SWAS 2012

Construction of global Galois representations associated with modular forms

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Contents

We will discuss the construction of Galois representations associated to modular forms/automorphic representations. This is the easy direction of the (conjectural) global Langlands correspondence. The subject started with work of Eichler and Shimura, who associated a Galois representation (on the Tate-module of an elliptic curve) to a newform of weight 2 on a modular curve $X_0(N)$. This was generalized by Kuga, Ihara and Deligne to classical modular forms of weight ≥ 2. We will mainly focus on this starting point, following Deligne's Bourbaki talk from 1969 [4] closely.
In the third meeting, there will be a general (easy) discussion of the global and local Langlands program (1 talk) and then a survey presentation of Kottwitz' method (following ideas also due to Langlands and Rapoport) to construct Galois representations associated with modular forms on (compact) unitary Shimura varieties.
Remark: There is a 'translation' of Deligne's article [4] available, but I guess with a minimum of knowledge of French you are better off with the original. At

least formulas should be crosschecked and bear in mind that the translation of "(la) pointe" (as opposed to "(le) point") is "cusp" and not "point".

1 1st meeting

1.1 Modular curves over $\mathbb{C}$

Main references: [4, 15, 7, 19], see also [2].

- Modular curves as quotients of the upper half plane.
- Recall its parametrization of elliptic curves with level structures (so far just as point sets over $\mathbb{C}$), in particular $X(N)$ and $X_0(p)(N)$.
- Adelic interpretation as in Deligne [4, §2], description of the resulting $\mathrm{GL}_2(\mathbb{A}_f)$-operation on the projective limit.
- Analytic construction of the compactification $\overline{X}$ (as Riemann surface).
- Construction and understanding of the local systems $U^1 = R^1\pi_*\mathbb{Q}_E$, where $\pi : E \to X$ is the universal elliptic curve.
- The coherent sheaves ω, Ω^1 and their relations on X and $\overline{X}$.
- The Hodge filtration of $U^1 \otimes \mathcal{O}_X$.
- Interpretation of sections $H^0(X, \omega^n)$, $H^0(X, U^k \otimes \omega^n)$, etc. as functions on $GL_2(\mathbb{R})$ with satisfy certain differential equations, see [19, §1]. (you might combine this with the adelic description, but this is not necessary) Here U^k is $\mathrm{Sym}^k(U^1)$.

1.2 The isomorphism of Shimura

Main references: [4, 19].
Let $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ be a congruence subgroup and let $X = \mathbb{H}/\Gamma$.

- Introduction of $H^1(\Gamma, U^k)$ (Galois cohomology) and $\widetilde{H}^1(\Gamma, U^k)$ (cocyles, where the restriction to all unipotent subgroups is trivial). Here we denote by U^k just $\mathbb{Q}^{k+1}$ with the standard representation of $\mathrm{GL}_2(\mathbb{Q})$, i.e. again $U^k = \mathrm{Sym}^k(U^1)$.
- Prove the isomorphism $H^1(\Gamma, U^k) \cong H^1(X, U^k)$ and
$$\widetilde{H}^1(\Gamma, U^k) \cong \mathrm{image}(H^1_c(X, U^k) \to H^1(X, U^k)).$$
- Understanding of the spaces $H^0(\overline{X}, \Omega^1(U^k) \otimes \omega^n)$.
- The map to cohomology $\delta : H^0(\overline{X}, \Omega^1(U^k)) \to \widetilde{H}^1(X, U^k \otimes \mathbb{C})$ — explain why the image lands in $\widetilde{H}$. This map arises from the long exact sequence associated with the de Rham resolution of $U^k \otimes \mathbb{C}$.

- The map $H^0(\overline{X}, \omega^k \otimes \Omega^1) \hookrightarrow H^0(\overline{X}, \Omega^1(U^k))$. The spaces are denoted $S_{k+2}(\Gamma)$ and $S_2(\Gamma, k)$ respectively in [19, §2]. Present [19, 3.2.1].

- The goal of this talk is to understand and prove the isomorphism

$$H^0(\overline{X}, \Omega^1 \otimes \omega^k) \oplus \overline{H^0(\overline{X}, \Omega^1 \otimes \omega^k)} \to \widetilde{H}^1(X, U^k \otimes \mathbb{C})$$

which might be seen as a Hodge structure of type $(k+1, 0), (0, k+1)$ on the space on the RHS.

- Present [19, Théorème (3.2.5)].

- Present [19, Théorème (3.3.1)].

- Finally present [19, Théorème (4.2.4/4.2.6)] — but already in the formulation of [4, Théorème 2.10].

1.3 Integral model of the compactification

Main references: [4, §3], [6].
First part: Present the algebraic theory of the modular curve, explain how $X(N)$ and $X_0(p)(N)$ represent a moduli problem over spec($\mathbb{Z}[1/N]$).
Second part: Present the results and some ideas from [6].

2 2nd meeting

2.1 Algebraic preliminaries

Main References: [9, 5, 16]
The purpose of this talk is a review of etale cohomology and the Weil conjectures.

- Example of the Tate module in the case of an Abelian variety. Come back to this example as often as possible making it possible to understand this talk without previous knowledge in etale cohomology.

- Explain the category of l-adic sheaves over a variety over a field. Present cohomology with and without compact support (also briefly the relative version and the Leray spectral sequence).

- For local system over varieties over $\mathbb{C}$, explain the comparison isomorphism between usual sheaf cohomology and etale cohomology (with and without compact support).

- Recall the absolute and relative Frobenius morphisms.

- Mention already the modular description of the Frobenius on the modular curve.

- Recall the Grothendieck-Lefschetz fixed point formula.

- State the Weil conjectures (Theorem of Deligne [5]) without proof.
- Explain the "trace morphism" on cohomology for a finite morphism (cf. [4, Lemme 4.6] for what we have to understand in the second talk).
- Explain briefly Poincaré duality for l-adic cohomology.

2.2 The congruence relation

Main Reference: [4, §3, §4].

- Hecke operators on X/Γ as correspondences. [4, §3]. The modular descriptions of $X(N)$, $X_0(p)(N)$, etc. should be only recalled from the 3rd talk of the first meeting.
- Explicit description of the Hecke correspondences, resp. of the adelic operation on the local systems U^k.
- Explicit description of the Hecke correspondences on modular forms.
- The structure of the Hecke algebra (only for maximal compact open) as in [4, 3.12]. Reinterpretation of the operation on modular forms as convolution w.r.t. the adelic operation with a function (left and right invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$).
- The relation between Hecke eigenvalues and Fourier coefficients. You may first present something about the algebraic theory of the Fourier development [6, VII]. The relation to Hecke theory may be found in [7, 5.8].
- Compatibility of the Shimura isomorphism with the various actions of Hecke operators [4, 3.19].
- The main part of this talk should be the presentation of [4, §4], in particular the description of the Hecke correspondence T_p in the fibre above $\mathbb{F}_p$ in terms of Frobenius and Verschiebung [4, 4.8/4.9].

2.3 Weil implies Ramanujan/Petersson

Firstly interpret the congruence relation as a result about existence of Galois representations associated with modular forms. The Galois action and the action of the Hecke algebras at all p on ${}_nW_l := \widetilde{H}^1_{et}(X(n), U^k \otimes \mathbb{Q}_l)$[1] for $(n,p) = (n,l) = 1$ commute. The Shimura isomorphism

$$H^0(\overline{X}(n), \Omega^1 \otimes \omega^k) \oplus \overline{H^0(\overline{X}(n), \Omega^1 \otimes \omega^k)} \to \widetilde{H}^1(X(n), U^k_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

is, furthermore, compatible with the action of the Hecke algebra (cf. previous talk). The idea is therefore to look at the maximal Eigenspaces for the Hecke

[1] For $n \gg 1$; otherwise defined as in [4].

algebra on this. Firstly one splits off the space of "newforms"[2] on each side via orthogonality w.r.t. the Petersson inner product (LHS) and the scalar product described in [4, (3.20)] for the etale side ("RHS", but tensored with $\mathbb{Q}_l$). Both come from a scalar product on $\widetilde{H}^1(X(n), U^k_{\mathbb{Q}})$ with values in $(2\pi i)^{-k-1}\mathbb{Q}$. Explain this in detail. Because of multiplicity one (e.g. [7, 5.7]) the maximal Eigenspaces are 1-dimensional with Eigenvalues given by Fourier coefficients of the corresponding newform (a cusp form of weight $k+2$ for $\Gamma_0(n)$). Hence, we get a 2-dimensional subspace $V \subseteq \widetilde{H}^1(X(n), U^k_{\mathbb{Q}}) \otimes K$, for some number field K (the field generated by the Fourier coefficients), such that the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on $V \otimes_K K_\nu$ for any prime $\nu | l$. This Galois representation is unramified outside l and n. The congruence relation then shows (as in [4, p.171]) that Frobenius at p satisfies

$$\det(1 - F_p X | V \otimes_K K_\nu) = 1 - a_p X + p^{k+1} X^2$$

where a_p is the p-th Fourier coefficient of the Eigenform. The **Petersson/ Ramanujan conjecture** (in its refined form due to Serre) claims that the roots of this equation have complex absolute value $p^{\frac{k+1}{2}}$ for any complex embedding of K. Thus it suffices to show that, in general, the Eigenvalues of Frobenius on $\widetilde{H}^1_{et}(X(n), U^k \otimes \mathbb{Q}_l)$ have absolute value $p^{\frac{k+1}{2}}$. This is shown in [4, §5]. The main point is that $\widetilde{H}^1_{et}(X(n), U^k \otimes \mathbb{Q}_l)$ over $\mathbb{F}_p$ can be related via the Leray spectral sequence to the cohomology of a smooth rational variety of dimension $k+1$, namely to a smooth compactification of the k-th power universal elliptic curve $E^k \to X(n)$. The smooth compactification is constructed ad. hoc. in [4, §5], but you may cite the modern and more general result from [8] about the existence of toroidal compactifications in arbitrary characteristic. It would be nice to state explicitly the consequence for the modular form Δ of weight 12 as in the introduction to [4] (Ramanujan's original conjecture).

3 3rd meeting

3.1 Introduction to the Langlands program

Main references: [11], [3, especially chapter 7 and 10–11], and also [2, 1.8, 3.9] and [1].

- Start by presenting the correspondence between classical automorphic forms and automorphic representations for GL_2 [3, chapter 7], [11, §7]. Nice would be also a description of the local constituents of an automorphic representation for GL_2 and their associated local Galois representations. How does the weight and type (holomorphic/Maass) of a classical eigenform correspond to the representation and parameter at ∞. [2, p. 93. (3)], [11, §8].

[2] say for $\Gamma_0(n)$ for simplicity — this means that we take invariants under the group of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ everywhere. In this case the operator I_p^* operates trivially.

- Explain the local Weil-group.
- Local Langlands correspondence for GL_n.
- This is the most important part of the talk: General local Langlands correspondence for reductive groups. We need the notion of L-group and the classification of spherical representations in terms of Satake parameters [3, §11, 2.1], [1, chapter III]. Explain their associated unramified Galois representations.
- Global Langlands correspondence. This can be presented in a very rough and philosophical way. See e.g. Taylor's ICM talk [18, §4], cf. also [2, chapter 3.9]. Note that the topic of this seminar is to establish certain instances of the inclusion of parts of (AF) $\subset$ (lR) in the notation of [18, §4]. Namely, for $G = \mathrm{GL}_2$ in the first 2 meetings, for G a unitary group associated with a compact Shimura variety in this meeting. We have identified and will identify however only the local Galois representations at unramified primes.
- Return to the case $G = \mathrm{GL}_2$ and explain the connection between on the one hand side the Fourier coefficients of a classical newform, the Euler product of its L-function and the Satake parameters of the local constituents of the associated automorphic representation.

3.2 The conjecture of Langlands-Rapoport and Kottwitz' formula

In this talk we would like to understand the formula of Kottwitz for the number of points — or more generally, the trace of Frobenius twisted by a Hecke correspondence on the etale cohomology — of the (good) reduction mod p of Shimura varieties. The conceptual explanation of this formula, which goes back to Langlands and Rapoport [14], lies in the theory of motives over finite fields in its Tannakian formulation. The most elegant way to derive the formula uses the standard conjectures as well as the Tate and Hodge-conjecture. For certain PEL-type Shimura varieties the formula has been derived unconditionally in [13] and there are now many variants including primes of bad reduction (involving the theory of vanishing cycles) and even Shimura varieties with boundary (cf. e.g. [17]). However, in this talk we will assume the validity of the above mentioned conjectures to understand the philosophy behind the formula. We will closely follow Milne's articles.

3.3 Construction of l-adic Galois representations associated with automorphic representations of unitary Shimura varieties

The purpose of this talk is to present the paper of Kottwitz [12]. The method remains ubiquitous in the subject and contains already many of the main ideas, especially:

- The trace formula and its stabilization.
- The "Fundamental Lemma".

The generalizations of those ideas to arbitrary reductive groups were completed only recently.

4 Further reading

The following is a (very incomplete) list of some collected works, resp. recent books on the more advanced aspects of the subject:

1. Automorphic Forms, Representations, and L-functions I, II, Corvallis 1977
2. Automorphic Forms, Shimura Varieties, and L-functions, Ann Arbor 1988
3. Representation Theory and Automorphic Forms, Edinburgh 1996
4. The Geometry and Cohomology of Some Simple Shimura varieties, Harris and Taylor, 2002 [10]
5. Harmonic Analysis, The Trace Formula, And Shimura Varieties, Clay Math. Proceedings, Toronto 2003
6. An Introduction to the Langlands program 2004 [3]
7. On the cohomology of certain noncompact Shimura varieties, Morel, 2010 [17]
8. Paris book project, part I: On the stabilzation of the Trace Formula, 2011

5 References

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