, $Lv_0 = \lambda v_0$ so that $Bv_0 = 0$. Let $v_1 = Av_0$, $v_2 = Av_1$ and so on until

$v_k = Av_{k-1}$, such that $Av_k = 0$. Then $Lv_j = (\lambda + 2j)v_j$. Let U be the span of $v_0, \dots, v_k$. We want to show that U is a $\mathfrak{g}$-stable subspace. Let μ be the Casimir eigenvalue on V. One has

$$\mu v_j = Cv_j = (L^2 + 2AB + 2BA)v_j = (\lambda + 2j)^2 v_j + 2ABv_j + 2Bv_{j+1},$$

where we formally set $v_{-1} = v_{k+1} = 0$. For $j = 0$ we have $Bv_j = 0$ and we conclude that $Bv_1 \in \mathbb{C}v_0 \subset U$. For $j = 1$ it follows that $ABv_j \in \mathbb{C}v_j$ and hence $Bv_2 \in \mathbb{C}v_1$, too. This can be repeated and one gets the following diagram

$$\begin{tikzcd}
V_\lambda \arrow[r, bend left, "A"] & V_{\lambda+2} \arrow[l, bend left, "B"] \arrow[r, bend left, "A"] & V_{\lambda 42} \arrow[l, bend left, "B"] \arrow[r, bend left, "A"] & \cdots \arrow[l, bend left, "B"] \arrow[r, bend left, "A"] & V_{\lambda+2k-2} \arrow[l, bend left, "B"] \arrow[r, bend left, "A"] & V_{\lambda+2k} \arrow[l, bend left, "B"] .
\end{tikzcd}$$

In particular, the space U is stable under A, B and L. Hence U is a stable subspace.

We need to determine the eigenvalue λ. Look at the equation

$$\mu v_j = (\lambda + 2j)^2 v_j + 2ABv_j + 2Bv_{j+1}.$$

As $AB = [A, B] + BA = L + BA$ it follows $2ABv_j = 2(\lambda + 2j)v_j + 2Bv_{j+1}$ and so

$$\mu v_j = (\lambda + 2j)^2 v_j + 2(\lambda + 2j)v_j + 4Bv_{j+1}.$$

For $j = k$ we get

$$\mu = (\lambda + 2k)(\lambda + 2k + 2)$$

and for $j = 0$ one has

$$\mu v_0 = \lambda^2 v_0 + 2\lambda v_0 + 4Bv_1.$$

We have

$$4Bv_1 = 4BAv_0 = 4 \underbrace{[B, A]}_{-L} v_0 + 4 \underbrace{ABv_0}_{0} = -4\lambda v_0.$$

For $j = 0$ it emerges

$$\mu = \lambda^2 - 2\lambda.$$

Dahertogether we infer

$$(\lambda + 2k)(\lambda + 2k + 2) = \lambda^2 - 2\lambda.$$

The only solution of this equation is $\lambda = -k$.

For the converse: Assume that V has the announced form. We need to show that it is irreducible. Let $W \neq 0$ be a stable subspace and let $0 \leq j_0 \leq k$ be the smallest index, such that there is a vector $0 \neq v = \lambda_0 v_1 + \cdots + \lambda_k v_k \in W$ with $\lambda_{j_0} \neq 0$. Then $A^{k-j_0} v = \lambda_{j_0} v_k$, and hence $v_k \in W$. As $a^{k-j-1} v = \lambda_{j_0} v_{k-1} + \lambda_{j_0+1} v_k$ it follows that $v_{k-1} \in W$ and so on. Therefore, $v_{j_0}, \dots, v_k \in W$. We infer that on the $\mathfrak{g}$-module V/W the operator L only has the eigenvalues $-k, -k+2, \dots, 2j_0 - k + 2$, which by the first part implies that $j_0 = 0$ and so $W = V$. Part (b) is proven. Part (c) also follows, since the operators L, A and B are determined by k. □

Theorem 2.5.4. *For the group* $G = \mathrm{SL}_2(\mathbb{R})$ *and every* $k \in \{0, 1, 2, \dots\}$ *there is exactly one irreducible representation of dimension* $k+1$.

Proof. Let V_k be the set of all homogeneous polynomials of degree k in the variables x, y, i.e.,

$$V_k = \mathbb{C}x^k \oplus \mathbb{C}x^{k-1}y \oplus \cdots \oplus \mathbb{C}y^k.$$

Let $\rho_k : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(V_k)$ be defined by

$$\rho_k(A) f(x, y) = f((x, y)A),$$

i.e.,

$$\rho_k\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x, y) = f(ax + cy, bx + dy).$$

For $a = e^t$ one plugs in the matrix $\left(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\right)$ and one sees that it has the vectors vectoren $x^k, x^{k-1}y, \dots, y^k$ as eigenvectors with the eigenvalues $a^k, a^{k-1}, \dots, a^{-k}$, which for the Lie algebra in the derived representation corresponds to the eigenvalues $-k, 2-k, \dots, k$. This implies that the representation of the Lie algebra is irreducible and so is the one of $\mathrm{SL}_2(\mathbb{R})$. The uniqueness follows from the uniqueness of the Lie algebra representation. □

Corollary 2.5.5. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex Lie algebra, spanned by an SL(2)-Tripel* H, E, F. *Then every finite-dimensional representation of* $\mathfrak{g}_{\mathbb{C}}$ *is a direct sum of* H*-eigenspaces and all eigenvalues are integers.*

* * *

3 Algebraic groups

3.1 Compact groups

Definition 3.1.1. A real polynomial f in n^2 variables $(X_{i,j})_{1\le i,j\le n}$ can be understood as map $f : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$. A subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ is called **linear algebraic group**, if there are polynomials $f_1, \dots, f_r$, such that

$$G = \Big\{x \in \mathrm{GL}_n(\mathbb{R}) : f_1(x) = \dots = f_r(x) = 0\Big\}.$$

Examples 3.1.2.

(a) $\mathrm{GL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{R}) = \Big\{\det(x) = 1\Big\}$ are examples.

(b) If $b : \mathbb{R}^n \to \mathbb{R}$ is a bilinear form, then

$$\mathrm{O}(b) = \Big\{g \in \mathrm{GL}_n(\mathbb{R}) : b(gx, gy) = b(x, y)\ \forall_{x,y\in\mathbb{R}^n}\ \Big\}$$

is a linear algebraic group. If b is symmetric and non-degenerae, one calls $\mathrm{O}(b)$ the **orthogonal group** of b. If b is antisymmetric, i.e., $b(y, x) = -b(x, y)$, then one calls $\mathrm{O}(b)$ the **symplectic group**.

Lemma 3.1.3. *Every compact Lie group has an injective finite-dimensional representation.*

Proof. We show that for every finite-dimensional representation ρ and given $x \ne y$ in K with $\rho(x) = \rho(y)$ there is a finite-dimensional representation τ with $\rho \subset \tau$, such that $\tau(x) \ne \tau(y)$. On the Hilbert-space $L^2(K)$ one has $R_x \ne R_y$. As $L^2(K)$ is a direct sum of irreducibles, there is an irreducible representation η with $\eta(x) \ne \eta(y)$. The representation $\tau = \rho \oplus \eta$ then has the desired property.

Now let ρ be a finite-dimensional representation and let d be the dimension of the kernel of ρ. If $d > 0$, then choose a $y \ne 1$ in the unit-component of $\ker(\rho)$ and $\tau \supset \rho$ as above with $\tau(y) \ne 1$. Then $\ker(\tau)$ is of propely smof all dimension. Iteration brings us to the case $d = 0$, which means that the kernel is a discrete sub group of the compact group K, hence finite. By finitely many application of the above one gets the desired injectivity. $\square$

Theorem 3.1.4. (a) *Every compact group* $K \subset \mathrm{GL}_n(\mathbb{R})$ *is algebraic.*

(b) *Every comact Lie group is isomorphic to an algebraic group.*

Proof. (a) Let $K \subset \mathrm{GL}_n(\mathbb{R})$ be compact and let $I \subset \mathcal{R} = \mathbb{R}[X_{1,1}, \dots, X_{n,n}]$ be the set of all polynomials f with $f(K) = 0$. Then I is an ideal in the polynomial ring $\mathcal{R}$. We claim:

(i) K is the joint zero-set of I.

(ii) The ideal I is finitely generated.

(i): Let Z be the joint zero-set. Then $K \subset Z$. The group K acts on $\mathcal{R}$ by $R_k f(x) = f(xk)$. Let $y \in \mathrm{GL}_n(\mathbb{R})$, $y \notin K$. The polynomial $f(x) = \sum_{i,j}(x_{i,j} - y_{i,j})^2$ vanishes in $x = y$ and is > 0 everywhere else. As K is compact, there is $c > 0$, such that $f(K) \geq c$. Let

$$g(x) = \int_K f(xk)\, dk.$$

This poynomial is K-invariant and $g(K) = \mu$ for a $\mu \geq c > 0$. Then $h(x) = g(x) - \mu$ is a polynomial, which vanishes on K, but not in y, so $y \notin Z$ and hence $K = Z$.

(ii) The Ring $\mathcal{R}$ is noetherian and therefore I is finitely-generated.

So let $f_1, \dots, f_r$ be generators of I. Then K is the joint zero set of $f_1, \dots, f_r$.

(b) A given compact Lie group K has an injective representation $\rho : K \to \mathrm{GL}_n(\mathbb{R})$, hence it is isomorphic to the image, which is an algebraic group. □

Corollary 3.1.5. *The metaplektic groups* $\mathrm{SL}_2(\mathbb{R})_n$ *with* $n \geq 2$ *are not algebraic.*

Proof. We show that $G_n = \mathrm{SL}_2(\mathbb{R})_n$ has no injective finite-dimensional representation, more precisely, we show that every finite-dimensional representation factors through the projection $p : G_n \to G = \mathrm{SL}_2(\mathbb{R})$. If ρ is a finite-dimensional representation, then it induces a representation ρ_* of the Lie algebra. We have found all representations of the Lie algebra and we have seen that they all are induced by representations of $G = \mathrm{SL}_2(\mathbb{R})$. So there is a representation π of G, such that $\rho_* = \pi_* \circ p_*$, where $p : G_n \to G$ the projection ist. Then ρ and $\pi \circ p$ as G_n is connected, they agree. □

3.2 Complex groups

Definition 3.2.1. Let $U \subset \mathbb{C}^n$. A function $f : U \to \mathbb{C}$ is called **holomorphic**, if the function $z \mapsto f(w_1, \dots, w_{j-1}, z, w_{j+1}, \dots, w_n)$ is holomorphic for every choice of $1 \le j \le n$ and w_k, $k \neq j$. The function is called **anti holomorphic**, if $\overline{f}$ is holomorphic. A map $f : U \to \mathbb{C}^m$ is called **holomorphic/antiholomorphic**, if every coordintae is holomorphic/antiholomorphic.

Definition 3.2.2. A **complex manifold** is a manifold M with a $\mathbb{C}^n$-valued atlas, whose coordinate changes are holomorphic.

Lemma 3.2.3. *Every covering of a complex manifold $p : X \to M$ admits exactly one holomorphic structure, which makes p holomorphic.*

Proof. As the projection is a local homeomorphism, we can pull back the holomorphic charts. The coordinate changes remain the same. □

Definition 3.2.4. A **complex Lie group** is a Lie group, which is a complex manifold, such that multiplication and inversion are holomorphic maps.

Examples 3.2.5. $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ are examples.

Lemma 3.2.6. *Let $p : X \to G$ be a connected covering of a complex Lie group G. Every choice $x_0 \in X$ of a preimage of the unit induces exactly one structure of a complex group on X with x_0 as unit.*

Proof. As in the case of ordinary Lie groups. □

Definition 3.2.7. Let G be a real Lie group. A **complexification** of G is an embedding of real Lie groups $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a complex Lie group, together with an antiholomorphic automorphism $\kappa : G_{\mathbb{C}} \to G_{\mathbb{C}}$, such that $\kappa \circ \kappa = \mathrm{Id}$ and

$$G_{\mathbb{C}}^{\kappa} = \left\{ z \in G_{\mathbb{C}} : \kappa(z) = z \right\} = G.$$

Further we insist that G meets every connec ted component of $G_{\mathbb{C}}$.

Theorem 3.2.8. (a) *A complexification of a Lie group is uniquely determined.*

(b) *If $G \subset \mathrm{GL}_n(\mathbb{R})$ is algebraic, then G posesess a complexification, which is algebraic, too.*

(c) *Every compact Lie group K posesses an algebraic complexification.*

(d) *If $G_\mathbb{C}$ is the complexification of G, then the Lie algebra $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$ is the complexification*

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$$

of the Lie algebra $\mathfrak{g}$ of G.

Proof. (a) Let $G_\mathbb{C}$ and $H_\mathbb{C}$ complexifications with complex conjugations κ and η. Let $e \in G$ be the unit. The differential κ_* acts on the tangent space $T_eG_\mathbb{C}$ and satisfies $\kappa_*^2 = \mathrm{Id}$, hence κ_* is diagonalisable with the eigenvalues ± 1. The multiplication with $i = \sqrt{-1}$ yields a linear map J on $T_xG_\mathbb{C}$ with $J^2 = -1$, where the eigenwert $\pm i$ appear in the same multiplicity. As κ is antiholomorphic, one has $\kappa_* J = -J\kappa_*$ or $\kappa_* = J\kappa_* J$. This shows that $\kappa_* v = v$ implies $\kappa_*(Jv) = -Jv$, so the eigenvalues ± 1 of κ_* appear in the same multiplicity. Every eigenvector to the eigenvaluet 1 lies in T_xG and so $\dim_\mathbb{C} G_\mathbb{C} = \dim G = \dim_\mathbb{C} H_\mathbb{C}$. For open unit neighbourhoods $U \subset G_\mathbb{C}$, $V \subset H_\mathbb{C}$ and $\phi : U \to U' \subset \mathbb{C}^N$, $\psi : V \to V' \subset \mathbb{C}^N$ holomorphic charts the map $\alpha : \psi\phi^{-1}$ is defined on $\psi(G \cap U)$ and analytic, hence extends to a complex neighbourhood. The map α satisfies $\alpha(xy) = \alpha(x)\alpha(y)$ for $x, y \in G$ in a unit neighbourhood and this identity holds in the complex as well. Hence $G_\mathbb{C}$ and $H_\mathbb{C}$ are locally isomorphic. After iterated application of multiplication one gets holomorphic group homomorphisms $\alpha : G_\mathbb{C} \to H_\mathbb{C}$ and $\beta : H_\mathbb{C} \to G_\mathbb{C}$, which satisfy $\beta \circ \alpha = \mathrm{Id}$ on G. By the Identity Theorem this holds everywhere.

This also implies (d).

(b) For a subset $A \subset \mathrm{GL}_n(\mathbb{C})$ let

$$\mathcal{I}(A) = \left\{ f \in \mathcal{R}_\mathbb{C} = \mathbb{C}[X_{i,j} : 1 \le i, j \le n] : f(A) = \{0\} \right\}$$

the **vanishing ideal** of A. If $A \subset B$, then $\mathcal{I}(A) \supset \mathcal{I}(B)$. If A is stable under complex conjugation, then one has for $f \in \mathcal{I}(A)$, that $f^* \in \mathcal{I}(A)$, where $f^*(z) = \overline{f(\overline{z})}$. Because of $f = \frac{1}{2}(f + f^*) - i\frac{1}{2}(if + (if)^*)$ it follows $\mathcal{I}(A) = \mathbb{C} \cdot \mathcal{I}_\mathbb{R}(A)$, where $\mathcal{I}_\mathbb{R}(A) = \mathcal{I}(A) \cap \mathbb{R}[X_{i,j}]$.

Conversely let $J \subset \mathcal{R}_\mathbb{C}$. Denote by $\mathcal{V}(J)$ the joint zero set of J. If $I \subset J$, then $\mathcal{V}(I) \supset \mathcal{V}(J)$.

Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be an algebraic Lie group and let I be the vanishing ideal of G. Let $G_\mathbb{C}$ denote the zero set of I in $\mathrm{GL}_n(\mathbb{C})$. We claim, that $G_\mathbb{C}$ is a complexification. The complex conjugation takes the role of κ. We have to show, that $G_\mathbb{C}$ is a group, i.e., closed under multiplication and inversion. Let $g \in G$, for every $x \in gG$ we have $f(g^{-1}x) = 0$, $f \in I$. As G is a group, the set of all these functions $f(g^{-1}x)$ is I again and so $GG_\mathbb{C} \subset G_\mathbb{C}$ as well as $G_\mathbb{C}G \subset G_\mathbb{C}$. Let V be the set of all $z \in \mathrm{GL}_n(\mathbb{C})$ with $zG_\mathbb{C} \subset G_\mathbb{C}$. That means that V is the zero set of the ideal

$$J = \mathcal{I}(V) = \Big\{ f(z \cdot -) : f \in \mathbb{C}I \Big\}.$$

Since $G \subset V$ it follows $\mathcal{I}(G) \supset \mathcal{I}(V)$ and so

$$G_\mathbb{C} = \mathcal{V}(\mathcal{I}(G)) \subset \mathcal{V}(\mathcal{I}(V)) = V.$$

Therefore $G_\mathbb{C}$ is closed under multiplication. For the inversion one argues similarly.

(c) This follows from (b), since every compact Lie group is algebraic. □

Theorem 3.2.9. *Let G be a connected Lie group with complexification $G_\mathbb{C}$. Then every finite-dimensional complex representation (π, V_π) of G extends to a uniquely determined representation $(\pi_\mathbb{C}, V_\pi)$ of $G_\mathbb{C}$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ is the Lie algebra of $G_\mathbb{C}$. The $\mathbb{R}$-linear map π' can be extended to a $\mathbb{C}$-linear map on $\mathfrak{g}_\mathbb{C}$ and becomes a representation of the Lie algebra of $G_\mathbb{C}$ and thus a local representation, or a representation of the universal covering $\tilde{G}_\mathbb{C}$. Let $H \subset \tilde{G}_\mathbb{C}$ be the preimage G and let $Z \subset H$ be the kernel of the projection $H \to G$. Then Z is a central subgroup of H and hence a normal subgroup in the complexification $\tilde{G}_\mathbb{C}$ of H. (Since the equation $hzh^{-1} = z$ holds for given $z \in Z$ and all $h \in H$ and is an equality of holomorphic maps in h.) Therefore $\tilde{G}_\mathbb{C}/Z$ is a complexification of G and hence must coincide with $G_\mathbb{C}$. As the representation π is defined on G, it is trivial on Z and consequently extends to $G_\mathbb{C}$ fort. The uniqueness is clear on the Lie algebra and follows for the group $G_\mathbb{C}$ locally and hence globally. □

4 Maximal tori

4.1 The mapping degree

Definition 4.1.1. A **manifold with boundary** is a separable Hausdorff-space M, such that every point has an open neighbourhood U, for which there is a homeomorphism $\phi : U \to D \subset \mathbb{R}^n$, for an open subset D of

$$H = \left\{ x \in \mathbb{R}^n : x_1 \geq 0 \right\}$$

ist.

As in the case of manifolds one defines differentiable structures, where at the boundary only one-sided derivatives are used.

Theorem 4.1.2 (Theorem of Stokes). *Let M be a smooth, n-dimensional oriented manifold with boundar and let* $\omega \in \Omega^{n-1}(M)$ *with compact support. Then one has*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Proof. The proof is in the Analysis book and works by reducing the statement to the Fundamental Theorem of Analysis. □

Corollary 4.1.3. *Let* $f, g : M \to N$ *smooth maps between smooth, n-dimensional manifolds, such that f and g smoothly homotopic, i.e., there is a smooth map* $H : [0,1] \times M \to N$, *such that* $H(0,x) = f(x)$, $H(1,x) = g(x)$. *For every differential form* $\omega \in \Omega_c^n(N)$ *one has*

$$\int_M f^*\omega = \int_M g^*\omega.$$

Proof. Write $I = [0,1]$. By the Theorem of Stokes one has

$$\int_{\partial I \times M} H^*\omega = \int_{I \times M} dH^*\omega.$$

Further, $dH^*\omega = H^*d\omega = 0$ and $\int_{\partial I \times M} H^*\omega = \int_M f^*\omega - \int_M g^*\omega$. □

Theorem 4.1.4 (The mapping degree). *Let M, N connected, compact, oriented smooth manifolds of dimension n. For every smooth map $f : M \to N$ there is an integer* $\deg(f) \in \mathbb{Z}$ *such that for every $\omega \in \Omega^n(N)$ one has*

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

This number $\deg(f)$ *is called the* **mappind degree** *of f.*
Let $q \in N$ be a point, such that the preimage $f^{-1}\{q\}$ consists of $k + m$ points $p_1, \dots, p_{k+m}$. Assume that

(a) *the differential Df is invertible in every point p_j,*

(b) *Df keeps the orientation at $p_1, \dots, p_m$ and*

(c) *inverts the orientation at $p_{m+1}, \dots, p_{k+m}$.*

Then one has $\deg(f) = k - l$. *In particular: If* $\deg(f) \neq 0$, *then is f surjective.*

Proof. If the image of f is a set of measure zero, then the degree is zero and there is nothing to show. So assume that the measure of the image is positive. A point $q \in N$ is called a **regular point** of f, if the differential $Df(p)$ is invertible for every point p in the preimage $f^{-1}\{q\}$. Otherwise the point q is a **singular point**. The **Theorem of Sard** says that every $q \in N$ has only finitely many preimages and that the singular points form a closed set of measure zero in N. A proof can be found for instance in Broecker-Jaehnich: Introduction to Differential Topology.

In particular, there are regular points. So let q be a regular point and let $f^{-1}\{q\} = \{p_1, \dots p_{k+l}\}$ as in the Theorem. By the Theorem of the local inverse, the map f is, locally around every p_j, a homeomorphism. Therefore there are open sets $B \ni q$ and $B_j \ni p_j$, all diffeomorphic to an open ball, such that $f|_{B_j} \xrightarrow{\cong} B$ is a diffeomorphism and $f^{-1}(B)$ is the disjoint union of the B_j. If $\operatorname{supp}(\omega) \subset B$, then the theorem follows. In general, it follows by partition of unity. □

* * *

4.2 Maximal tori

Definition 4.2.1. A **torus** T is a Lie group, which is isomorphic to $\mathbb{T}^r$ for some $r \in \mathbb{N}_0$.

Let K be a compact Lie group and let $T \subset K$ be a maximal torus in K. Let

$$N_K(T) = \left\{k \in K : k^{-1}Tk = T\right\}$$

denote the **normaliser** of T in K. Then T is normal in $N_K(T)$. The quotient group

$$W = N_K(T)/T$$

Is called the **Weyl group** of T.

Example 4.2.2. Let $K = \mathrm{U}(n)$. Then the set T of all diagonal matrices in K is a maximal torus. The group $N_K(T)$ is the group of all **monomial matrices** in K. Here a matrix A is called monomial, if A possesses exactly one non-zero entry in every row and every column. If A is in $\mathrm{U}(n)$, then all these entries lie in $\mathbb{T}$. Therefore it follows, that $W(T) = N(T)/T$ is isomorphic to the group of all permutation matrices in other words, in this case we have

$$W_K(T) \cong \mathrm{Per}(n).$$

Lemma 4.2.3. *Let $\phi : \mathbb{T} \to \mathbb{T}$ be a continuous group homomorphism. Then there is a $k \in \mathbb{Z}$, such that $\phi(z) = z^k$ for every $z \in \mathbb{T}$.*

Proof. Exercise. □

Proposition 4.2.4. *Let K be a connected compact Lie group and let $T \subset K$ be a maximal torus. Then the map*

$$q : K/T \times T \to K, \qquad (k, t) \mapsto ktk^{-1}$$

has degree $|W|$, where W is the Weyl group. In particular, q is surjective.

The proof of this proposition will consume a good part of this section.

Proof. Choose an inner product $\langle ., . \rangle$ on the real vector space $\mathfrak{k} = \mathrm{Lie}(K)$, which is invariant under the adjoint representation of the compact group K. This means

$$\langle \mathrm{Ad}_k X, \mathrm{Ad}_k Y \rangle = \langle X, Y \rangle .$$

The Lie algebra $\mathfrak{k} = T_eK$ decomposes into the tangent space $\mathfrak{t} = \mathrm{Lie}(T)$ of T and its orthocomplement $\mathfrak{t}^{\perp}$

$$\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{t}^{\perp}.$$

This decomposition is stable under the action of T. Since T is maximal, $\mathfrak{t}^\perp$ does not contain any T-invariant vector except zero.

We write the action of T on $\mathfrak{t}^\perp$ as

$$\mathrm{Ad}_{K/T} : T \to \mathrm{Aut}(\mathfrak{t}^\perp).$$

We choose compatible orientations on K, T and K/T. Then there ar uniquely determined top-degree differential forms dk, dt and dkT with Integral 1 on K, T and K/T. Further there is exactly one left-invariant differetial form $d\tau \in \Omega^r(K)$ with $r = \dim T$, such that $d\tau|_T = dt$. Let $\pi : K \to K/T$ be the projection. We have $d\tau \wedge \pi^*(dkT) = dk$, since these top-differential form must be proportional and have the same integral. On $K/T \times T$ we have the top-Differentialform $\alpha = p_1^*(dkT) \wedge p_2^*(dt)$, where p_1, p_2 are the two projections. Identifying $\mathfrak{k}$ with $\mathfrak{t}^\perp \oplus \mathfrak{t}$ and evaluating at the neutral element, one gets $\alpha_{(eT,e)} = dk_e$.

Lemma 4.2.5. *We define the determinant* $\det(q) : K/T \times T \to \mathbb{R}$ *of q by the Equation*

$$q^*(dk) = \det(q)\alpha.$$

Then one has

$$\det(q)(kT, t) = \det(\mathrm{Ad}_{K/T}(t^{-1}) - \mathrm{Id}),$$

where Id *the Identitaet on* $\mathfrak{t}^\perp$ *ist.*

Proof. The forms dk and dkT are K-invariant and dt is T-invariant. Therefore one can compute the determinant at the point (kT, t), by first applying the left translation in $K/T \times T$ and K, and then evaluates the determinant at the point (eT, e). For given $k \in K$ and $t \in T$ we consider the differential of the composition

$$\begin{array}{ccccccc} K/T \times T & \xrightarrow{l_{(k,t)}} & K/T \times T & \xrightarrow{q} & K & \xrightarrow{l_{kt^{-1}k^{-1}}} & K \\ (x, y) & \mapsto & (kx, ty) & \mapsto & \underbrace{(kx)(ty)(kx)^{-1}}_{=a} & \mapsto & kt^{-1}k^{-1} \cdot a = kt^{-1}xtyx^{-1}k^{-1}, \end{array}$$

i.e., of the map

$$(x, y) \mapsto c_k\left(c_{t^{-1}}(x) \cdot y \cdot x^{-1}\right).$$

The determinant in question is the determinante of the differential of this map at the point (e, e), restricted to the subspace $\mathfrak{t}^\perp \oplus L(T)$ of $L(K) \oplus L(T) = L(K \times T)$.

The map Ad_k is orthogonal and K is connected. Therefore $Lc_k = \mathrm{Ad}_k$ has determinante 1. Further the differential of the product is the sum of the single differentials, which implies that the determinant of q equals the determinant of the linear endomorphism

$$(X, Y) \mapsto \mathrm{Ad}_{K/T}(t^{-1})X + Y - X$$

of $\mathfrak{t}^\perp \oplus L(T)$. In matrices this is

$$\begin{pmatrix} \mathrm{Ad}_{K/T}(t^{-1} - \mathrm{Id}) & \\ & \mathrm{Id}_T \end{pmatrix},$$

implying the claim of the lemma. □

We continue the proof of Proposition 4.2.4. By Lemma 1.11.3 the torus $T \cong \mathbb{T}^r$ is topologically generated by one element.

Lemma 4.2.6. *Let τ be a topological geenrator of T. Then one has*

(i) $q^{-1}(\tau)$ *has* $|W|$ *points and*

(ii) $\det(Dq) > 0$ *holds in all of these points.*

The dimension of K/T is even.

Proof. (i) Let N be the normaliser of T in K and let $k \in K$. There is exactly one $s \in T$ with $q(kT, s) = \tau$, also $ksk^{-1} = \tau$, or $s = k^{-1}\tau k \in T$. This is the case iff $k \in N$. We conclude

$$q^{-1}(\tau) = \Big\{(kT, k^{-1}\tau k) : k \in N\Big\}$$

and so $|q^{-1}(\tau)| = |W|$.

(ii) We consider the determinant of $\left(\mathrm{Ad}_{K/T}(\tau^{-1}) - \mathrm{Id}\right)$ on $\mathfrak{t}^\perp$. We show that this endomorphism has no real eigenvalues. We then cobclude that the eigenvalues come in complex conjugate pairs and the determinant is positive. As a side result we get that the dimension of $\mathfrak{t}^\perp$, i.e., the dimension of K/T is even.

Assume, that $\left(\mathrm{Ad}_{K/T}(\tau^{-1}) - \mathrm{Id}\right)$ has a real eigenvalue. Then $\mathrm{Ad}_{K/T}(\tau^{-1})$ also has one. The torus T acts on $\mathfrak{t}^\perp$ by orthogonal maps, hence every of these is diagonalisable in the complexification of $\mathfrak{t}^\perp_{\mathbb{C}}$ with eigenvalues of absolute value 1. Therefore $\mathrm{Ad}_{K/T}(\tau)$ has an eigenvalue ±1. As T is abelian, all of T can be diagonalised simultaneously. So there are continuous group homomorphisms $\chi_j : T \to \mathbb{T}$ such that

$$\mathfrak{t}_{\mathbb{C}}^{\perp} = \bigoplus_{j=1}^{n} \mathfrak{t}_{\mathbb{C}}^{\perp}(\chi_j),$$

where

$$\mathfrak{t}_{\mathbb{C}}^{\perp}(\chi) = \left\{X \in \mathfrak{t}_{\mathbb{C}}^{\perp} : \mathrm{Ad}(t)X = \chi(t)X \ \forall_{t \in T}\right\}.$$

It follows that there is a $\chi = \chi_j : T \to \mathbb{T}$ such that $\chi(\tau) \in \mathbb{R}$, hence $\chi(\tau) = \pm 1$. As τ is a topological generator, it follows $\chi^2 = 1$, which can't be true, as every χ is of the form $\chi(z_1, \dots, z_r) = z^{\nu_1} \cdots z_r^{\nu_r}$ for some $\nu \in \mathbb{Z}^r$ ist. So if $\chi^2 = 1$, then one has $\chi = 1$, which contradicts the maximality of the torus T! The lemma and Proposition 4.2.4 are proven. □

Theorem 4.2.7. *Let K be a connected compact Lie group.*

(a) *All maximal tori are conjugate.*

(b) *Every element of K lies in a maximal torus.*

(c) *The exponential map* $\exp : \mathfrak{k} \to K$ *is surjective.*

Proof. Let S, T maximal tori. Let σ be a topological generator of S. By Proposition 4.2.4 the map q is surjective, therefore there is a $k \in K$, such that $\sigma \in kTk^{-1}$ and since σ generates the torus topologically, we get $S \subset kTk^{-1}$. Equality follows form maximality of S. □

Definition 4.2.8. The **rank** a compact Lie group K is the dimension of a maximal torus.

Theorem 4.2.9. *Let K be a connected compact Lie group and T a maximal torus.*

(a) *One has $Z(T) = T$, so T is a maximal abelian subgroup of K.*

(b) *If $S \subset K$ is a connected abelian subgroup, then the centraliser $Z(S)$ is the union of all maximal tori T with $T \supset S$.*

(c) *Let $Z(K)^0$ be the connected component of the center. Then one has*

$$Z(K)^0 = \bigcap_T T,$$

where the intersection runs over all maximal tori.

Proof. (a) follows from (b) with $S = T$.

(b) The closure $\overline{S}$ of S is compact, connected and abelian, hence a torus. Further one has $Z(S) = Z(\overline{S})$, such that we can assume, that S is a torus. Let $x \in Z(S)$ and let B be the closure of the group generated by S and x. Then B is compact and abelian, and so the one-component B^0 is a torus. As xB^0 generates the group B/B^0, it follows $B/B^0 \cong \mathbb{Z}/m$ for some $m \in \mathbb{N}$. Further one has $B \cong B^0 \times \mathbb{Z}/m$. By Lemma 1.11.3 it follows, that B is the closure of a cyclic group $\langle x_0 \rangle$ for some $x_0 \in xB^0$.

But since x_0 lies in a maximal torus T, we get $x \in T$ and $S \subset T$.

(c) If x lies in the center center, by (a) it lies in every maximal torus. Conversely let x lie in every maximal torus. As every element in some maximal torus, the element x commutes with every element of K, hence lies in the center. □

Theorem 4.2.10. *The Weyl group is finite.*

Proof. Let N^0 be the one-component of the group $N = N_K(T)$. Let $T \cong \mathbb{T}^r$. Consider the automorphism group

$$\mathrm{Aut}(T) = \mathrm{Aut}(\mathbb{T}^r) \cong \mathrm{GL}_r(\mathbb{Z}).$$

The isomorphism on the right maps a m rechts bildet a matrix $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$ to the automorphism

$$(z_1, \dots, z_r) \mapsto (z_1^{a_{11}} \cdots z_r^{a_{1r}}, \dots, z_1^{a_{r1}} \cdots z_r^{a_{rr}}).$$

In ordrr to see, that every automorphism is of this form, let $\phi : \mathbb{T}^r \to \mathbb{T}^r$ be any automorphism. For $1 \le i, j \le r$ let $\alpha_i : \mathbb{T} \to \mathbb{T}^r, z \mapsto (1, \dots, 1, z, 1, \dots, 1)$ the i-th embedding and $\beta_j : \mathbb{T}^r \to \mathbb{T}$ the j-th projection. Then $\beta_j \circ \phi \circ \alpha_i$ is an endomorphism of $\mathbb{T}$, hence of the form $z \mapsto z^k$ for be a $k \in \mathbb{Z}$. This means that ϕ is given by a matrix $A \in \mathrm{M}_n(\mathbb{Z})$. As ϕ is invertible, we infer $A \in \mathrm{GL}_n(\mathbb{Z})$.

The map $N = N_K(T) \to \mathrm{Aut}(T), n \mapsto \left(t \mapsto ntn^{-1}\right)$ is continuous, but the set of the the possible values is discrete. Therefore the one-component N^0 of N acts trivially on T. This means that N^0 commutes with every $t \in T$, which in tirn menas that the group generated by T and N^0 is abelian. However the group is T is maximally abelian by Theorem 4.2.9, which implies that $N^0 = T$. Therefore $W = N/T = N/N_0$ is discrete and compact, hence finite. □

Theorem 4.2.11 (Weyl integral-formula)**.** *Let K be a connected compact Lie group, T a maximal torus and $f \in L^1(K)$. Then one has*

$$\int_K f(x)\,dx = \frac{1}{|W|} \int_T \left(\det\left(1 - \mathrm{Ad}_{K/T}(t^{-1})\right) \int_K f(ktk^{-1})\,dk \right) dt.$$

Proof. Let $f_t : K/T \to \mathbb{C}, k \mapsto f(ktk^{-1})$, so $f(ktk^{-1}) = f_t \circ \pi(k)$, where $\pi : K \to K/T$ is the canonical projection. The inner integral on the right is $\int_K f_t\,dk = \int_{K/T} f_t\,dkT$ and after change of order of integration the right hand side equals

$$\begin{aligned} \int_{K/T} \left(\int_T f \circ q(kT, t) \cdot \det(q)(kT, t)\,dt \right) d(kT) &= \int_{K/T \times T} f \circ q(kT, t) \cdot q^*(dk) \\ &= \deg(q) \int_K f = |W| \int_K f(k)\,dk. \end{aligned}$$ □

Lemma 4.2.12. *Let T be a maximal torus in the compact Lie group K.*

(a) *The Weyl group $W = W(T)$ acts effectively on T, i.e., the homomorphism*

$$W \to \mathrm{Aut}(T)$$

is injective.

(b) *Two elements of T are K-conjugate if and only if they lie in the same Weyl orbit.*

Proof. (a) Let $w = xT$ be in the kernel of this homomorphism, then x commutes with every $t \in T$. By Theorem 4.2.9 the element x lies in T.

(b) Let $x, y \in T$ and $kxk^{-1} = y$ for some $k \in K$. Let $Z(x)$ and $Z(y)$ be the centralisers of x and y. conjugation with k induces an homomorphism $Z(x) \to Z(y)$. Then T and kTk^{-1} are maximal tori in the unit-component $Z(y)^0$ of $Z(y)$, so they are conjugate in $Z(y)^0$, i.e., there is an $h \in Z(y)^0$, such that $T = hkTk^{-1}h^{-1}$, hence $hk \in N(T)$ and $(hk)x(hk)^{-1} = hkxk^{-1}h^{-1} = hyh^{-1} = y$. This means that x and y lie in the same Weyl orbit. □

Definition 4.2.13. If $k \in K$, we write $K.k$ for the K-conjugation class of k, i.e.,

$$K.k = \left\{xkx^{-1} : x \in K\right\}.$$

Let $[K]$ the set of all conjugation classes in K. The natural map $K \to [K]$ induces a topology on $[K]$.

Proposition 4.2.14. *The canonical map*

$$\kappa : T/W \to [K], \qquad Wt \mapsto K.t$$

is a homeomorphism.

Proof. The map is well-defined, bijective and continuous. Both spaces are compact Hausdorff spaces, so for $A \subset T/W$ one has

$$A \text{ cloed} \Rightarrow A \text{ compact} \Rightarrow \kappa(A) \text{ compact} \Rightarrow \kappa(A) \text{ abgeschlossen}.$$

Hence κ is a homeomorphism. □

Theorem 4.2.15. *Let $\phi : K \to L$ be a surjective continuous homomorphism of connected Lie groups.*

(a) *If $T \subset K$ is a maximal torus, then $\phi(T) \subset L$ is a maximal torus.*

(b) *The kernel $\ker\phi$ lies in T, if and only if $\ker\phi \subset Z(K)$. In this case, ϕ induces an isomorphism of the Weyl groups.*

(c) *If $\dim K = \dim L$, then $\ker\phi \subset Z(K)$.*

Proof. (a) Let $S \subset L$ be a maximal torus and let $s \in S$ be a topological generator. Choose a $t \in \phi^{-1}(s)$. Then t lies in a maximal torus T of K and $\phi(T) \supset S$ is abelian, hence

$\phi(T) = S$. Every maximal torus of K is conjugate to T and therefore the image is a maximal torus.

(b) Let T be a maximal torus of K with $\ker\phi \subset T$. As $\ker\phi$ is a normal subgroup, it lies in every conjugate of T, i.e., in every maximal torus and hence in the center of K. The converse direction is clear.

We assume that $\ker\phi$ is central and we consider the Weyl groups. It is always true, that $\phi(N(T)) \subset N(\phi(T))$. If additionally $\ker(\phi)$ is central and $\phi(T) = S$, then one has

$$N(T) = \phi^{-1}(N(S)).$$

Proof of this statement. We already know "$\subset$". So let $k \in K$ with $\phi(k) \in N(S)$. Then for a topological generator $\tau \in T$, one has $\phi(k\tau k^{-1}) = \phi(k)\phi(\tau)\phi(k)^{-1} \in S$ which means $k\tau k^{-1} = tz$ for some $t \in T$ and some $z \in Z(K) \subset T$. We infer that $k \in N(T)$. □

(c) Finally let $\dim K = \dim L$. Then $H = \ker\phi$ is a discretee normal subgroup, so has to be central, as for every $\gamma \in H$ the map

$$K \to \mathrm{Aut}(H), \qquad k \mapsto k\gamma k^{-1}$$

is continuouse and maps the connected group K to the discrete group H, hence is constant. □

Examples 4.2.16. (a) If $\dim K = \dim L$, then any surjective homomorphism $\phi : K \to L$ is a covering.
For this one needs to know, that the differential $D\phi_e$ is invertible in the unit element, which it is, as it is surjective. Then the Theorem on local inverses gives the claim.

(b) The power map $z \mapsto z^n$ from $\mathbb{T}$ to $\mathbb{T}$ is an example. The kernel is the group of n-th roots of unity $\{e^{2\pi i j/n} : j \mod n\}$.

(c) Let $K = \mathrm{SU}(2) = \{g \in \mathrm{SL}_2(\mathbb{C}) : gg^* = 1\}$, where $g^* = \overline{g}^t$. Then the Lie algebra of this matrix group equals

$$su(2) = \left\{X \in \mathrm{M}_2(\mathbb{C}) : \overline{X}^t = -X,\ \mathrm{tr}(X) = 0\right\}.$$

It is a real subspace of $\mathrm{M}_2(\mathbb{C}) \subset \mathrm{M}_4(\mathbb{R})$. The vectors

$$E = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad F = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad G = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

form a basis of $su(2)$. We consider the bilinear form $b : su(2) \times su(2) \to \mathbb{R}$,

$$b(X, Y) = \operatorname{Re}\left(\operatorname{tr}(X^* Y)\right).$$

It is symmetric, since

$$b(X, Y) = \operatorname{Re}\left(\operatorname{tr}(-XY)\right) = \operatorname{Re}\left(\operatorname{tr}(-YX)\right) = b(Y, X).$$

One computes, that E, F, G are pairwise orthogonal and

$$b(E, E) = b(F, F) = b(G, G) = 2.$$

Further this form is invariant under the adjoint representation, which in matrix groups is given by conjugation:

$$B(gXg^{-1}, gYg^{-1}) = \operatorname{Re}(\operatorname{tr}(gX^*g^*gYg^*)) = \operatorname{Re}(\operatorname{tr}(gX^*Yg^{-1})) = b(X, Y).$$

This means that the adjoint representation is a homomorphism

$$\mathrm{SU}(2) \to \mathrm{SO}(b) \cong \mathrm{SO}(3).$$

The kernel of the adjoint representation is $\{\pm 1\}$, so the image has dimension 3, the same as SO(3). Since SO(3) is connected, the adjoint representation is surjective. Therefore the group SU(2) is a twofold covering of SO(3) and

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\pm 1.$$

Theorem 4.2.17. *The covering*

$$\mathrm{SU}(2) \to \mathrm{SO}(b) \cong \mathrm{SO}(3)$$

ist the universelle covering of SO(3).

Proof. We only need to know that SU(2) is simply connected. This has been shown in Proposition 1.9.8. □

Definition 4.2.18. An element $k \in K$ of a connected compact Lie group K is called **regular**, if the one-component $Z(k)^0$ of its centraliser $Z(k)$ is a torus. Then this torus is automatically maximal

Example 4.2.19. Let $K = \mathrm{SO}(3)$. The element

$$s = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

is regular, since its centraliser is

$$Z(s) = \left\{ \begin{pmatrix} u & \\ & \det(u) \end{pmatrix} : u \in \mathrm{O}(2) \right\}.$$

The centraliser has two connected components and $Z(s)^0$ is isomorphic to SO(2).

Theorem 4.2.20. *Let K be a connected compact Lie group.*

(a) *The set K^{reg} of regular elements is open and dense in K.*

(b) *For $k \in K$ the following are equivalent:*

  (i) *k is regular,*

  (ii) *there is exactly one maximal torus containing k.*

Proof. (a) Let T be a maximal torus and let $\mathfrak{t} \subset \mathfrak{k}$ be its Lie algebra. Let $f : T \to \mathbb{R}$ denote the smooth map

$$f(t) = \det\Big(\mathrm{Ad}(t) - 1 \Big|\, \mathfrak{k}/\mathfrak{t} \Big).$$

We claim that $t \in T$ is regular if and only if $f(t) \neq 0$.

Proof of this statement. Let $t \in T$ with $f(t) = 0$. Then there exists $X \in \mathfrak{g} \setminus \mathfrak{t}$ such that $\mathrm{ad}(t)X = X$. Then $\exp(\lambda X)$ centralises t for every $\lambda \in \mathbb{R}$, hence the centraliser of t is strictly bigger then T, i.e., t is not regular. For the converse, assume t also lies in another torus $S \neq T$. Then $\mathrm{Lie}(S) = \mathfrak{s} \neq \mathfrak{t}$ and there is a vector $v \in \mathfrak{s}$, which does not lie in $\mathfrak{t}$, hence $f(t) \neq 0$. □

This implies that $T^{\mathrm{reg}} = K^{\mathrm{reg}} \cap T$ is open in T. On the other hand, the set T^{top} of all topological generators of T lies in T^{reg} and is dense in T by Lemma 1.11.3.

Finally we consider the map $q : K/T \times T \to K$, $(kT, t) \mapsto ktk^{-1}$ of Proposition 4.2.4. One has $K^{\mathrm{reg}} = q(K/T \times T^{\mathrm{reg}})$ and since q is surjective, the claim follows.

(b): (i)⇒(ii): Let k be regular and let T be a maximal torus containing k. Then is T is contained in $Z(k)$ and since T is maximal, it equals the torus $Z(k)^0$.
(ii)⇒(i): Let T be the unique maximal torus containing k. Then $T \subset Z(k)^0$. Let M be the connected compact Lie group $Z(k)^0$. Then k lies in the center of M. By Theorem 4.2.9, the center of M is the intersection of all maximal tori in M, so k lies in every maximal torus of M, so there is only one and M must be a torus. □

Corollary 4.2.21. *The map*

$$\begin{aligned} q^{\mathrm{reg}} : K/T \times T^{\mathrm{reg}} &\to K^{\mathrm{reg}}, \\ (kT, t) &\mapsto ktk^{-1} \end{aligned}$$

ist a covering of degree $|W|$.

Proof. In the proof of Proposition 4.2.4 we have shown that it is covering in a neighbourhood of an arbitrary topological generator. Therefore the map is a covering on an open dense subset of K^{reg}. On K^{reg} every point has exactly $|W|$ preimages. As also q^{reg} is a smooth map, considering local coordinates one gets that q^{reg} is a covering everywhere on K^{reg}. □

Examples 4.2.22. (a) Let $K = \mathrm{SO}(2n)$. For $\theta \in \mathbb{R}$ let

$$k(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

and

$$T = \{\mathrm{diag}(k(\theta_1), \dots, k(\theta_n)) : \theta_1, \dots, \theta_n \in \mathbb{R}\},$$

Then T is a maximal torus in K, hence the rank of $\mathrm{SO}(2n)$ equals n. Let A be the subgroup of $\{\pm 1\}^n$ consisting of all elements, for which the product of all entries is 1. The Weyl group is isomorphic to $A \rtimes \mathrm{Per}(n)$.

(b) Let $K = \mathrm{SO}(2n+1)$. Then

$$T = \{\mathrm{diag}(k(\theta_1), \dots, k(\theta_n), 1) : \theta_1, \dots, \theta_n \in \mathbb{R}\}$$

is a maximal torus and $W \cong \{\pm 1\}^n \rtimes \mathrm{Per}(n)$.

* * *

4.3 Weights

Definition 4.3.1. Let V be a real vector space of dimension $n \in \mathbb{N}$. A **lattice** in V is a discrete subgroup $\Gamma \subset (V, +)$, containing a basis of V. By Lemma 1.11.1 the group Γ is of the form

$$\Gamma = \mathbb{Z}v_1, \oplus \cdots \oplus \mathbb{Z}v_n,$$

with a Basis $v_1, \dots, v_n$.

Definition 4.3.2. Let T be a compact torus with Lie algebra $\mathfrak{t}$. Then every continuous group homomorphism $\chi : T \to \mathbb{T}$ is of the form

$$\chi\,(\exp(X)) = e^{2\pi i \lambda(X)}$$

for a unique linear map $\lambda : \mathfrak{t} \to \mathbb{R}$. More precisely, one has

$$\lambda(X) = \frac{1}{2\pi i} \left.\frac{d}{dt}\right|_{t=0} \chi(\exp(tX))$$

We say that λ is the derivative of χ and we write

$$\chi(t) = t^{\lambda}.$$

Lemma 4.3.3. *A linear map $\lambda : \mathfrak{t} \to \mathbb{R}$ is derivative of a character if and only if*

$$\lambda(\ker(\exp)) \subset \mathbb{Z},$$

where $\ker(\exp)$ *is the set of all $X \in \mathfrak{t}$, for which* $\exp(X) = 1$ *holds. The map $\chi \mapsto \lambda$ is a bijection*

$$\widehat{T} \quad \leftrightarrow \quad \Big\{\lambda \in \mathfrak{t}^* : \lambda\,(\ker(\exp)) \subset \mathbb{Z}\Big\}.$$

Proof. Suppose that λ maps $\ker(\exp)$ to $\mathbb{Z}$. Then the map $\mathfrak{t} \to \mathbb{T}$, $X \mapsto e^{2\pi i \lambda(X)}$ factors through the exponential map $\exp : \mathfrak{t} \to T$ and the ensuing map $\chi : T \to \mathbb{T}$, given by $\chi(\exp(X)) = e^{2\pi i \lambda(X)}$ is a character. As the exponential map is surjective, this character is uniquely determined. The inverse map is given by taking the derivative of a character. $\square$

Definition 4.3.4. Let K be a connected compact Lie group and let T be a maximal torus in K. Let $\mathfrak{t} \subset \mathfrak{k}$ denote their Lie algebras. Consider a finite-dimensional complex representation

$$\pi : K \to \mathrm{GL}(V_\pi).$$

A linear map

$$\lambda : \mathfrak{t} \to \mathbb{R}$$

is called **weight** of the representation π, if there is a vector $0 \neq v \in V_\pi$, such that

$$\pi\left(\exp(X)\right) v = e^{2\pi i \lambda(X)} v$$

holds for every $X \in \mathfrak{t}$. The set

$$V_\pi(\lambda) = \left\{ v \in V_\pi : \pi\left(\exp(X)\right) v = e^{2\pi i \lambda(X)} v \right\}$$

is a subspace of V_π, the **weight space** of the weight λ.

Lemma 4.3.5. *A finite-dimensional representation (π, V_π) of K decomposes as a direct sum of its weight spaces*

$$V_\pi = \bigoplus_\lambda V_\pi(\lambda),$$

where the sum runs over all weights of the representation.

Proof. This is the isotypical decomposition of $\pi|_T$. □

* * *

4.4 Roots

Definition 4.4.1. Let K be a connected compact Lie group and $T \subset K$ be a maximal torus. The **non-trivial** weights of the adjoint representation

$$\mathrm{Ad} : K \to \mathrm{Aut}_{\mathbb{C}}(\mathfrak{k}_{\mathbb{C}})$$

are called **roots**. Note that $\mathfrak{k}$ in general needs to be complexified for roots to exist.

Proposition 4.4.2. *The complexified Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is a direct sum*

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{k}_\alpha,$$

where Φ *is the set of roots and for every root* $\alpha : \mathfrak{t} \to \mathbb{R}$, *the* **root space** *is*

$$\mathfrak{k}_\alpha = \left\{ X \in \mathfrak{k}_\mathbb{C} : \mathrm{Ad}\left(\exp(Y)\right) X = e^{2\pi i \alpha(Y)} X \right\}.$$

Proof. This is just Lemma 4.3.5 applied to the adjoint representation. It only needs to be shown, that the 0-root space $\mathfrak{k}_{\mathbb{C},0}$ of the representation is exactly $\mathfrak{t}_\mathbb{C}$. The inclusion $\mathfrak{t}_\mathbb{C} \subset \mathfrak{k}_{\mathbb{C},0}$ is clear. Conversely let $Y \in \mathfrak{k}_{\mathbb{C},0}$, so $[Y, \mathfrak{t}_\mathbb{C}] = 0$. As $\mathfrak{t}_\mathbb{C}$ is the complexification of $\mathfrak{t}$, the equation $[Y, \mathfrak{t}_\mathbb{C}] = 0$ is equivalent to $[Y, \mathfrak{t}] = 0$. If $Y = A + Bi$ with $A, B \in \mathfrak{k}$, then it follows $[A, \mathfrak{t}] = \mathrm{Re}([Y, \mathfrak{t}]) = 0$ and likewise for B. Then $\exp(\mathfrak{t} + \mathbb{R}A)$ is an abelian subgroup of K and since T is maximal abelian, we get $A \in \mathfrak{t}$ and likewise for B. □

Lemma 4.4.3. *For the root spaces one has*

$$[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta}.$$

Here the cases $\alpha + \beta = 0$, $\alpha = 0$ *or* $\beta = 0$ *are included, where we write*

$$\mathfrak{k}_0 = \mathfrak{t}.$$

Proof. Let $X \in \mathfrak{k}_\alpha$ and $Y \in \mathfrak{k}_\beta$. For $t \in T$ one has

$$\begin{aligned} \mathrm{Ad}(t)[X, Y] &= [\mathrm{Ad}(t)X, \mathrm{Ad}(t)Y] \\ &= [e^{2\pi i \alpha(t)} X, e^{2\pi i \beta(t)} Y] = e^{2\pi i (\alpha+\beta)(t)} [X, Y]. \end{aligned}$$

□

Example 4.4.4. Let $K = \mathrm{SU}(3)$. Then

$$T = \left\{ \mathrm{diag}(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) : a, b, c \in \mathbb{R},\ a + b + c = 0 \right\}$$

is a maximal torus. The real Lie algebra of K is

$$\mathfrak{su}(3) = \left\{ X \in \mathrm{M}_3(\mathbb{C}) : X^* = -X,\ \mathrm{tr}\, X = 0 \right\}.$$

The complexification can be identified with

$$\mathfrak{sl}_\mathbb{C}(3) = \left\{ X \in \mathrm{M}_3(\mathbb{C}) : \mathrm{tr}\, X = 0 \right\}.$$

The dimension of K is 8, the dimension of a maximal torus is 2, so there are maximally

6 roots. Indeed, the roots are

$$\begin{aligned}\alpha(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}) &= a-b,\\ \beta(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}) &= a-c,\\ \gamma(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}) &= b-c\end{aligned}$$

and their negatives. The root spaces are spanned by

$$X_\alpha = E_{1,2}, \quad X_\beta = E_{1,3}, \quad X_\gamma = E_{2,3}.$$

* * *

5 Root systems

5.1 Groups of rank 1

Let K be a connected compact Lie group and let T be a maximal torus in K. Recall that for a root α the root space is defined as

$$\mathfrak{k}_\alpha = \left\{ X \in \mathfrak{k}_{\mathbb{C}} : \mathrm{Ad}\,(\exp(Z))\, X = e^{2\pi i\alpha(Z)} X \ \ \forall_{Z\in\mathfrak{t}} \right\}.$$

As the derivative of Ad is the Lie-representation ad, we can take derivatives and conclude

$$\mathfrak{k}_\alpha = \left\{ X \in \mathfrak{k}_{\mathbb{C}} : [Y, X] = 2\pi i\alpha(Y)\, X \quad \forall_{Y\in\mathfrak{t}} \right\}$$

Theorem 5.1.1. *Let K be a connected compact Lie group of rank 1 with* $\dim K > 1$. *Then one has $K \cong \mathrm{SO}(3)$ or $K \cong \mathrm{SU}(2)$.*
In particular, we have $\dim(K) = 3$ *and* $|W(K)| = 2$.

Proof. Let T be a maximal torus with Lie algebra $\mathfrak{t}$. For $H \in \mathfrak{t} \setminus \{0\}$ there is $n_\alpha \in \mathbb{R}$ such that $[H, X_\alpha] = in_\alpha X_\alpha$ for every $X_\alpha \in \mathfrak{k}_\alpha$. Then one has $n_{-\alpha} = -n_\alpha$. Let $X \mapsto X^c$ be the complex conjugation on $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$.

For $0 \neq X_\alpha \in \mathfrak{k}_\alpha$ one has $X_\alpha^c \in \mathfrak{k}_{-\alpha}$, since the eigenvalue is complex conjugate. Assume, that $[X_\alpha, X_\alpha^c] = 0$. Then the sub algebra of $\mathfrak{k}$, generated by $X_\alpha + X_\alpha^c$ and $i(X_\alpha - X_\alpha^c)$, is abelian, which is impossible, since K has rank 1.

Thus we conclude that $[X_\alpha, X_\alpha^c] \neq 0$. On the other hand one has $[X_\alpha, X_\alpha^c] \in \mathfrak{k}_0 = \mathfrak{t}_\mathbb{C}$, hence there is $\lambda \in i\mathbb{R}$ with $[X_\alpha, X_\alpha^c] = \lambda H$. Let $\Phi^+ = \left\{\alpha : n_\alpha > 0\right\}$ and let n_β be the smallest of the numbers n_α, $\alpha \in \Phi^+$. Choose $X_\beta \in \mathfrak{k}_\beta \smallsetminus \{0\}$ and set $X_{-\beta} := cX_\beta$. The subspace V of $\mathfrak{k}_\mathbb{C}$, generated by $\mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha$ and $X_{-\beta}$, is stable unter $\operatorname{ad}(H) : X \mapsto [H, X]$. On V, the endomorphism $\operatorname{ad} H$ has trace

$$\sum_{\alpha\in\Phi^+\smallsetminus\{\beta\}} in_\alpha \dim \mathfrak{k}_\alpha + in_\beta(\dim \mathfrak{k}_\beta - 1).$$

As $[\mathfrak{k}_\alpha, \mathfrak{k}_\gamma] \subset \mathfrak{k}_{\alpha+\gamma}$, the space V also is stable under X_β and $X_{-\beta}$. One has $[X_\beta, X_{-\beta}] = \lambda H$ for some $\lambda \neq 0$ and so

$$\operatorname{ad} H = -\frac{1}{\lambda}\left(\operatorname{ad} X_\beta \operatorname{ad} X_{-\beta} - \operatorname{ad}_{-\beta} \operatorname{ad} X_\beta\right).$$

This means, that the trace of $\operatorname{ad} H$ vanishes. It follows $\mathfrak{k}_\alpha = 0$ if $\alpha \in \Phi^+$, $\alpha \neq \beta$ and $\dim L_\beta = 1$, which means that

$$\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{k}_\beta \oplus \mathfrak{k}_{-\beta}.$$

So the dimension equals 3 and the Weyl group has order 2. Choose an $\operatorname{Ad}(K)$-invariant inner product on $\mathfrak{k}$. Then Ad is a homomorphism

$$\operatorname{Ad} : K \to \mathrm{SO}(3).$$

We claim, that the differential of Ad at the point e has fiull rank. **Assume**, this is not the case. Then there is an $X \in \mathfrak{k}$ with $D\operatorname{Ad}(e)(X) = 0$. The 1PSG $\{\exp(tX) : t \in \mathbb{R}\}$ then is central, therefore K has a center $Z(K)$ of dimension > 0. As $K/Z(K)$ also contains a maximal torus, every maximal torus of K has dimension > 1, **contradiction!**

It follows that $\operatorname{Ad} : K \to \mathrm{SO}(3)$ is a covering. As the covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is the universal covering and has degree 2, there are exactly 2 coverings, SU(2) and $SO(3)$. □

* * *

5.2 Weyl-chambers

The root space for a root α

$$\mathfrak{k}_\alpha = \left\{X \in \mathfrak{k}_{\mathbb{C}} : [Y, X] = 2\pi i \alpha(Y) X \ \forall_{Y \in \mathfrak{t}}\right\}$$

can also be written as

$$\mathfrak{k}_\alpha = \left\{X \in \mathfrak{k}_{\mathbb{C}} : \mathrm{Ad}(t) X = t^\alpha X \ \forall_{t \in T}\right\}.$$

Definition 5.2.1. Let Φ be the set of all roots. For $\alpha \in \Phi$ write

$$U_\alpha = \left\{t \in T : t^\alpha = 1\right\}.$$

Proposition 5.2.2. *Let K a connected compact Lie group,*

(i) *Let $y \in T$ and let $Z(y) \subset K$ the centraliser . For the Lie algebra one has*

$$\mathfrak{z}(y) = \mathrm{Lie}\,(Z(y)) = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Phi \\ y^\alpha = 1}} \mathfrak{k}_\alpha.$$

(ii) *The center of K is*

$$Z(K) = \bigcap_{\alpha \in \Phi} U_\alpha.$$

(iii) *The set $\bigcup_{\alpha \in \Phi} U_\alpha$ equals the set $T \smallsetminus T^{\mathrm{reg}}$ of all singular points.*

Proof. (i) Let $X \in \mathfrak{z}(y)$. Then one has $\exp(tX) \in Z(y)$ for every $t \in \mathbb{R}$. We have $\exp(tX) = y \exp(tX) y^{-1} = \exp(t\,\mathrm{Ad}(y)X)$ and since this holds for all t, it follows $\mathrm{Ad}(y)X = X$. Writing X as sum of its root space components, one gets the claim. Coversely, let $X \in \mathfrak{k}_\alpha$ for some α with $y^\alpha = 1$. Then $\mathrm{Ad}(y)X = X$ and man gets $\exp(tX) \in Z(y)$ for every $t \in \mathbb{R}$, so $X \in \mathfrak{z}(y)$.

(ii) Let $x \in Z(K)$. Then $x \in T$ and $x^\alpha = 1$ for every α. Conversely, let x be in the intersection of all U_α. The rot space decomposition gives $\mathrm{Ad}(x)X = X$ for every $X \in \mathfrak{k}$, so $x \exp(X) x^{-1} = \exp(X)$. The subgroup H of all $y \in K$ with $xyx^{-1} = y$ contains an open unit neighbourhood, hence is open, and so it equals K, as K is connected.

(iii) Let $y \in U_\alpha$. Then $\exp(\mathfrak{k}_\alpha)$ lies in the centraliser $Z(y)$. Further T lies in this centraliser, hence $Z(y)^0$ is strictly bigger than T and can therefore not be a torus.

Hence y is singular. Conversely let $y \notin \bigcup_\alpha U_\alpha$, then $y^\alpha \neq 1$ for every α. By (i) one has $\mathfrak{z}(y) = \mathfrak{t}$ and therefore $Z(y)^0 = T$ because of maximality of T. □

Example 5.2.3. Let $K = \mathrm{SU}(2)$, $T = \{\left(\begin{smallmatrix}\varepsilon & \\ & \overline{\varepsilon}\end{smallmatrix}\right) : \varepsilon \in \mathbb{T}\}$. Then one has

$$\begin{pmatrix}\varepsilon & \\ & \overline{\varepsilon}\end{pmatrix}\begin{pmatrix}1 & z \\ -\overline{z} & 0\end{pmatrix}\begin{pmatrix}\overline{\varepsilon} & \\ & \varepsilon\end{pmatrix} = \begin{pmatrix}0 & \varepsilon^2 z \\ -\overline{\varepsilon}^2\overline{z} & 0\end{pmatrix}.$$

It follows that the roots of SU(2) are $\pm\alpha$ with

$$\left(\begin{smallmatrix}\varepsilon & \\ & \overline{\varepsilon}\end{smallmatrix}\right)^\alpha = \varepsilon^2, \qquad \left(\begin{smallmatrix}\varepsilon & \\ & \overline{\varepsilon}\end{smallmatrix}\right)^{-\alpha} = \varepsilon^{-2}.$$

One concludes $U_\alpha = \{1, -1\}$.

Lemma 5.2.4. *Let u be a topological generator of U_α^0, the connected component of U_α. Let $r = \operatorname{rank}(K) = \dim T$. Then one has dim$(U_\alpha) = r - 1$ and*

$$\dim Z(u) = r + 2.$$

If α is a root, then so is $-\alpha$.

Proof. The dimenson of U_α is $r - 1$, since U_α is the kernel of $(.)^\alpha$. By Theorem 4.2.9 the group $Z(u) = Z(U_\alpha^0)$ is connected. Consider the diagram

$$\begin{tikzcd} Z(u) \arrow[r] & Z(u)/U_\alpha^0 \\ T \arrow[r] \arrow[u, hook] & T/U_\alpha^0 \arrow[u, hook] \end{tikzcd}$$

As T is a maximal torus in K, it is a maximal torus in $Z(u)$ and therefore T/U_α^0 is a maximal torus in $Z(u)/U_\alpha^0$. But $\dim T/U_\alpha^0 = 1$ and $\dim Z(u) \geq r + 2$ by Proposition 5.2.2 (i). Then it follows from Theorem 5.1.1, that $\dim Z(u)/U_\alpha^0 = 3$, i.e., $\dim Z(u) = r + 2$.

Finally for $-\alpha$: We have that $-\alpha$ is a root of $Z(u)/U_\alpha^0$ by Theorem 5.1.1. Hence it is a root of K. □

Proposition 5.2.5. *Let U_α^0 be the connected component of U_α.*

(a) *If $U_\alpha^0 = U_\beta^0$, then $\alpha = \pm\beta$.*

(b) *Every rot space $\mathfrak{k}_\alpha$ is of complex dimension 1.*

(c) $\dim(K) = \operatorname{rank}(K) + 2m$, *where* $2m = |\Phi|$ *is the number of roots.*

(d) *The Weyl group of* $Z(U_\alpha) = Z(U_\alpha^0)$ *has order 2.*

(e) *Let* α, β *be roots with* $\mathbb{R}\alpha = \mathbb{R}\beta$*. Then* $\beta = \pm\alpha$*.*

(f) *One has* $Z(U_\alpha) = Z(U_\alpha^0)$*. Let* w *in the Weyl group of* $Z(U_\alpha)$*. Then* $w(u) = u$ *for every* $u \in U_\alpha$*.*

Proof. (a) and (b) follow from Proposition 5.2.2 and the lemma. This also implies (c).

(d) By Theorem 4.2.15 the Weyl group of $(Z(u_\alpha), T)$ is isomorphic to the Weyl group of $Z(u_\alpha)/U_\alpha^0$. The claim follows from Theorem 5.1.1.

(e) Let $\alpha = c\beta$ with $c \in \mathbb{R}^\times$. Then $\ker(\alpha) = \ker(\beta) = LU_\alpha^0 = LU_\beta^0$. It follows $U_\alpha^0 = U_\beta^0$ and the claim follows by Proposition 5.2.5 (a).

(f) One has $Z(U_\alpha) \subset Z(U_\alpha^0)$. As $T \subset Z(U_\alpha$ and $\exp(\mathfrak{k}_{\pm\alpha}) \subset Z(U_\alpha)$, we get $\dim Z(u) \geq k + 2 = \dim Z(U_\alpha^0)$. The group $Z(U_\alpha^0)$ is connected, whence $Z(U_\alpha) = Z(U_\alpha^0)$. □

Corollary 5.2.6. *For every root* α *the group* $U_\alpha \subset T$ *has at most two connected components.*

Proof. Let w_α be the generator of the Weyl group of $Z(U_\alpha)$. The action of w_α induces a topological group isomorphism

$$\phi : S^1 \cong T/U_\alpha^0 \xrightarrow{w_\alpha} T/U_\alpha^0 \cong S^1.$$

This map has order 2, i.e., $\phi \circ \phi = \mathrm{Id}$ and is non-trivial. It follows $\phi(e^{2\pi i t}) = e^{-2\pi i t}$. The map has exactly two fixed points± 1.

Every connected component of U_α gives a fixed point of ϕ: A connected component is a coset $t_0 U_\alpha^0$ with $t_0 \in U_\alpha$.

The element w_α lies in $Z(U_\alpha)$ and since t_0 lies in U_α, it is centralised by w_α, i.e., $w_\alpha t_0 w_\alpha^{-1} = t_0$ and so $t_0 U_\alpha^0$ is a fixed point of ϕ. As ϕ has 2 fixed points, there are at most 2 components. □

Definition 5.2.7. Choose an element $H_0 \in \mathfrak{t}$, such that $\alpha(H_0) \neq 0$ for every $\alpha \in \Phi$. This is possible, because the set the roots is finite. Then set

$$\Phi^+ = \Big\{\alpha \in \Phi : \alpha(H_0) > 0\Big\}.$$

It follows $\Phi = \Phi^+ \dot\cup (-\Phi^+)$. For $\alpha \in \Phi$ ee define

$$M_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha},$$

then one has

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} M_\alpha.$$

Instead of $\alpha \in \Phi^+$ one also writes $\alpha > 0$ and calls Φ^+ the set of the **positive roots**.

Definition 5.2.8. The finitely many subspaces

$$\mathcal{H}_\alpha = \mathrm{Lie}(U_\alpha) = \ker \alpha, \qquad \alpha \in \Phi^+$$

decompose $\mathfrak{t}$ into finitely many convex domains of the form

$$\left\{ v \in \mathfrak{t} : \mathrm{sign}(\alpha(v)) = \varepsilon_\alpha,\ \alpha \in \Phi^+ \right\},$$

where $\varepsilon \in \{\pm 1\}^{\Phi^+}$. A domain of this form is called a **Weyl chamber**. If C is a Weyl chamber, then the boundary $\partial C = \overline{C} \smallsetminus C$ consists of finitely many hypersurfaces. These are called the **walls** of the chamber.

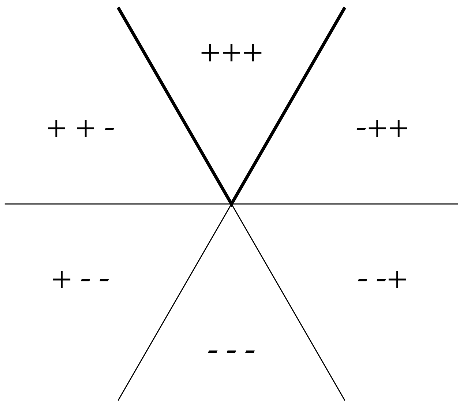
Example 5.2.9. Let $K = \mathrm{SU}(3)$, as in Example 4.4.4. The Lie algebra of the torus is

$$\mathfrak{t} = \left\{ \mathrm{diag}(x, y, z) : x, y, z \in i\mathbb{R},\ x + y + z = 0 \right\}.$$

The complexified Lie algebra is the abelian Lie algebra of all diagonal matrices of trace zero. A system of positive roots is given by

$$\alpha(x, y, z) = x - y, \quad \beta(x, y, z) = x - z, \quad \gamma(x, y, z) = y - z.$$

There are 6 Weyl chambers:

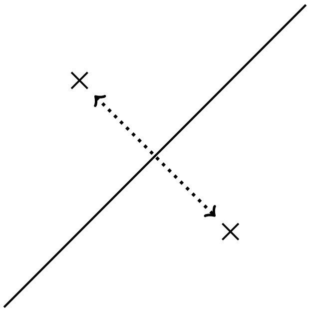


The signs are the signs of the three roots α, β, γ and the two walls of the chamber $+++$ have are drawn thicker.

Definition 5.2.10. On the Lie algebra $\mathfrak{k}$ we install an $\mathrm{Ad}(K)$-invariant inner product $\langle .,. \rangle$. We identify $\mathfrak{t}^*$ with $\mathfrak{t}$ by means of this inner product. Every root α yields a **reflection** $s_\alpha : \mathfrak{t} \to \mathfrak{t}$, given by

$$s_\alpha(v) = v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

The fixed point set of this reflection is the hypersurface $\mathcal{H}_\alpha$.



Theorem 5.2.11.

(a) *Let $\alpha \in \Phi$. The reflection s_α lies in the Weyl group W. This means, there is be a $w_\alpha \in W$, such that $w(X) = s_\alpha(X)$ for every $X \in \mathfrak{t}$. In particular, every reflection s_α maps roots to roots.*

(b) *Let C be a Weyl chamber. The reflections at the walls of C generate the Weyl group.*

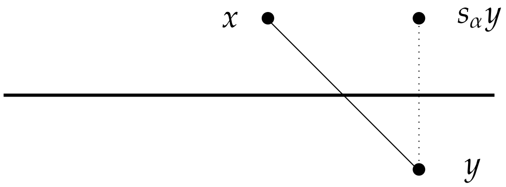
(c) *The Weyl group acts simply transitively on the set of Weyl chambers. In particular, there are exatcly $|W|$ Weyl chambers.*

Proof. (a) The inclusion $Z(U_\alpha) \subset K$ induces an inclusion of the Weyl groups. The generator w_α of the Weyl group of $Z(U_\alpha)$ fixes $\mathcal{H}_\alpha$ pointwise, preserves the inner product and has order 2. Hence $w_\alpha = s_\alpha$.

Part (b) and (c) follows from the next two lemmas.

Lemma 5.2.12. *Let C be a fixed Weyl chamber and let $W' \subset W$ be the subgroup generated by all reflections at the walls of C. Then W' acts transitively on the set of Weyl chambers.*

Proof. Let D be another Weyl chamber and let $x \in C$, $y \in D$. One has that $y \notin C$. Consider the line segment L connecting x and y. This L passes through at least one wall $\mathcal{H}_\alpha$ and then the point $s_\alpha y$ is closer to x as y.



We repeat this with $s_\alpha y$ instead of y and since there are only finityely many Weyl chambers, this process stops with one $w \in W'$, such that wy lies in C. This means $wD \cap C \neq \emptyset$ and since the Weyl group maps chambers to chambers, we get $wD = C$. □

Lemma 5.2.13. *Let C be a Weyl chamber and $w \in W$ such that $wC = C$. Then one has $w = 1$.*

Proof. Let n be the order of the element $w \in W$ and let $x \in C$. The point

$$y = \frac{1}{n}\sum_{j=1}^{n} w^j x$$

lies in C, since C is convex. One has $wy = y$, so w has a fixed point in C. As w acts linearly, it fixes the line $\mathbb{R}y$ pointwise. Let $g \in N(T)$ be aa element with $w = gT$. Then conjugation with g fixes the 1PSG $P = \exp(\mathbb{R}y)$ pointwise. The group generated by P and g, is abelian and lies in a maximal torus S. The group P lies in K^{reg}, as y lies in the interior of the Weyl chamber. Therefore P lies in exactly one maximal torus and so we get $S = T$ which implies $g \in T$ and hence $w = 1$. □

The theorem is proven. □

Proposition 5.2.14. *For any two roots α, β one has*

$$\beta - s_\alpha(\beta) \in \mathbb{Z}\alpha.$$

Proof. Let $E \in \mathfrak{k}_\alpha$ and $F \in \mathfrak{k}_{-\alpha}$ nonvanishing elements. Let $H = [E, F]$. Then H lies in $\mathfrak{t}$. One has $[H, E] = \alpha(H)E$. **ssume** that $\alpha(H) = 0$. Then $[E, F]$ is in the kernel of α. It folows that $\exp(\mathbb{R}E)$ and $\exp(\mathbb{R}F)$, together with U_α generate an abelian group of dimension $\geq \dim U_\alpha + 2 = \dim T + 1$, which is not possible by maximality of the torus T. **Contradiction!**

So one has $\alpha(H) \neq 0$. We can choose E in a way that $\alpha(H) = 2$. Then (H, E, F) is an sl2-Tripel. The complex Lie algebra $s\ell_2(\mathbb{C})$, generated by this triple, acts on $\mathfrak{k}_{\mathbb{C}}$ by the adjoint representation and by Corollary 2.5.5 it has H integer eigenvalues. For $Z \in \mathfrak{k}_\beta$ one has

$$[H, Z] = \beta(H)Z$$

and it follows $\beta(H) \in \mathbb{Z}$. Therefore we have

$$(\beta - s_\alpha(\beta))\,(H) = \beta(H) - \beta(\underbrace{s_\alpha(H)}_{-H}) = 2\beta(H).$$

As $\beta - s_\alpha(\beta) \in \mathbb{R}\alpha$ and $\alpha(H) = 2$, this implies

$$\beta - s_\alpha(\beta) = \beta(H)\alpha. \qquad \square$$

* * *

5.3 Abstract root systems

Definition 5.3.1. Let E be a finite-dimensional real vector space with an inner product $\langle .,. \rangle$. A **root system** in E is a finite set $\Phi \subset E \smallsetminus \{0\}$, such that

(i) Φ spans E,

(ii) for every $\alpha, \beta \in \Phi$ with $\mathbb{R}\alpha = \mathbb{R}\beta$ one has $\beta = \pm\alpha$,

(iii) for $\alpha \in \Phi$ one has $s_\alpha(\Phi) = \Phi$,

(iv) for every $\beta \in \Phi$ it holds $s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$.

Definition 5.3.2. The **Weyl group** o the root system Φ is the subgroup of $\mathrm{GL}(E)$, generated by all reflections s_α, $\alpha \in \Phi$.

$$s_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

For a root α let

$$\hat{\alpha} = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$$

be the **co-root** of α. Then $\widehat{\Phi} = \left\{\hat{\alpha} : \alpha \in \Phi\right\}$ also is a root system.

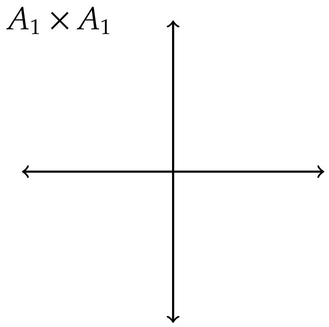
Remark 5.3.3. Condition (iv) of 5.3.1 can also be expressed as $\{\beta : \alpha\} \in \mathbb{Z}$ mit

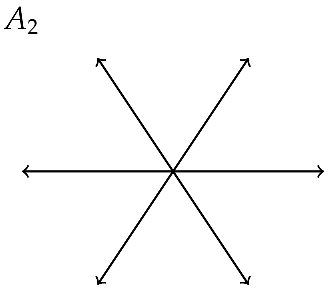
$$\{\beta : \alpha\} = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

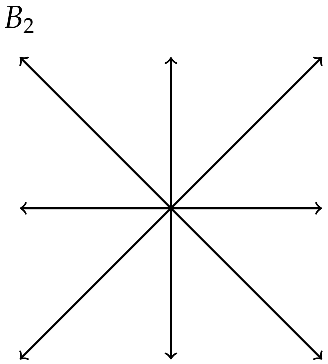
These numbers are called the **Cartan numbers** of the root system.

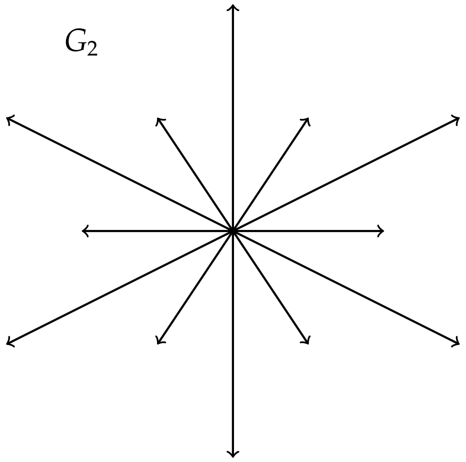
The dimension of E is called the **rank** of the root system.

Her a liost of all root systems of rank 2:









Lemma 5.3.4. *For two roots $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$, the angle θ between them can only take*

the values $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$*. The number* $\{\alpha : \beta\}$ *can only take the following values:*

$\{\alpha : \beta\}$	$\{\beta : \alpha\}$	θ	$\left\|\beta\right\|^2 / \|\alpha\|^2$
0	0	$\frac{\pi}{2}$	*
1	1	$\frac{\pi}{3}$	1
−1	−1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
−1	−2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
−1	−3	$\frac{5\pi}{6}$	3

Proof. One has $\|\alpha\| \left\|\beta\right\| \cos\theta = \langle \alpha, \beta \rangle$. Therefore $\{\beta : \alpha\} = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta$ and $\{\alpha : \beta\}\{\beta : \alpha\} = 4\cos^2\theta$. This last number is a positive integer, but $0 \le \cos^2\theta \le 1$ and the numbers $\{\alpha : \beta\}$ and $\{\beta : \alpha\}$ have the same sign, yielding the given list. □

Lemma 5.3.5. *Let* α, β *be two non-proportional roots. If the angle* θ *is maller then* $\pi/2$*, so* $\langle \alpha, \beta \rangle > 0$*, then* $\alpha - \beta$ *is a root, too. If* $\langle \alpha, \beta \rangle < 0$*, then* $\alpha + \beta$ *is a root.*

Proof. The second assertion follows from the first by replacing β with $-\beta$. So assume $\langle \alpha, \beta \rangle > 0$. By the list of the last lemma, one of the two numbers $\{\alpha : \beta\}, \{\beta : \alpha\}$ must be 1. If $\{\alpha : \beta\} = 1$, then $s_\alpha(\beta) = \alpha - \beta \in \Phi$. Otherwise, $s_\beta(\alpha) = \beta - \alpha \in \Phi$. □

* * *

5.4 Simple roots

Definition 5.4.1. Let $\Phi \subset E$ be an abstract root system and let $x_0 \in E^{\mathrm{reg}}$. Let

$$\Phi^+ = \left\{ \alpha \in \Phi : \left\langle \phi, x_0 \right\rangle > 0 \right\}.$$

We call the roots in Φ^+ the **positive roots**.

The set Φ^+ only depends on the Weyl chamber which contains x_0. So there are $|W|$ many different choices of sets of positive roots.

Definition 5.4.2. A subset $\Delta \subset \Phi$ is called **basis** of the root system, if

(a) Δ is a basis of the real vector space E and

(b) every root β can be writen as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where either all $k_\alpha \in \mathbb{Z}_{\geq 0}$ or all $k_\alpha \in \mathbb{Z}_{\leq 0}$.

Theorem 5.4.3. *Every root system Φ has a Basis.*
More precisely, for every choice Φ^+ of positive roots there is exactly one basis Δ, which is contained in Φ^+. In this case, the positive roots are positive linear combinations.

Proof. Choose a vector $x_0 \in E^{\text{reg}}$ and the corresponding choice Φ^+. We sagen, $\alpha \in \Phi^+$ is **decomposable**, if there are $\beta_1, \beta_2 \in \Phi^+$ gibt, such that $\alpha = \beta_1 + \beta_2$. Otherwise, α is **non-decomposable**. Let Δ be the set of all non-decompsable roots in Φ^+.

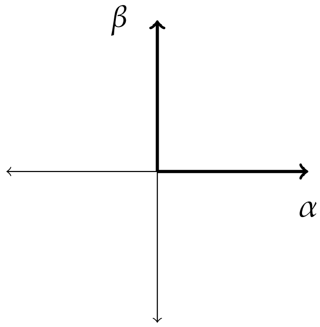
We show that every positive root is a positive $\mathbb{Z}$-linear combination of elements of Δ ist.
Assume that there is an $\alpha \in \Phi^+$ which cannot be written in this way and choose α such that $\langle x_0, \alpha\rangle$ has the smallest value among all such α. Now α itself is decomposable and so $\alpha = \beta_1 + \beta_2$ and $\langle x_0, \alpha\rangle = \langle x_0 \beta_1\rangle + \langle x_0, \beta_2\rangle$, where both inner products are > 0. So both are $< \langle x_0, \alpha\rangle$, hence β_1, β_2 both are $\mathbb{Z}_+$-linear combinations of elements of Δ. Therefore, α is, too. **Contradiction!**

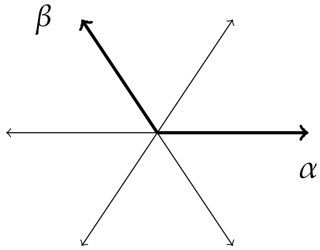
Next we show that for $\alpha, \beta \in \Delta$ one has $\langle \alpha, \beta\rangle \leq 0$ except in the case when $\alpha = \beta$. Otherwise $\alpha - \beta$ is a root, so $\alpha - \beta$ or $\beta - \alpha$ lies in Φ^+. In the first case $\alpha = \beta + (\alpha - \beta)$ and α is decomposable, which is a contradiction. The other case is analogous.

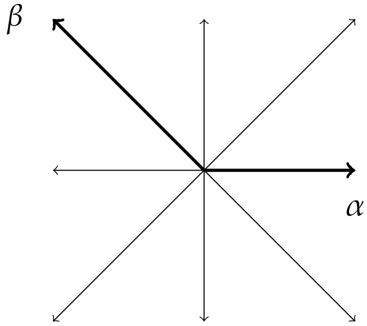
It remains to show, that Δ is linealy independent. So let $\sum_{\alpha\in\Delta} c_\alpha \alpha = 0$. We separate indices with $c_\alpha < 0$ from those with $c_\alpha > 0$, we can write $\sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$ with $t_\alpha, s_\beta > 0$, where the α and β run through two disjoint sets of roots. Let $e = \sum_\alpha s_\alpha \alpha$. Then one has $\langle e, e\rangle = \sum_{\alpha,\beta} t_\alpha s_\beta \langle \alpha, \beta\rangle \leq 0$, so $e = 0$ and all c_α sind Null. □

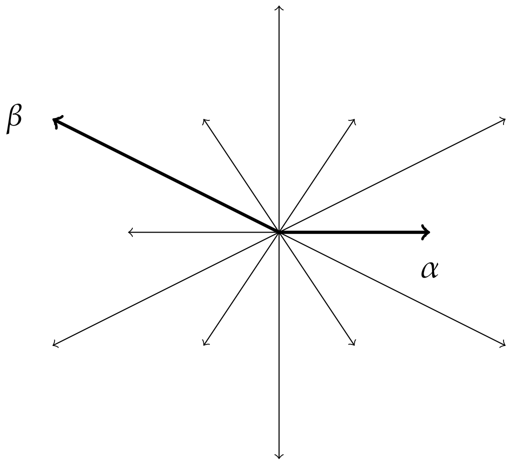
Corollary 5.4.4. *If Δ is a basis and $\alpha, \beta \in \Delta$, then one has $\langle \alpha, \beta\rangle \leq 0$ and $\alpha - \beta$ is not a root.*

Example 5.4.5. In each of the two-dimensional cases, we give a posible choice of a basis $\{\alpha, \beta\}$.









Theorem 5.4.6. (a) *If Δ is a basis, then*

$$C_\Delta = \left\{ x \in E : \langle x, \alpha \rangle > 0 \ \forall_{\alpha \in \Delta} \right\}$$

is a Weyl chamber. The map $\Delta \mapsto C_\Delta$ is a bijection between the set of bases and the set of Weyl chambers.

(b) *The Weyl group acts be simply transitively on the set the Weyl chambers.*

Proof. (a) As every element of Φ^+ is a positive linear combination of $\alpha \in \Delta$, we get

$$C_\Delta = \left\{ x \in E : \langle x, \alpha \rangle > 0 \ \forall_{\alpha \in \Phi^+} \right\}.$$

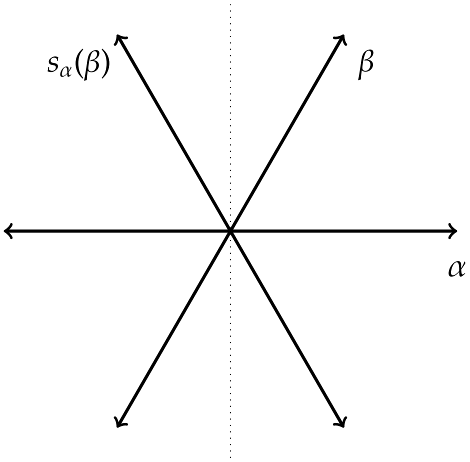
If $\beta \in \Phi^- = \Phi \smallsetminus \Phi^+$ and $x \in C_\Delta$, then one has $\langle x, -\beta \rangle > 0$, so together one gets that C_Δ lies in E^{reg} hence inside a Weyl chamber, since C_Δ is convex. Every boundary point x of C_Δ satisfies $\langle x, \alpha \rangle = 0$ for some root α, so it's not in E^{reg}. that means that C_Δ is indeed a Weyl chamber.

If conversely, a Weyl chamber C is given, then every point $x_0 \in C$ defines a choice Φ^+ of positive roots and this choice defines a basis Δ.

(b) The proof is similar to the case of root systems of compact groups, Theorem 5.2.11. □

If not otherwise specified, we will always assume a fixed choice of a set of positive roots is given.

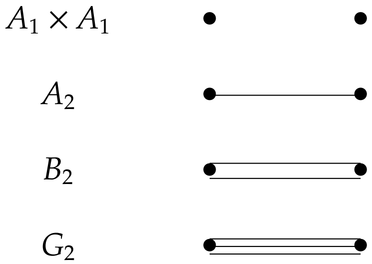
Lemma 5.4.7. *If α is a simple root and is $\beta \neq \alpha$ a positive root, then is $s_\alpha(\beta)$ is again a positive root. In this way the positive root $\alpha \in \Delta$ permutes the set $\Phi^+ \smallsetminus \{\alpha\}$.*



In this picture the positive roots are $\alpha, \beta, s_\alpha(\beta)$.

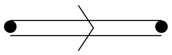
Proof. Let $\alpha \in \Delta$ and let $\beta \neq \alpha$ be a positive root. Let C be the Weyl chamber, given by Φ^+ or Δ. Then $s_\alpha(C)$ is a Weyl chamber again, which still lies on the positive side of the hyperplane $\mathcal{H}_\beta$, which means that $s_\alpha(\beta)$ remains a positive root. □

Definition 5.4.8. For two different positive roots α, β the number $\{\alpha : \beta\}\{\beta : \alpha\}$ equals $0, 1, 2$ or 3. Let $\Delta = \alpha_1, \dots, \alpha_l$. Define the **Coxeter graph** as the graph (with multiple edges) with l nodes, where the i-th and the j-th are connected by $\{\alpha_i : \alpha_j\}\{\alpha_j : \alpha_i\}$ edges. Examples:

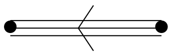


If 2 or 3 edges occur, then one of the two roots is longer than the other. In this case we give the edges a direction, pointing from the longer to the shorter root. The result is the **Dynkin diagram**:

B_2



G_2



* * *

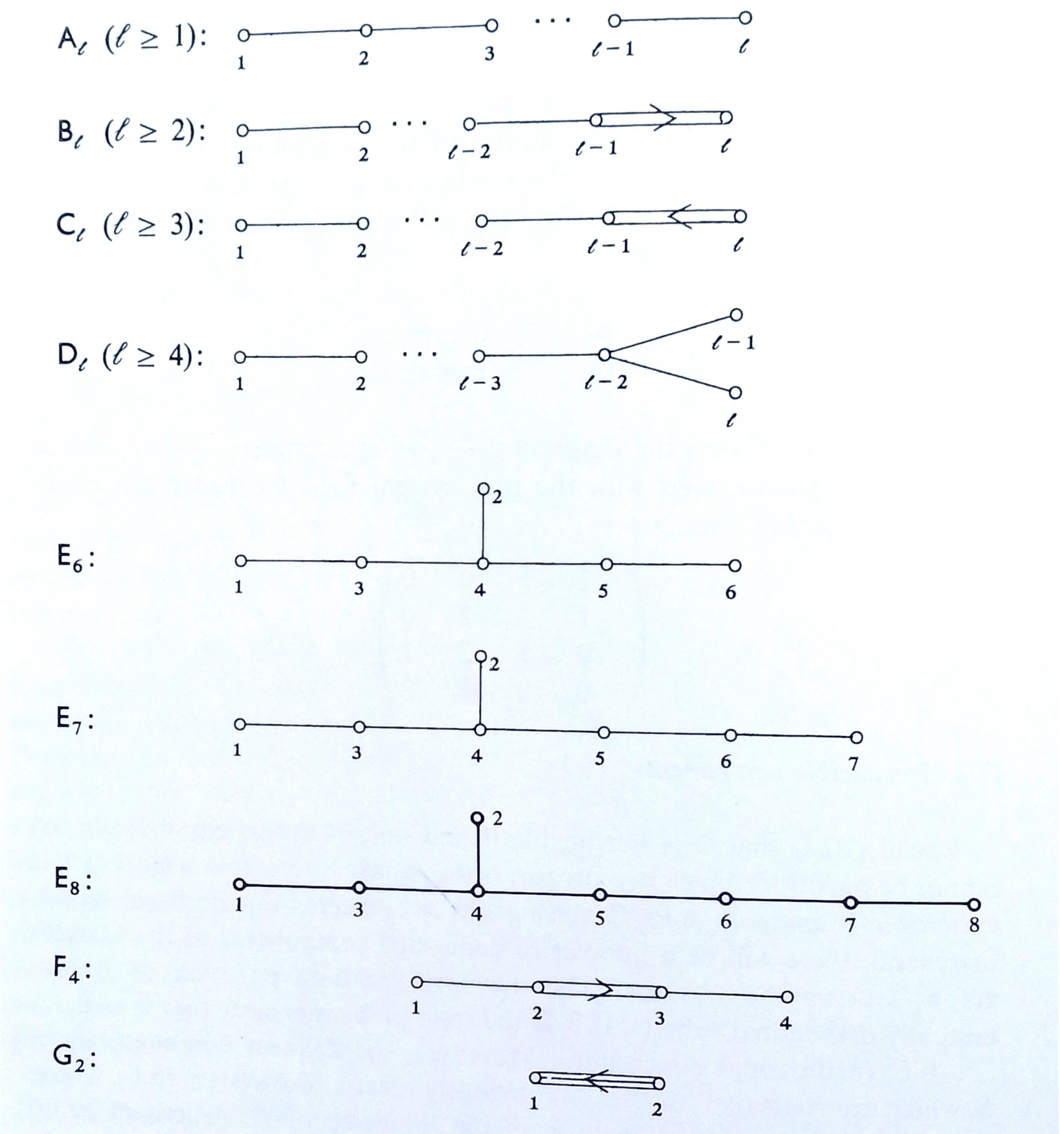
5.5 Classification

Definition 5.5.1. A root system Φ is called **reducible**, if $\Phi = A \sqcup B$ with $A \perp B$. Otherwise it is called **irreducible**.

Lemma 5.5.2. *Every root system is an orthogonal sum of irreducibles.*

Proof. Clear. $\square$

Theorem 5.5.3. *Two irreducible root systems with the same Dynkin-Diagram are isomorphic.*
If Φ is an irreducible root system of rank $l \in \mathbb{N}$, then its Dynkin-Diagramm is one of the following.



Proof. Humphreys: Introduction to Lie Algebras and Representation Theory. □

Finally: every abstract root system is the root system a compact Lie group. The root system determines the Lie algebra up to isomorphy. To show this, one has to construct a Lie group for every irreducible root systems.

* * *

6 Highest weight theory

6.1 Highest weights

Let K be a connected compact Lie group and T a maximal torus in K. The **character lattice** $X^*(T) = \mathrm{Hom}(T, \mathbb{R}/\mathbb{Z})$ can be viewed as a discrete subgroup of $\mathfrak{t}^*$. This means, we identify $X^*(T)$ with

$$\Big\{\lambda : t \to \mathbb{R} : \lambda\,(\ker(\exp)) = 0\Big\}.$$

Choose a Weyl chamber $C_0 \subset \mathfrak{t}$ and so a set Φ^+ of positive roots, i.e., those, which take positive values on C_0. Let (π, V) be a finite-dimensional irreducible representation of the Lie algebra $\mathfrak{k}$. Every Operator of the form $\pi(X)$ with $X \in \mathfrak{k}_\alpha$ and $\alpha > 0$ is called a **going-up operator** or simply **up-operator**. In the case $\alpha < 0$ it is called a **down-operator**.

Lemma 6.1.1. *For be a weight $\chi \in \mathfrak{t}^*$ of π and a root α one has*

$$\pi(\mathfrak{k}_\alpha)V(\chi) \subset V(\alpha + \chi).$$

Proof. Let $X \in \mathfrak{k}_\alpha$ and $v \in V(\chi)$. For every $H \in \mathfrak{t}$ we have $[H, X] = 2\pi i\alpha(H)X$ and $\pi(H)v = 2\pi i\chi(H)v$. Let $w = \pi(X)v$. Then one has

$$\begin{aligned}
\pi(H)w &= \pi(H)\pi(X)v \\
&= [\pi(H), \pi(X)]v + \pi(X)\pi(H)v \\
&= \pi([H, X])v + 2\pi i\pi(X)\chi(H)v \\
&= \pi(2\pi i\alpha(H)X)v + 2\pi i\chi(H)\pi(X)v = 2\pi i(\alpha(H) + \chi(H))\pi(X)v \\
&= 2\pi i(\alpha + \chi)(H)w,
\end{aligned}$$

hence $w \in V(\alpha + \chi)$. □

Choose a K-invariant inner product $\langle .,. \rangle$ on $\mathfrak{k}$. For every weight $\chi \in \mathfrak{t}^*$ there is a unique $H_\chi \in \mathfrak{t}$, such that $\chi(H) = \langle H, H_\chi \rangle$ for every $H \in \mathfrak{t}$.

Lemma 6.1.2. *We extend $\langle .,. \rangle$ as a complex bilinear form $\langle .,. \rangle_{\mathbb{C}}$ on $\mathfrak{k}_{\mathbb{C}}$.*

(a) *One has $\langle [X, Y], Z \rangle_{\mathbb{C}} = \langle X, [Y, Z] \rangle_{\mathbb{C}}$ for all $X, Y, Z \in \mathfrak{k}_{\mathbb{C}}$.*

(b) *For a root α and $X \in \mathfrak{k}_\alpha$ as well as $Y \in \mathfrak{k}_\beta$ we have $\langle X, Y\rangle_{\mathbb{C}} = 0$, except when $\beta = -\alpha$. In the latter case,*

$$\langle X, Y\rangle_{\mathbb{C}}\, H_\alpha = [X, Y].$$

Proof. (a) As both sides of the formula are trilinear, it suffices to show the claim for $X, Y, Z \in \mathfrak{k}$. Since $\langle.,.\rangle$ is invariant under K, the adjoint representation $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{k})$ is orthogonal. Therefore $\mathrm{ad} : \mathfrak{k} \to \mathrm{End}(\mathfrak{k}_{\mathbb{C}})$ is a $*$-representation, i.e., one has

$$\langle \mathrm{ad}(-Y)X, Z\rangle = \langle X, \mathrm{ad}(Y)Z\rangle\,,$$

which is equivalent to the claim.

(b) For every $H \in \mathfrak{t}$ one has

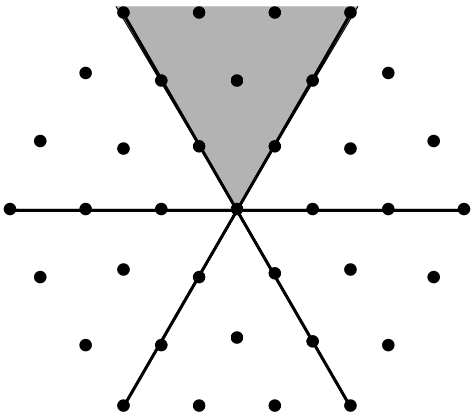
$$\begin{aligned} 2\pi i\alpha(H)\, \langle X, Y\rangle_{\mathbb{C}} &= \langle [H, X], Y\rangle_{\mathbb{C}} \\ &= -\,\langle X, [H, Y]\rangle = -2\pi i\beta(H)\,\langle X, Y\rangle_{\mathbb{C}}\,. \end{aligned}$$

This means that $\langle X, Y\rangle$ is non-zero only if $\beta = -\alpha$. In the case $\beta = -\alpha$ one has $[X, Y] \in \mathfrak{t}$ and for $H \in \mathfrak{t}$,

$$\begin{aligned} \langle [X, Y], H\rangle_{\mathbb{C}} &= \langle X, [Y, H]\rangle_{\mathbb{C}} = \alpha(H)\,\langle X, Y\rangle_{\mathbb{C}} \\ &= \langle H_\alpha, H\rangle_{\mathbb{C}}\,\langle X, Y\rangle_{\mathbb{C}} = \langle \langle X, Y\rangle_{\mathbb{C}}\, H_\alpha, H\rangle_{\mathbb{C}}\,. \end{aligned}$$

As this holds for all $H \in \mathfrak{t}$, it follows that $[X, Y] = \langle X, Y\rangle_{\mathbb{C}}\, H_\alpha$. □

Definition 6.1.3. A character $\chi : \mathfrak{t} \to \mathbb{R}$ is called **dominant**, if H_χ lies in the closure $\overline{C_0}$ of the prositive Weyl chamber.



Definition 6.1.4. A weight χ of the representation π is called **highest weight**, if

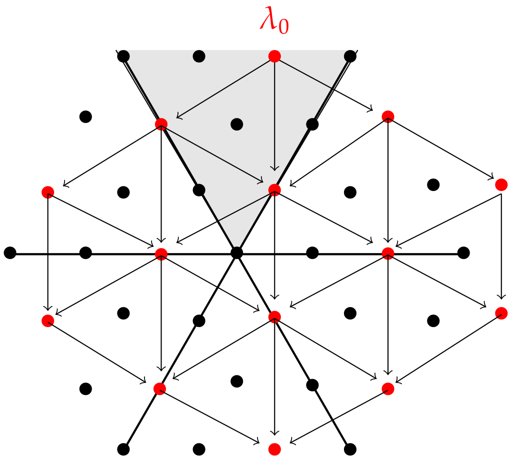
$\pi(\mathfrak{k}_\alpha)V(\chi) = 0$ for every $\alpha > 0$. This means that the weight cannot be increased by and up-operator.

Theorem 6.1.5. *Let (π, V_π) be an irreducible finite-dimensional representation of K. Then there is exactly one highest weight $\lambda = \lambda_\pi$. This is a dominant weight. The* **highest weight space** *$V_\pi(\lambda)$ is one-dimensional. Every dominant character is highest weight of a unique irreducible representation. The map $\pi \mapsto \lambda_\pi$ is a bijection*

$$\widehat{K} \quad \xrightarrow{\cong} \quad \Big\{ \textit{dominant characters} \Big\}.$$

If λ is the highest weight of π, then all weight spaces are obtained from $V_\pi(\lambda)$ by iterated application of the down-operators $\pi(X)$, $X \in \mathfrak{k}_\alpha$, $\alpha < 0$.

The following picture shows an example of the weights attached to a representation. The weights are marked red, the other characters are black. The highest weight is λ_0 and the down-operators are indicated by arrows.



Proof. The set the weights of π is stable unter the Weyl group. Therefore the closure of every Weyl chamber contains a weight and so does the positive chamber. As there are only finitely many weights, the positive chamber must contain a highest weight λ_0. Let $0 \neq v_0 \in V_\pi(\lambda_0)$ be a weight vector and let S be the space spanned by all vectors of the form Av_0, where A is a composition of descent operatoprs. Then is S stable under all down-operators and all $\pi(H)$ with $H \in \mathfrak{t}$. If we can show, that S also is stable under all up-operators, then it is stable under the whole Lie algebra and hence $S = V_\pi$.

So we show that S is stable under all up-operators. We set $v = D_k D_{k-1} \cdots D_1 v_0$ where every D_j is a down-operator and we show that $Uv \in S$ for every up-operator U. We use induction in k.

For $k = 0$ one has $v = v_0$ and since λ_0 is a highest weight, it follows $Uv_0 = 0$. Now assume the claim proven for k and let $v = D_{k+1}D_k \cdots D_1 v_0$. Let $w = D_k \cdots D_1 v_0$ and $D = D_{k+1}$, so $v = Dw$. Then D is a down-Operator, say $D = \pi(Y)$ for some $Y \in \mathfrak{k}_\beta$ with a negative root β. Let $X \in \mathfrak{k}_\alpha$ for a positive root α. By induction hypothesis one has

$$\begin{aligned} \pi(X)v &= \pi(X)Dw = \pi(X)\pi(Y)w \\ &= [\pi(X), \pi(Y)]w + \underbrace{\pi(Y)\underbrace{\pi(X)w}_{\in S}}_{\in S}. \end{aligned}$$

Let $Z = [X, Y] \in \mathfrak{k}_{\alpha+\beta}$. If $\alpha + \beta$ is not a root, then Z is zero or in $\mathfrak{t}$ and $\pi(Z)v$ lies in S. If $\alpha + \beta$ is a positive root, then $\pi(Z)w$ lies in S by indiuction hypothesis. If finally $\alpha + \beta$ is a negative root, then $\pi(Z)w$ lies in S, since S is stable under down-operators. Tigether this yields $\pi(X)v \in S$ as claimed.

The uniqueness of the highest weight follows, since no highest weight can be obtained from an other by down-operators.

For the injectivity of the map $\widehat{K} \to \big\{\text{dominant characters}\big\}$. let (π, V_π) and (η, V_η) be two irreducible representations with the same highest weight λ and corresponding highest weight vectors v_π and v_η. We define a linear map $\phi : V_\pi \to V_\eta$ by

$$\phi\Big(\pi(X_k) \cdots \pi(X_1)v_\pi\Big) = \eta(X_k) \cdots \eta(X_1)v_\eta,$$

where the X_j are negative root vectors. With an easy induction one shows that this map is well-defined. By definition it is equivariant with repect to down-operators. By another induction one shows that ϕ also commutes with up-operators. That means that ϕ is a non-vanishing $\mathfrak{k}$, hence K-homomorphism, hence an isomorphism.

Finally we show that every dominant character λ is a highest weight. Let J be the left ideal of $U(\mathfrak{k})$ generated by $\bigoplus_{\alpha>0} \mathfrak{k}_\alpha$ and all $H - \lambda(H)$ for $H \in \mathfrak{t}$. By the Theorem of Poincaré-Birkhoff-Witt it follows, that J cannot be all of $U(\mathfrak{k})$, in particular it does not contain the unit 1. Let $M = U(\mathfrak{k})/J$. Then the $U(\mathfrak{k})$ module M is generated by the vector $v_0 = 1 + J$. We write the action of $U(\mathfrak{k})$ in the form $\pi : U(\mathfrak{k}) \to \mathrm{End}(M)$. The vector v_0 by construction is a highest weight vector for the weight λ. If U is a proper sub-module of M, then U cannot contain the vector v_0. By Zorn's lemma there is a maximal sub-module U, not containing v_0. Then M/U has highest weight λ and is irreducible ny maximality of U. This representation then yields an irreducible representation of K. $\square$

* * *

6.2 The Weyl character formula

Let K be a connected compact Lie group and $T \subset K$ a maximal torus. The determinant defines a map $\det : W \subset \mathrm{Aut}(\mathfrak{t}) = \mathrm{GL}(\mathfrak{t}) \to \mathbb{R}^\times$. As the Weyl group W is finite, the group $\det(W)$ is finite, so it lies in $\{\pm 1\}$. The Weyl integral formula (Theorem 4.2.11) says

$$\int_G f(x)\,dx = \frac{1}{|W|}\int_T \Big(\det\Big(1 - \mathrm{Ad}_{K/T}(t^{-1})\Big)\Big)\int_K f(ktk^{-1})\,dk\,dt.$$

For $t \in T$ we have the determinant formula

$$\det\Big(1 - \mathrm{Ad}_{K/T}(t^{-1}\Big) = \prod_{\alpha\in\Phi}(1 - t^{-\alpha}).$$

Let Φ^+ be a System of positive roots and let $\rho : \mathfrak{t} \to \mathbb{C}$ be defined by

$$\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

The map $t \mapsto t^{2\rho}$ is a character, but $t \mapsto t^\rho$ is only a local homomorphism in general. The function

$$D(t) := t^\rho \prod_{\alpha\in\Phi^+}(1 - t^{-\alpha}),$$

satisfies

$$\begin{aligned}
D(t)^2 &= t^{2\rho}\prod_{\alpha>0}(1 - t^{-\alpha})^2 \\
&= t^{\sum_{\alpha>0}\alpha}\prod_{\alpha>0}(1 - t^{-\alpha})^2 \\
&= \prod_{\alpha>0}(t^\alpha - 1)(1 - t^{-\alpha}) \\
&= (-1)^{|\Phi^+|}\prod_{\alpha>0}(1 - t^\alpha)(1 - t^{-\alpha}) \\
&= (-1)^{|\Phi^+|}\det\Big(1 - \mathrm{Ad}_{K/T}(t^{-1}\Big).
\end{aligned}$$

Theorem 6.2.1. *Let π an irreducible representation with highest weight λ. In a neighbourhood of the unit one has*

$$\operatorname{tr}\pi(t) = \frac{1}{D(t)}\sum_{w\in W}\det(w)t^{w(\lambda+\rho)},$$

if $D(t)\neq 0$. The denominator $D(t)$ can be written in the form

$$D(t) = \sum_{w\in W}\det(w)t^{w\rho}.$$

The proof requires some constructions.

Definition 6.2.2. A function $f : \mathfrak{t} \to \mathbb{C}$ is called **even**, if

$$f\left(w(X)\right) = f(X)$$

for every $X \in \mathfrak{t}$ and f is called **odd**, if

$$f\left(w(X)\right) = \det(w)f(X)$$

for all $X \in \mathfrak{t}$, $w \in W$.

If f is odd, then $f(X) = 0$ for every X with $\alpha(X) = 0$ for a root α.

Definition 6.2.3. Let $\lambda \in \mathfrak{t}^*$ be a complex valued linear form. The **alternating sum** is defined by

$$\begin{aligned} A(\lambda) : \mathfrak{t} &\to \mathbb{C}, \\ H &\mapsto \sum_{w\in W}\det(w)e^{2\pi i\lambda(wH)}. \end{aligned}$$

Lemma 6.2.4. (a) *The function $A(\lambda)$ is odd.*

(b) *For every $w \in W$ one has $A(\lambda)\circ w = A(\lambda\circ w)$.*

(c) *One has*

$$\begin{aligned} A(\lambda) = 0 &\Leftrightarrow \lambda w = w \text{ for be a } w \in W \smallsetminus \{1\} \\ &\Leftrightarrow H_\lambda \text{ does not lie in any Weyl chamber.} \end{aligned}$$

Proof. (a) is trivial. For (b) compute

$$\begin{aligned} A(\lambda)\circ w(H) &= \sum_{v\in W} \det(v) e^{2\pi i\lambda(vwH)} \\ &= \sum_{v\in W} \underbrace{\det(wvw^{-1})}_{=\det(v)} e^{2\pi i\lambda(wvH)} = A(\lambda\circ w)(H). \end{aligned}$$

(c) The second equivalence follows as W acts simply transitively on the set chambers.

For the converse assume, that $A(\lambda) = 0$. Then the maps $\lambda(w\cdot)$ cannot be distinct, because then the characters $e^{2\pi i\lambda(w\cdot)}$ would be linearly independent. □

Definition 6.2.5. Choose a system Φ^+ of positive roots. Let $\delta : \mathfrak{t} \to \mathbb{C}$ be defined by $\delta(X) = D(\exp(X))$, so

$$\delta(X) = \prod_{\alpha>0} \left(e^{\pi i\alpha(X)} - e^{-\pi i\alpha(X)} \right).$$

Lemma 6.2.6. *The function δ is odd.*

Proof. Let Δ be a basis of the root system and let $\beta\in\Delta$. As s_β permutes the positive roots $\neq\beta$ (Lemma 5.4.7), we get

$$\delta(s_\beta X) = \left(e^{-\pi i\beta(X)} - e^{\pi i\beta(X)} \right) \prod_{0<\alpha\neq\beta} \left(e^{\pi i\alpha(X)} - e^{-\pi i\alpha(X)} \right) = -\delta(X).$$

The reflection s_β generate the Weyl group, so the claim follows. □

Proof of the theorem. Let $\mathfrak{t}^{\mathrm{reg}} = \mathfrak{t} \smallsetminus S$, where

$$S = \bigcup_{\alpha>0} \alpha^{-1}(\mathbb{Z}).$$

Then the map δ has no zeros in $\mathfrak{t}^{\mathrm{reg}}$. As δ has simple zeros along the walls, the Taylor Theorem implies that the function

$$c(\lambda) = \frac{A(\lambda)}{\delta}, \qquad \lambda\in\mathfrak{t}^*,$$

extends to a continuous function on $\mathfrak{t}$. This function is even.

Lemma 6.2.7. *The function $\delta\overline{\delta}$ factors through* $\exp : \mathfrak{t} \to L$. *One has*

$$\delta\overline{\delta} = \eta\circ\exp,$$

where $\eta(t) = \det(\mathrm{Ad}_{G/T}(t^{-1}) - 1)$.

Proof. We compute

$$\begin{aligned}\delta\overline{\delta} &= e^{2\pi i\rho(X)}\prod_{\alpha>0}(1-e^{2\pi i\alpha(X)})e^{-2\pi i\rho(X)}\prod_{\alpha>0}(1-e^{-2\pi i\alpha(X)})\\ &= \prod_{\alpha\in\Phi}(1-e^{2\pi i\alpha(X)}) = \prod_{\alpha\in\Phi}(e^{2\pi i\alpha(X)}-1).\end{aligned}$$ □

Definition 6.2.8. A linear form $\lambda : \mathfrak{t} \to \mathbb{C}$ is called **integral**, if it vanishes on the preimage $\exp^{-1}(1)$ of the unit. If this is the case, then there is a character $\chi = \chi_\lambda : T \to \mathbb{T}$, such that

$$\chi\Big(\exp(X)\Big) = e^{2\pi i\lambda(X)}.$$

The map $\lambda \mapsto \chi_\lambda$ is a bijection

$$\Big\{\text{integral forms}\Big\} \quad\leftrightarrow\quad \widehat{T}.$$

Lemma 6.2.9. *The additive group of all odd,* $\mathbb{C}$*-valued functions on* $\mathfrak{t}$ *of the form* $X \mapsto \sum_{j=1}^N k_j e^{2\pi i\lambda_j(X)}$ *with* $\lambda_j \in \mathfrak{t}^*$, $k_j \in \mathbb{Z}$ *is a free abelian group with generators* $A(\lambda)$, $\lambda \in C^*$, *where* C^* *is a fixed Weyl chamber.*

Proof. Let $g = \sum_{j=1}^N k_j e^{2\pi i\lambda_j(X)}$ with distinct λ_j. We can write g as a finite linear combination

$$g = \sum_{\lambda\in\mathfrak{t}^*} k_\lambda e^{2\pi i\lambda}$$

with $k_\lambda \in \mathbb{Z}$. As g is odd and the characters $e^{2\pi i\lambda}$ are linearly independent, one has $k_{w\lambda} = \det(w)k_\lambda$. It follows

$$g = \sum_{\lambda\in C^*} k_\lambda \sum_{w\in W}\det(w)e^{2\pi i\lambda} = \sum_{\lambda\in C^*} k_\lambda A(\lambda).$$

The linear independence of the characters implies the linear independence of the $A(\lambda)$, $\lambda \in C^*$. □

Let $\phi \in C(K)$ be a conjugation inveriant function. By Weyl's integral formukla one has

$$|W|\int_K \phi(x)\,dx = \int_T \phi(t)|\delta(t)|^2\,dt.$$

Let $f : \mathfrak{t} \to \mathbb{C}$ be a linear combination of functions of the form $X \mapsto e^{2\pi i\lambda(X)}$ with

$\lambda \in \mathfrak{t}^*_{\mathbb{R}}$. We want to define an "integral" of f as follows: We choose an isomorphism $\mathfrak{t} \cong \mathbb{R}^k$, which maps the kernel of $\exp : \mathfrak{t} \to T$ to $\mathbb{Z}^k$, such that it induces an isomorphism $T \cong \mathbb{R}^k/\mathbb{Z}^k$. We define

$$\int_{\mathfrak{t}} f := \lim_{N\to\infty} \frac{1}{(2N)^k} \int_{-N}^{N} \cdots \int_{-N}^{N} f(x_1, \dots, x_k)\, dx_1 \dots dx_k.$$

For $\alpha \in \mathfrak{t}^*_{\mathbb{R}}$ it is easy to see that

$$\int_{\mathfrak{t}} e^{2\pi i\alpha} = \begin{cases} 1, & \alpha = 0, \\ 0 & \alpha \neq 0. \end{cases}.$$

So the functions $e^{2\pi i\alpha}$ with $\alpha \in \mathfrak{t}^*_{\mathbb{R}}$ form an orthonormal system for the inner product

$$\langle f, g\rangle := \int_{\mathfrak{t}} f \cdot \overline{g}.$$

Lemma 6.2.10. *Let π be an irreducible representation of K and χ_π its character. Let $\tilde{\chi}_\pi : \mathfrak{t} \to \mathbb{C}$ be defined as $\tilde{\chi}_\pi = \chi_\pi \circ \exp$. Then there is exactly one linear form $\gamma \in C^*$ with $\tilde{\chi}_\pi \cdot \delta = \pm A(\gamma)$.*

Proof. By the Weyl Integral formula one has

$$1 = \int_K \chi_\pi \overline{\chi}_\pi = \frac{1}{|W|} \int_T \chi_\pi \overline{\chi}_\pi \delta\overline{\delta},$$

hence

$$\langle \tilde{\chi}_\pi \delta, \tilde{\chi}_\pi \delta\rangle = |W|.$$

On the other hand, the function $\tilde{\chi}_\pi \delta$ is odd (Lemma 6.2.9) and therefore

$$\tilde{\chi}_\pi \delta = \sum_{j=1}^{N} k_j\, A(\gamma_j)$$

with $\gamma_j \in C^*$. As the $e^{2\pi i\alpha}$ form an ONB, it follows for $\gamma, \lambda \in C^*$, that

$$\langle A(\gamma), A(\lambda)\rangle = \begin{cases} 0 & \geq\neq \lambda, \\ |W| & \gamma = \lambda. \end{cases}$$

So all k_j are zero, except for one, which is ±1. □

Top conclude the proof of the theorem, we need to show that the sign is +1 and that $\gamma = \lambda + \rho$, where λ is the highest weight of π.

Lemma 6.2.11. *Let $\chi = \chi_\pi$ for a $\pi \in \widehat{K}$. Let $\gamma \in C^*$ be the unique linear form with $(\chi \circ \exp) \cdot \delta = \pm A(\gamma)$. Write $\gamma = \eta + \rho$. Then η is integral and lies in the closure of C^*, i.e., is a dominant character. Indeed, η is the highest weight of π. The sign is $+$.*

Proof. By assumption, the function $A(\gamma)/\delta$ factors through exp. If $Y \in \mathfrak{t}$ lies in the kernel of exp and if $X \in \mathfrak{t}$ is arbitrary, then one has

$$A(\gamma)(X+Y) \cdot \delta(X) = A(\gamma)(X) \cdot \delta(X+Y).$$

Explicitely this means that

$$\begin{aligned}\left(\sum_{w\in W} \det(w) e^{2\pi i(\eta+\rho)(w(X+Y))}\right) \cdot e^{2\pi i\rho(X)} \cdot \prod_{\alpha>0}\left(1-e^{-2\pi i\alpha(X)}\right)&\\ = \left(\sum_{w\in W} \det(w) e^{2\pi i(\eta+\rho)(w(X))}\right) \cdot e^{2\pi i\rho(X+Y)} \cdot \prod_{\alpha>0}\left(1-e^{-2\pi i\alpha(X+Y)}\right).&\end{aligned}$$

By Lemma **??** the functional $w\rho - \rho$ is integral. As $Y \in \ker \exp$, it follows $e^{2\pi i\rho(wY)} = e^{2\pi i\rho(Y)}$. It also follows, that $e^{-2\pi i\alpha(Y)} = 1$ for $\alpha > 0$. This means that for $X \in \mathfrak{t}^{\mathrm{reg}}$ both sides are divisible by

$$e^{2\pi i\rho(X+Y)} \cdot \prod_{\alpha>0}\left(1-e^{-2\pi i\alpha(X)}\right).$$

We then get

$$\left(\sum_{w\in W} \det(w) e^{2\pi i(\eta+\rho)(w(X))} e^{2\pi i\eta(wY)}\right) = \left(\sum_{w\in W} \det(w) e^{2\pi i(\eta+\rho)(w(X))}\right).$$

Becasue the characters are linearly independent, it follows $e^{2\pi i\eta(wY)} = 1$ for every $Y \in \ker \exp$ and therefore η is integral. It remains to show that η is dominant, i.e., that $\eta \in \overline{C^*}$. As $\gamma = \eta + \rho$ lies in the open Weyl chamber, one has $\langle \eta + \rho, \alpha\rangle > 0$ for every simple root α and so $\langle \eta, \alpha\rangle \geq 0$ for every $\alpha \in \Delta$.

Finally we show that η is the highest weigth of π. In the formula

$$\tilde{\chi}_\pi \delta = \pm A(\gamma)$$

we write both sides as linear combinations of characters and then we compare coefficients. This implies the claim. □

The lemma implies Theorem 6.2.1. □

Theorem 6.2.12 (Weyl dimension formula)**.** *Let K be a connected compact Lie group and let $\pi \in \widehat{K}$ with highest weight λ. Then the dimension of V_π equals*

$$\dim \pi = \prod_{\alpha > 0} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle}.$$

Corollary 6.2.13. (a) *If K is not abelian, then there are irreducible representations of arbitrary large dimension.*

(b) *If* $\dim Z(K) = 0$*, then there are only finitely many representations of any given dimension.*

Proof of the theorem. One has $\dim \pi = \operatorname{tr} \pi(1) = \lim_{X \to 0} \operatorname{tr} \pi(\exp(X))$ and so

$$\dim \pi = \lim_{t \to 0} \frac{A(\lambda + \rho)(t\rho)}{\delta(t\rho)}.$$

The character formula implies

$$A(\rho) = \delta.$$

By the W-invariance of the inner product it follows

$$\begin{aligned} A(\lambda + \rho)(t\rho) &= \sum_{w \in W} \det(w) e^{2\pi i t \langle \lambda + \rho, w\rho \rangle} \\ &= \sum_{w \in W} \det(w) e^{2\pi i t \langle \rho, w(\lambda + \rho) \rangle} \\ &= A(\rho)(t(\lambda + \rho)) = \delta(t(\lambda + \rho)). \end{aligned}$$

Therefore

$$\begin{aligned}
\dim \pi &= \lim_{t\to 0} \frac{\delta(t(\lambda+\rho))}{\delta(t\rho)} \\
&= \lim_{t\to 0} \frac{\prod_{\alpha>0}\left(e^{\pi i t\langle \alpha,\lambda+\rho\rangle} - e^{-\pi i t\langle \alpha,\lambda+\rho\rangle}\right)}{\prod_{\alpha>0}\left(e^{\pi i t\langle \alpha,\rho\rangle} - e^{-\pi i t\langle \alpha,\rho\rangle}\right)} \\
&= \lim_{t\to 0} \frac{\prod_{\alpha>0}(2\pi i t\langle \alpha,\lambda+\rho\rangle + O(t^2))}{\prod_{\alpha>0}(2\pi i t\langle \alpha,\rho\rangle + O(t^2))} \\
&= \prod_{\alpha>0} \frac{\langle \alpha,\lambda+\rho\rangle}{\langle \alpha,\rho\rangle}. \qquad \square
\end{aligned}$$

As an example, we describe the representations of $K = \mathrm{SU}(3)$. The group

$$T = \left\{ \operatorname{diag}(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) : a,b,c \in \mathbb{R},\ a+b+c=0 \right\}$$

is a maximal torus. The real Lie algebra of K is

$$\mathfrak{su}(3) = \left\{ X \in \mathrm{M}_3(\mathbb{C}) : X^* = -X,\ \operatorname{tr} X = 0 \right\}.$$

The complexification can, as above, be identified with

$$\mathfrak{sl}_{\mathbb{C}}(3) = \left\{ X \in \mathrm{M}_3(\mathbb{C}) : \operatorname{tr} X = 0 \right\}.$$

The real dimension of K is 8. We choose positive roots

$$\begin{aligned}
\alpha(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) &= a-b, \\
\beta(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) &= a-c, \\
\gamma(e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) &= b-c
\end{aligned}$$

The root spaces are spanned by

$$X_\alpha = E_{1,2}, \quad X_\beta = E_{1,3}, \quad X_\gamma = E_{2,3}.$$

A positive Weyl chamber in $\mathfrak{t} = \left\{(a,b,c) \in \mathbb{R}^3 : a+b+c=0\right\}$ is given by $\mathfrak{t}^+ = \left\{(a,b,c) \in \mathfrak{t} : a > b > c\right\}$. As $c = -a-b$, this set equals

$$\mathfrak{t}^+ = \left\{(a,b,-a-b) \in \mathfrak{t} : a > b, 2b+a > 0\right\}$$

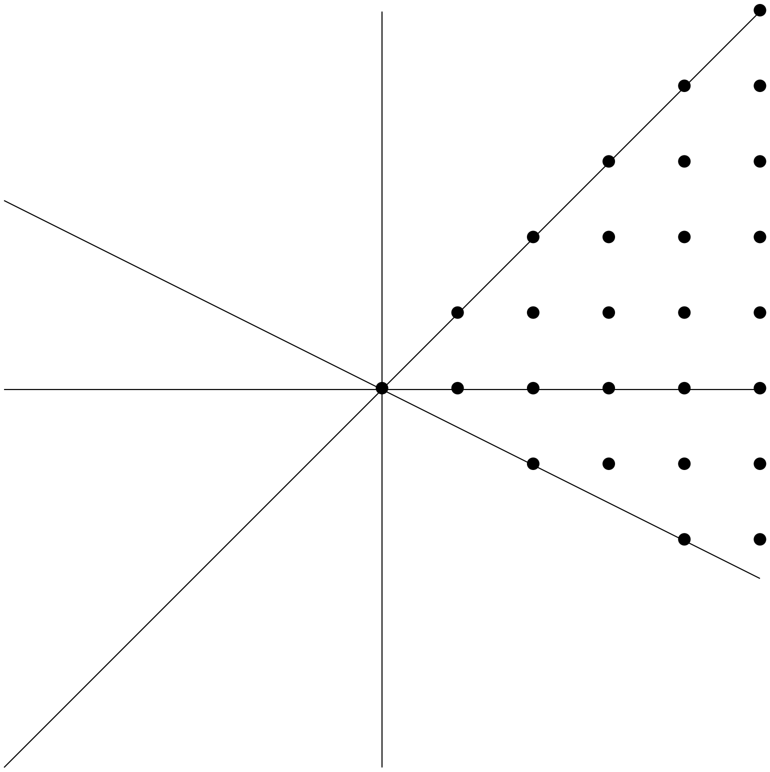
The character lattice is generated by

$$\chi_1(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}) = a, \quad \chi_2(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}) = b.$$

With the inner product

$$\langle (a,b,c),(a',b',c')\rangle = aa' + bb' + cc'$$

the character χ_1 is represented by $H_1 = (1,0,0)$ and χ_2 by $H_2 = (0,1,0)$. Therefore a character $\chi = a\chi_1 + b\chi_2$, $a, b \in \mathbb{Z}$ is dominant if and only if $a \geq b$ and $2a + b \geq 0$ holds.



* * *

6.3 The representation ring

Let G be a Lie group. The **representations ring** $\mathrm{Rep}(G)$ is the free abelian group generated by all isomorphism classes of finite-dimensional representations modulo the relations

$$[B] - [A] - [C] = 0$$

for every exact sequence of G homomorphisms

$$0 \to A \to B \to C \to 0.$$

This additively written group becomes a ring with the product

$$[A][B] := [A \otimes B].$$

Example 6.3.1. Let A be an abelian Lie group. The irreducible representations are one dimensional and are given by its dual group $\widehat{A}$. Every finite-dimensional representation is a direct sum of one dimensionals and the character of a tensor product is the product of the characters. Hence in this case the representation ring equals the group algebra,

$$\mathrm{Rep}(A) \cong \mathbb{Z}[\widehat{A}].$$

In particular, for aa torus $A = \mathbb{T}^r$ the group $\widehat{A}$ is the free abelian group generated by the basis characters $\varepsilon_1, \dots, \varepsilon_r$ and so $\mathrm{Rep}(\mathbb{T}^r)$ is isomorphic to the polynomial ring $\mathbb{Z}[X_1, \dots, X_r]$ in r variables.

Proposition 6.3.2. *Let G be a topological group. For every finite-dimensional representation π we write the character as χ_π, so $\chi_\pi(k) = \mathrm{tr}\,\pi(k)$. If $0 \to \pi \to \eta \to \gamma \to 0$ is an exact sequence of finite-dimensional representations, then one has*

$$\chi_\eta = \chi_\pi + \chi_\gamma.$$

Further,

$$\chi_{\pi\otimes\gamma} = \chi_\pi \chi_\gamma.$$

This means, that the map $\pi \mapsto \chi_\pi$ identifies the ring $\mathrm{Rep}(G)$ with the subring of $C(G)$, generated by all characters.

Proof. Clear. □

Example 6.3.3. For a finite group G there are only finitely many irreducible representations, say $\pi_1, \dots, \pi_n$. Then

$$\mathrm{Rep}(G) = \mathbb{Z}\pi_1 \oplus \cdots \oplus \mathbb{Z}\pi_n$$

as additive group. The multiplication is obtained by decomposing the tensor product into irreducible summands. As an example we determine the representation ring of the group $G = \mathrm{Per}(3)$. We claim

$$\begin{aligned}\mathrm{Rep}(G) &\cong \mathbb{Z}[X,Y]/\left\langle X^2-1,(X-1)Y,Y^2-Y-X-1\right\rangle \\ &= \mathbb{Z}\oplus\mathbb{Z}X\oplus\mathbb{Z}Y.\end{aligned}$$

In the sicond presentation the multiplication is $\mathbb{Z}$-linear and satisfies the rules $X^2 = 1, XY = Y, Y^2 = Y + X + 1$.

Proof. We determine the dual $\widehat{G}$. In dimension one we have the trivial representation and the determinant. We have an irreducible 2-dimensional representation π_2, defined as follows: $\pi_2(1,2)$ is a reflection at the hyperplane $y = 0$, where (x, y) are the Koordinaten and $\pi(1,2,3)$ is a rotation by the angle $2\pi/3$.

By the Peter-Weyl-Theorem, the 6-dimensiona space $L^2(G)$ contains the representationss $\mathrm{triv}\otimes\mathrm{triv}$, $\det\otimes\det$ and $\pi_2\otimes\pi_2$. the dimensiona add up to 6, therefore these are all representations, i.e.,

$$\widehat{G} = \left\{\mathrm{triv}, \det, \pi_2\right\}.$$

We have $\mathrm{triv}\otimes\mathrm{triv} = \mathrm{triv}$ and $\det\otimes\det = \mathrm{triv}$. Further, since π_2 is the only 2-dimensional irreducible representation ist, it follows $\det\otimes\pi_2 = \pi_2$. We only need the decomposition of $\pi_2\otimes\pi_2$. For this we need to compute the character χ_{π_2}. The representation $\tau = \pi_2\oplus\mathrm{triv}$ is the standard matrix presentation of G. So $\operatorname{tr}\tau(1,2) = 1$, i.e., $\operatorname{tr}\pi_2(1,2) = 0$ and in the same way $\operatorname{tr}\pi_2(1,2,3) = -1$ and the same for all elements of order 2 or 3. Therefore the character χ_η of $\eta := \pi_2\otimes\pi_2$, which equals $\chi_{\pi_2}^2$, maps (1,2) to 0 and (1,2,3) to 1. Therefore

$$\left\langle\chi_{\mathrm{triv}},\chi_\eta\right\rangle = \frac{1}{6}(4+1+1) = 1.$$

Therefore the trivial representation appear exactly one time in $\pi_2\otimes\pi_2$ on. By the same token, $\left\langle\chi_{\det},\chi_\eta\right\rangle = 1$ and we conclude

$$\pi_2\otimes\pi_2 \cong \mathrm{triv}\oplus\det\oplus\pi_2.$$

The claim follows. □

Definition 6.3.4. Let $\phi : K \to L$ be a homomorphism of compact Lie groups. For a given representation π of L, the pullback $\phi^*(\pi) = \pi\circ\phi$ is a representation of K. The pullback is compatible with direct sums and tensor products, hence induces a ring homomorphism

$$\phi^* : \mathrm{Rep}(L) \to \mathrm{Rep}(K).$$

Theorem 6.3.5. *Let T be a maximal torus in the compact Lie group K. The Weyl group W acts on T and so on* $\mathrm{Rep}(T)$*. The inclusion $j : T \hookrightarrow K$ induces an isomorphism*

$$j^* : \mathrm{Rep}(K) \xrightarrow{\cong} \mathrm{Rep}(T)^W,$$

where the right hand side denote the subring of all W-invariant elements.
For the $\mathbb{Q}$-Algebra $\mathrm{Rep}(K)_{\mathbb{Q}} = \mathrm{Rep}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ *one has*

$$\mathrm{Rep}(K)_{\mathbb{Q}} \cong \mathbb{Q}[X_1, \dots, X_r]$$

with $r = \dim T$.

Proof. By highest weight theory the map j^* is injective. We show surjectivity. Let $f \in \mathrm{Rep}(T)^W$. The elements of $\mathrm{Rep}(T)$ are linear combinations of characters, hence van be viewed as functions on T. Let $g = f \circ \exp : \mathfrak{t} \to \mathbb{C}$. Then

$$g = \sum_{j=1}^{n} k_j e^{2\pi i \alpha_j}$$

with $\alpha_j \in DX^*(T)$, $k_j \in \mathbb{Z}$, where $DX^*(T) \subset \mathfrak{t}^*$ is the set of differentials of characters. This is a lattice in the real vector space $\mathfrak{t}^*$. The function

$$\delta = \prod_{\alpha > 0} \left(e^{\pi i \alpha} - e^{-\pi i \alpha} \right)$$

is odd (Lemma 6.2.6). Therefore

$$g \cdot \delta = e^{2\pi i \rho} \sum_{s} m_s e^{2\pi i \lambda_s}$$

with $m_s \in \mathbb{Z}$ and $\lambda_s \in DX^*(T)$. As $g \cdot \delta$ is odd, one has

$$g \cdot \delta = \sum_{l} r_l A(\beta_l)$$

with $r_l \in \mathbb{Z}$ and β_l in the positive Weyl chamber. Comparing coefficients one gets that every β_l is of the form $\rho + \lambda_{s(l)}$, so

$$f = \sum_l r_l c(\rho + \lambda_{s(l)}).$$

This lies in the image of j^*, since $c(\rho + \lambda)$ lies in the image of the representation of highest weight λ.

The addendum follows from the **Theorem of Chevalley**, which says that over every field of characteristic zero the invariants under a finite group, which is generated by reflections, form a polynomial ring. □

Definition 6.3.6. Let V be a $\mathbb{Q}$-vector space of dimension $r \in \mathbb{N}$. Let $V_{\mathbb{R}} = V \otimes \mathbb{R}$. A subset $C \subset V_{\mathbb{R}}$ is called a **rational cone** with r sides, if there are linearly independent linear functionals $\alpha_1, \dots, \alpha_r \in \mathrm{Hom}(V, \mathbb{Q})$, such that

$$C = \{v \in V_{\mathbb{R}} : \alpha_1(v) > 0, \dots, \alpha_r(v) > 0\}.$$

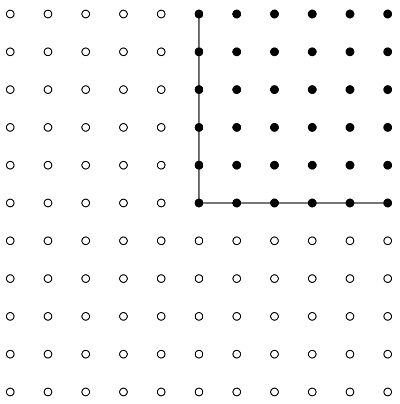
Lemma 6.3.7. *Let V be a finite-dimensional $\mathbb{Q}$-vector space and $\Lambda \subset V$ a lattice. Let $\alpha : V \to \mathbb{Q}$ be a non-zero linear map. Then $\alpha(\Lambda) = \mathbb{Z}r$ for an $r \in \mathbb{Q}$.*

Proof. Let $e_1, \dots, e_k$ be a basis of Λ. Replacing e_j by $-e_j$ if necessary, we can assume, that $\alpha(e_j) = m_j/n_j$ with coprime $m_j, n_j \in \mathbb{N}$. Let M be the smallest joint multiple of the denominators $m_1, \dots, m_k$ Then one has $M\alpha(e_j) \in \mathbb{N}$ and so $\alpha(\Lambda) \subset M\mathbb{Z}$. As $\alpha(\Lambda)$ is a subgroup of $\mathbb{Z}$, it follows $\alpha(\Lambda) = r\mathbb{Z}$ for some $r \in \mathbb{Q}$. □

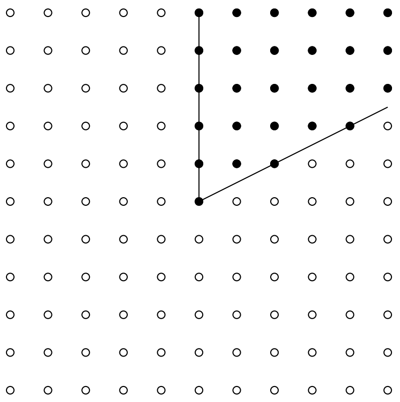
Lemma 6.3.8. *Let V be an r-dimensional $\mathbb{Q}$-vector space. Let $\Lambda \subset V$ be a lattice in V and let $C \subset V_{\mathbb{R}}$ be a rational cone of r sides. Then there are finitely many $\lambda_1, \dots, \lambda_n \in \Lambda$, such that*

$$\Lambda \cap \overline{C} = \left\{ \sum_{j=1}^n k_j \lambda_j : k_j \in \mathbb{N}_0 \right\}.$$

We call $\left\{\lambda_1, \dots, \lambda_n\right\}$ a **generating set** *of $\Lambda \cap \overline{C}$. We call it a* **free generating set**, *if $\lambda_1, \dots, \lambda_n$ is a Basis of V.*



A cone with a free generating set.



A cone, which admits no free generating set.

Proof. For $j = 1, \dots, r$ let $a_j \in \Lambda$ be the unique element with the property, that $\alpha_i(a_j) = 0$ for $i \neq j$ and $\alpha_j(a_j)$ is > 0 and minimal. Then $a_1, \dots, a_r$ is a basis of V, which lies in Λ, hence generates a sub-lattice $\Lambda' \subset \Lambda$. Let E be a set of representatives of Λ/Λ', such that every $v \in E$ lies in C, but for every $j = 1, \dots, r$ the vector $v - a_j$ is not in C. Let $\lambda_1, \dots, \lambda_n$ a numeration of the a_j and the elements of E. This one does the job. □

Theorem 6.3.9. *Let E be a generating set of $\overline{C} \cap X^*(T)$, where C is a Weyl chamber. Let $\mathbb{Z}[X_\lambda : \lambda \in E]$ be the polynomial ring in $|E|$ variables.*

(a) *The ring homomorphism*

$$\psi : \mathbb{Z}[X_\lambda : \lambda \in E] \to \mathrm{Rep}(K),$$

which maps X_λ to the irreducible representation π_λ of highest weight λ, is surjective.

(b) *If E is free, then ψ is an isomorphism.*

Proof. (a) Every dominant character χ is of the form $\chi = \sum_{\lambda\in E} k_\lambda \lambda$ with $k_\lambda \in \mathbb{N}_0$. By induction on $|\chi| = \sum_{\lambda\in E} k_\lambda$ we show, that every π_λ lies in the image of ψ. If $|\chi| = 1$, then $\chi = \lambda$ for some $\lambda \in E$ and then $\pi_\lambda = \psi(X_\lambda)$ by definiton. Now let $|\chi| > 1$. The representation π_χ has highest weight χ, as has the (not necessarily irreducible) representation

$$\eta_\chi := \psi\left(\prod_{\lambda\in E} X_\lambda^{k_\lambda}\right) = \bigotimes_{\lambda\in E} \pi_\lambda^{\otimes k_\lambda}.$$

The weight space in η_χ for the weight χ is one-dimensional and every weight $\chi' \neq \chi$ of η_χ satisfies $|\chi'| < |\chi|$. Therefore η_χ is isomorphic to

$$\eta_\chi \cong \pi_\chi \oplus R,$$

where R is a sum of irreducible representations $\pi_{\chi'}$ with $|\chi'| < |\chi|$. By induktion hypothesis R lies in the image of ψ and the same follows for π_χ.

(b) Let E be free. Then the coefficients k_λ in $\chi = \sum_{\lambda\in E} k_\lambda \lambda$ are uniquely determined. Let $f \in \ker(\psi)$. We write

$$f = \sum_{k\in\mathbb{N}_0^E} c_k \prod_{\lambda\in E} X_\lambda^{k(\lambda)} = \sum_{k\in\mathbb{N}_0^E} c_k X^k$$

with coefficients $c_k \in \mathbb{Z}$ of which almost all are zero. **Assume**, $f \neq 0$. Let d be the degree of f and let $k_0 \in \mathbb{N}_0^E$ be an element with $c_{k_0} \neq 0$ and $\deg(X^{k_0}) = d$. Let $\chi = \sum_{\lambda\in E} k_0(\lambda)\lambda$. Then χ is dominant and since all weights in π_χ have smaller degree, $\psi(f - c_{k_0}X^{k_0})$ lies in the span of the $\pi_{\chi'}$ with $\chi' \neq \chi$, in **contradiction** to $\psi(f - c_{k_0}X^{k_0}) = -c_{k_0}\psi(X^{k_0})$. □

* * *

7 Structure theory

7.1 Decomposition into simple groups

Lemma 7.1.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$.*

(a) *If $N \subset G$ is a normal closed subgroup, hence itself a Lie-subgroup, then the Lie algebra $\mathfrak{n}$ is an ideal of $\mathfrak{g}$.*

(b) *The Lie algebra of the center of G equals the center $\mathfrak{z}$ of $\mathfrak{g}$.*

Proof. (a) Let N be a closed normal subgroup. Let $X \in \mathfrak{g}$, $x \in G$ and $Y \in \mathfrak{n}$. By Proposition 1.9.5 one has

$$\exp\left(t\,\mathrm{Ad}(x)Y\right) = x\exp(tY)x^{-1}, \qquad t \in \mathbb{R}.$$

The right hand side lies in N, after taking deriatives we get that $\mathrm{Ad}(x)Y \in \mathfrak{n}$. By Lemma 1.8.1 it follows

$$[X, Y] = \mathrm{ad}(X)Y = \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}\left(\exp(sX)\right) Y \in \mathfrak{n}.$$

(b) Let $Z(G)$ be the center of G. The same argument as above shows, that for $X \in \mathrm{Lie}(Z(G))$ one has $\mathrm{ad}(X) = 0$, hence $\mathrm{Lie}(Z(G)) \subset \mathfrak{z}$. Conversely let $X \in \mathfrak{z}$, then $\frac{d}{ds}\big|_{s=0} \mathrm{Ad}\left(\exp(sX)\right) = 0$. Let $x_s = \exp(sX)$. Then the 1PSG $s \mapsto \mathrm{Ad}(x_s) \in \mathrm{Aut}(\mathfrak{g}) = \mathrm{GL}_n(\mathbb{R})$ with $n = \dim\mathfrak{g}$ als has the Derivative 0, hence equals $e \in \mathrm{GL}_n(\mathbb{R})$. As Ad is the diferential of the conjugation action int, it follows for $Y \in \mathfrak{g}$ and fixed $s \in \mathbb{R}$, that

$$\begin{aligned} Y &= \mathrm{Ad}(x_s)Y \\ &= \frac{d}{dt}\bigg|_{t=0} x_s \exp(tY)x_s^{-1}. \end{aligned}$$

The two 1PSGs $t \mapsto \exp(tY)$ and $t \mapsto x^s \exp(tY)x_s^{-1}$ have the same derivative at zero, hence they are equal. This means, that x_s commutes with $\exp(\mathfrak{g})$, since $\exp(\mathfrak{g})$ generates the group Gt, the element x_s is central and so $X \in \mathrm{Lie}(Z(G))$. □

Definition 7.1.2. A Lie algebra $\mathfrak{g}$ is called **simple**, if $\mathfrak{g}$ is not abelian and has no proper ideals.

An ideal $I \subset \mathfrak{g}$ is called **simple Ideal**, if it is a simple Lie algebra.

A connected compact Lie group K is called **simple**, if the Lie algebra of K is simple.

Theorem 7.1.3. *Let K be a connected compact Lie group.*

(a) *For the Lie algebra one has*

$$\mathfrak{k} = \mathfrak{z} \oplus I_1 \oplus \cdots \oplus I_n,$$

where $\mathfrak{z}$ is the center and $I_1, \dots, I_n$ are simple ideals.

(b) *The group $Z = \exp(\mathfrak{z})$ is the largest central torus in K, every $K_j = \exp(I_j)$ is a closed normal subgroup. The map*

$$\begin{aligned} Z \times K_1 \times \cdots \times K_n &\to K, \\ (z, k_1, \dots, k_n) &\mapsto z k_1 \cdots k_n \end{aligned}$$

is a surjective group homomorphism with finite kernel.

Proof. (a) The center $\mathfrak{z}$ is stable under the adjoint representation of K. Choose irreducible K-stable subspaces $I_1, \dots, I_n$ of $\mathfrak{k}$, such that $\mathfrak{k} = \mathfrak{z} \oplus I_1 \oplus \cdots \oplus I_n$. The Derivative of the adjoint representation is given by $\mathrm{ad} : \mathfrak{k} \to \mathrm{End}(\mathfrak{k})$ with

$$\mathrm{ad}(X)(Y) = [X, Y].$$

As I_j is stable under the adjoint representation, every I_j is an Ideal of the Lie algebra. The ideal I_j is simple, since it is irreducible under $\mathrm{ad}(\mathfrak{k})$.

(b) By Lemma 7.1.1 $Z = \exp(\mathfrak{z})$ is the connected component of the center $Z(K)$. Let K_i be the group generated by $\exp(I_i)$. Let $H_i \subset K$ be the centraliser of $\bigoplus_{j \neq i} I_j$, then H_i is a Lie-subgroup and $Z(K)\, K_i \subset H_i$. As this holds for every i, dimension reasons imply that $\mathrm{Lie}(H_i) = \mathfrak{z} \oplus I_i$.

We need to show, that K_i is closed, hence a submanifold. Let $Z(K)^0$ be the connected component of the center of K. The exact sequence

$$1 \to Z(K)^0 \to H_i \to H_i/Z(G)^0 \to 1$$

differentiates to

$$0 \to \mathfrak{z} \to \mathfrak{z} \oplus I_i \to I_i \to 0.$$

The projection $\mathfrak{z} \oplus I_i \to \mathfrak{z}$ induces a homomorphism $\phi : \tilde{H}_i \to Z(K)^0$ making the diagram

$$\begin{tikzcd}
& \tilde{H}_i \arrow[dl, "\phi"'] \arrow[d, "p"] \\
Z(K)^0 \arrow[r] & H_i
\end{tikzcd}$$

commutative, where the vertical arrow is the projection p. Then $K_i = p\left(\ker(\phi)\right)$ and we show, that K_i is closed. For this let $\alpha_\nu \to x_0$ be a sequence in K_i, converging to x_0 in H_i. As p is a covering, the sequence lifts to $\tilde{\alpha}_\nu \to \tilde{x}_0$. Then $\tilde{\alpha}_\nu$ lies in the kernel of ϕ and the same for $\tilde{x}_0$, so that $x_0 = p(\tilde{x}_0)$ lies in K_i.

Therefore K_i is closed, hence a compact Lie group. Let $k = \exp(X) \in K_i$ and $Y \in I_j$ with $i \neq j$. Then the derivative of

$$t \mapsto \exp(tY)k\exp(-tY) = \exp(\mathrm{Ad}(\exp(tY)X)$$

vanishes in $t = 0$. This implies that this group homomorphism is trivial, hence k and $\exp(Y)$ commute and the roduct map is a group homomorphism. □

* * *

7.2 Fundamental groups

Proposition 7.2.1. *The fudamental group of a torus $T \cong \mathbb{T}^n$ is $\mathbb{Z}^n$. It is naturally isomorphic to the kernel of the exponential map* $\exp : \mathfrak{t} \to T$.

Proof. The covering $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ has a simply connected domain, therefore it is the universal covering. □

Definition 7.2.2. A compact Lie group K is called **semisimple**, if the center $Z(K)$ is discrete. This is equivalent to saying that the Lie algebra $\mathfrak{k}$ has trivial center.

Let K be a semisimple compact Lie group and let $T \subset K$ be a maximal torus with Lie algebra $\mathfrak{t}$ and root system $\Phi \subset \mathfrak{t}^*$. For every $\alpha \in \Phi$ there is exactly one $X_\alpha \in \mathfrak{t}$, such that

(a) $\alpha(X_\alpha) = 2$,

(b) $\langle X_\alpha, Y \rangle = 0$ for every $Y \in \mathfrak{t}$ with $\alpha(Y) = 0$.

Here $\langle .,. \rangle$ is an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. The element X_α is called the **co-root** of α. Via the identification $\mathfrak{t} \xrightarrow{\cong} \mathfrak{t}^*$ given by the bilinear form $\langle .,. \rangle$, the co-root coincides with the co-root of Definition 5.3.1.

Long roots haben short co-roots and vice versa.

Lemma 7.2.3. *Suppose that K has trivial center. Then the co-roots lie in the kernel of the exponential map. They generate a sublattice Λ_K of $\pi_1(T) = \ker(\exp)$. This is called the* **c–root lattice**.

Proof. As the co-roots, like the roots, span the whole space, it suffices to show that $\Lambda_K \subset \pi_1(T)$. For this let $H_\alpha \in \mathfrak{t}$ be the representing vector of α, so $\alpha(X) = \langle X, H_\alpha \rangle$. Then $X_\alpha = cH_\alpha$ for some $c > 0$. One has

$$2 = \alpha(X_\alpha) = c\alpha(H_\alpha) = c \langle H_\alpha, H_\alpha \rangle = c \langle \alpha, \alpha \rangle .$$

Hence $c = 2/\langle \alpha, \alpha \rangle$ and so

$$X_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha.$$

For an arbitrary root β, we conclude that

$$\beta(X_\alpha) = \left\langle X_\alpha, H_\beta \right\rangle = \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \{\alpha, \beta\} \in \mathbb{Z}.$$

For $Y \in \mathfrak{k}_\beta$ we get $\mathrm{Ad}(\exp(X_\alpha))Y = e^{2\pi i \beta(X_\alpha)} Y = Y$. Then $\exp(X_\alpha)$ cummutes with $\exp(\mathfrak{k}_\beta)$ and since this holds for every β, the element $\exp(X_\alpha)$ lies in the center of K, so must be trivial. □

Theorem 7.2.4. *Let K be a semisimple compact Lie group. Then the fundamental group $\pi_1(K)$ is a finite abelian group. More precisely let $T \subset K$ be a maximal torus. The inclusion $\iota : T \hookrightarrow K$ induces a surjective map*

$$\iota_* : \pi_1(T) \twoheadrightarrow \pi_1(K).$$

The kernel of this map contains the co-root lattice Λ_K, so $\pi(K)$ is a quotient of the finite abelian group

$$\pi_1(T)/\Lambda_K.$$

One even has $\ker(\iota_*) = \Lambda_K$, but we won't show it here.

Proof. We first dhow that the co-root lattice Λ_K lies in $\pi_1(T)$. The center $Z(K)$ is discrete and therefore finite. This implies that $K \to K/Z(K) = K'$ is a finite covering. If γ lies in Λ_K, then γ is trivial in K', so it lies in $\pi_1(K)$.

Now let X_α be a co-root. As $\Lambda_K \subset \pi_1(K)$, it follows $\exp(X_\alpha) = 1$. Thus one has

$$\exp\Big((1-t)X_\alpha\Big) = \exp(-tX_\alpha) = \exp \circ s_\alpha(tX_\alpha).$$

It therefore suffices to show, that the paths

$$\begin{aligned} t &\mapsto \exp(tX_\alpha), \\ t &\mapsto \exp \circ s_\alpha(tX_\alpha), \end{aligned}$$

defined on $[0, 1/2]$, are homotopic with fixed ends in K. Their endpoint is $\exp(\frac{1}{2}X_\alpha)$. Because of

$$\alpha(X_\alpha) = \langle X_\alpha, H_\alpha \rangle = 2\frac{\langle H_\alpha, H_\alpha \rangle}{\alpha, \alpha} = 2$$

one has

$$\exp\left(\frac{1}{2}X_\alpha\right)^\alpha = e^{2\pi i \alpha\left(\frac{1}{2}X_\alpha\right)} = 1.$$

So the endpoint of these paths lies in U_α, the kernel of the global root $t \mapsto t^\alpha$. The reflection s_α is induced by the conjugation with an element g of the centraliser $Z(U_\alpha) = Z(U_\alpha^0)$, see Proposition 5.2.5. This centraliser is connected by Theorem 4.2.9. So let $s \to \gamma(s)$ be a path in $Z(U_\alpha)$, with $\gamma(0) = g$ and $\gamma(1) = 1$. Then

$$h(s,t) = \exp\left(t\,\mathrm{Ad}(\gamma(s))X_\alpha\right)$$

is the homotopy we have been searching for. $\square$