SWAS 2015: Quantum Unique Ergodicity

1. **Fundamentals**

   Introduce fundamentals of dynamics like mixing, ergodicity and entropy with examples (Chap 4 in [KH95] or other sources)
   In the remaining time, give a sketch of the proof that the geodesic flow of a negatively curved manifold is ergodic [Ano69].

2. **Entropy at work**

   Present the proof of the Fuerstenberg conjecture (under the assumption of positive entropy) in [Rud90]. For context look at [Lin05].

3. **Classification of invariant measures**

   Present the unipotent subgroup result of Ratner ([Rat91] or in simpler form in [Rat92])
   Present Lindenstrauss's main result (Theorem 1.1 in [Lin06]) and explain how it implies QUE. (You might want to look at [Ein10]).

4. **Early results**

   Give a short introduction to Pseudodifferential operators [Hör07], enough to be able to define micro local lifts as in [CdV85, Zel87]. (For a simplified version see [Lin01].)
   Present the results of [CdV85, Zel87].

5. **QUE for $\Gamma\backslash\mathbb{H} \times \mathbb{H}$**

   Present the contents of the paper [Lin01], in which it is shown, that weak* limits for Hilbert-Maaß forms are invariant under torus actions.

6. **Entropy of quantum limits**

   Show that any weak * limit of microlocal lifts of Hecke-eigenfunctions has positive entropy [BL03, BL04]. This is the place, where number theory actually plays a significant role.

7. **(G, T)-spaces and recurrence**

   Introduce (G, T)-spaces as in [Lin06], Section 2. In the following, it may suffice to present proofs only in the special case of a G-action (Example 2.2 of [Lin06]).
   Sketch the contents of Section 3-5 of [Lin06].

8. **The Einsiedler-Katok Lemma**

   Present Proposition 6.4. and its proof in [Lin06] and its predecessor Lemma 8.2 in [EK03] and what it is used for in [EK03].

9. **Proof of the main theorem**

   Show how Theorem 7.1 of [Lin06] implies Theorem 1.1 of the same paper and sketch the proof of Theorem 7.1

References

[Ano69] D. V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature.*, Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder, American Mathematical Society, Providence, R.I., 1969. MR0242194 (39 #3527)

[BL03] Jean Bourgain and Elon Lindenstrauss, *Entropy of quantum limits*, Comm. Math. Phys. **233** (2003), no. 1, 153–171, DOI 10.1007/s00220-002-0770-8. MR1957735 (2004c:11076)

[BL04] ———, *Corrections to* Entropy of quantum limits (2004), available at `http://www.ma.huji.ac.il/~elon/Publications/erratum-pos-entropy.pdf`.

[CdV85] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. **102** (1985), no. 3, 497–502 (French, with English summary). MR818831 (87d:58145)

[EK03] Manfred Einsiedler and Anatole Katok, *Invariant measures on G/Γ for split simple Lie groups G*, Comm. Pure Appl. Math. **56** (2003), no. 8, 1184–1221, DOI 10.1002/cpa.10092. Dedicated to the memory of Jürgen K. Moser. MR1989231 (2004e:37042)

[Ein10] Manfred Einsiedler, *Arirthmetic Quantum Unique Ergodicity foer* $\Gamma\backslash\mathbb{H}$ (2010), available at `http://math.arizona.edu/~swc/aws/2010/2010EinsiedlerNotes.pdf`.

[Hör07] Lars Hörmander, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007. Pseudo-differential operators; Reprint of the 1994 edition. MR2304165 (2007k:35006)

[KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza. MR1326374 (96c:58055)

[Lin01] Elon Lindenstrauss, *On quantum unique ergodicity for* $\Gamma\backslash\mathbb{H}\times\mathbb{H}$, Internat. Math. Res. Notices **17** (2001), 913–933, DOI 10.1155/S1073792801000459. MR1859345 (2002k:11076)

[Lin05] ———, *Rigidity of multiparameter actions*, Israel J. Math. **149** (2005), 199–226, DOI 10.1007/BF02772541. Probability in mathematics. MR2191215 (2006j:37007)

[Lin06] ———, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) **163** (2006), no. 1, 165–219, DOI 10.4007/annals.2006.163.165. MR2195133 (2007b:11072)

[Rat91] Marina Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), no. 3, 545–607, DOI 10.2307/2944357. MR1135878 (93a:22009)

[Rat92] ———, *Raghunathan's conjectures for* $\mathrm{SL}(2,\mathbf{R})$, Israel J. Math. **80** (1992), no. 1-2, 1–31, DOI 10.1007/BF02808152. MR1248925 (94k:22024)

[Rud90] Daniel J. Rudolph, $\times 2$ *and* $\times 3$ *invariant measures and entropy*, Ergodic Theory Dynam. Systems **10** (1990), no. 2, 395–406, DOI 10.1017/S0143385700005629. MR1062766 (91g:28026)

[Zel87] Steven Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), no. 4, 919–941, DOI 10.1215/S0012-7094-87-05546-3. MR916129 (89d:58129)