Quantum Unique Ergodicity (QUE)

This is a motivational text. Don't expect too much accuracy. For more precise statements see the papers in the reference list, in particular [2].

1 Unique ergodicity and ergodicity

Let X be a compact Hausdorff space and $T : X \to X$ a homeomorphism. Let μ be a Borel probability measure which is T-invariant.

Then (X, μ, T) is called *uniquely ergodic*, if μ is the only T-invariant Borel prob.measure on X.

Example. On the unbit circle $X = \{z \in \mathbb{C} : |z| = 1\}$ the rotation by an irrational angle $T(z) = e^{2\pi i\alpha}z$, where $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, is uniquely ergodic.

The triple (X, μ, T) is called *ergodic*, if every measurable, T-stable $A \subset X$ has measure 0 or 1. Then clearly

$$(X, \mu, T) \text{ uniquely ergodic} \quad \Rightarrow \quad (X, \mu, T) \text{ ergodic.}$$

2 Quantum world

In quantum physics, points are replaced with L^2-functions and unique ergodicity translates to the following: Let (ϕ_j) be an orthonormal base of $L^2(\mu)$ which are eigenfunctions of (a quantized version of) T, ordered by absolute value of the eigenvalue.

Then (X, μ, T) is called *quantum unique ergodic*, if the probability measure $\mu_j = |\phi_j|^2\mu$ converges to μ in the sense that

$$\int_X f\, d\mu_j \quad \overset{j\to\infty}{\longrightarrow} \quad \int_X f\, d\mu$$

holds for every $f \in C(X)$.

Further, (X, μ, T) is called *quantum ergodic* if there exists a subset $J \subset \mathbb{N}$ of density one, i.e.,

$$\lim_{n\to\infty} \frac{\#J \cap \{1, 2, \ldots, n\}}{n} = 1,$$

such that μ_j converges to μ when $j \to \infty$ is confined to J.

Quantum ergodicity was shown in the seventies and eighties to hold for closed Riemannian manifolds and for manifolds with boundary by several authors.

For QUE in contrast to QE, for long time nothing was known. In 1994 Rudnick and Sarnack [3] made the daring conjecture that QUE should hold for the sphere bundle of a closed hyperbolic surface. In 2006, Elon Lindenstrauss showed this for arithmetic surfaces. This breakthrough paper granted him the 2010 Fields Medal [1]. It brings together dynamical systems, ergodic theory, harmonic analysis and arithmetic in a deep and sophisticated way. The proof procedes by showing that the limit distribution is invariant under suitable groups and that this invaraince determines the measure uniquely. All this requires the entropy to be positive, which has to be veryfied separately, and at which point the arithmetic comes into play. It is this use of entropy which makes up a good deal of Lindenstrauss's feats. He understood that positive entropy along a 1-parameter subgroup is enough to conclude that an invariant measure is of algebraic character.

One of the seminal contributions of Lindenstrauss to this realm is his broadening of the notion of recurrence of a measure to a wide variety of situations, in particular, to situations where the measure is not invariant under a certain set of transformations. Quoting Lindenstrauss, "the only thing which is really needed is some form of recurrence which produces the complicated orbits which are the life and blood of ergodic theory."

In the special case of arithmetic hyperbolic surfaces, the so-called Hecke operators come into the picture and they act on the limiting measure arising from such a sequence of eigenfunctions. This action is recurrent and the tools developed by Lindenstrauss become applicable to this situation at hand, and lead elegantly to a solution of the problem. Solving the so-called arithmetic quantum unique ergodicity conjecture of Rudnick and Sarnak is exciting if for no other reason than that the conjecture has been established provisionally, based on the generalized Riemann hypothesis.

While this doesn't bring us closer to a solution of this famous question, this connection does testify to the depth of the mathematics involved.

Literatur

[1] Harry Furstenberg, *The work of Elon Lindenstrauss*, Proceedings of the International Congress of Mathematicians. Volume I, Hindustan Book Agency, New Delhi, 2010, pp. 51–54.

[2] Andrew Hassell, *What is quantum unique ergodicity?*, Austral. Math. Soc. Gaz. **38** (2011), no. 3, 158–167.

[3] Zeév Rudnick and Peter Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), no. 1, 195–213.

[4] Peter Sarnak, *Recent progress on the quantum unique ergodicity conjecture*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 2, 211–228.