Generalised Selberg zeta functions and a conjectural Lefschetz formula

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Abstract. A generalisation of the Selberg zeta function, or rather its logarithmic derivative, to higher rank is given. In the compact case one gets a Mittag-Leffler series expansion, the proof of which rests upon a dynamical Lefschetz formula. A modified version of the Lefschetz formula is then conjectured to hold in the non-compact case as well. A sketch of a possible proof is included.

Introduction

In the nineteenfifties, A. Selberg introduced, together with the trace formula, the Selberg zeta function for compact quotients of the hyperbolic plane. This was later generalised to rank one spaces [**19, 33**] and also to non-compact spaces (still of rank one) [**20**]. See also [**7, 8, 15, 16, 17, 18, 23, 30, 31, 32**].

In this paper we first survey results on a several variable Dirichlet series $L^j(s)$ which may be considered as a generalisation of the logarithmic derivative of the Selberg zeta function. The main result is that $L^j(s)$ admits a Mittag-Leffler series expansion. This can be used to derive asymptotical results as a prime geodesic theorem or class number asymptotics [**10**].

Hitherto the Mittag-Leffler formula has been shown for compact locally symmetric spaces only. In the second part we present a conjectural Lefschetz formula for the non-compact case. Such Lefschetz formulae for compact spaces are in [**13**] and [**25**]. We also give a sketch of a possible proof in the non-compact case, which, however, rests on further, more technical conjectures.

1. Generalised Selberg zeta functions

First recall the classical Selberg zeta function. For this fix a semi-simple Lie group G which is connected and has finite centre. Assume that the split-rank of G is one. Let $X = G/K$ be the associated symmetric space. Here $K \subset G$ is a

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maximal compact subgroup of G. Let $\Gamma \subset G$ be a discrete, torsion-free subgroup. Let $P = MAN$ be the Langlands decomposition of a non-trivial parabolic subgroup P (all such are conjugate in G). Then A is a split torus of dimension one and M is compact. Every $\gamma \in \Gamma$ with $\gamma \neq 1$ is G-conjugate to an element $a_\gamma m_\gamma \in A^- M$, where A^- is the negative Weyl chamber in A with respect to P. The element a_γ is uniquely determined and m_γ is unique up to M-conjugation. We say that $\gamma \in \Gamma$ is *primitive* if $\gamma = \sigma^n$, $\sigma \in \Gamma$, $n \in \mathbb{N}$, implies $n = 1$. The Selberg zeta function is defined by

$$Z_\Gamma(s) = \prod_{[\gamma]\ \mathrm{prim}} \prod_{k \geq 0} \det\left(1 - e^{-sl(\gamma)} (a_\gamma m_\gamma)^k | \mathfrak{n}\right),$$

where $\mathfrak{n}$ is the Lie algebra of N and $a_\gamma m_\gamma$ acts on $\mathfrak{n}$ by the adjoint representation. The first product is extended over the set of all primitive conjugacy classes $[\gamma]$ in Γ. The length of γ is defined by $l(\gamma) = |\log a_\gamma|$. Then the product converges for $\mathrm{Re}(s)$ large enough and $Z_\Gamma(s)$ extends to a meromorphic function on $\mathbb{C}$ [**7, 9, 25**].

Note that to every $\gamma \in \Gamma \smallsetminus \{1\}$ there is a unique primitive element γ_0 such that γ is a positive power of γ_0. Note that

$$\frac{Z_\Gamma'}{Z_\Gamma}(s) = \sum_{[\gamma] \neq 1} \frac{l(\gamma_0)}{\det(1 - a_\gamma m_\gamma | \mathfrak{n})} e^{-sl(\gamma)},$$

and thus for every $j \in \mathbb{N}$,

$$(-1)^{j+1} \left(\frac{\partial}{\partial s}\right)^{j+1} \frac{Z_\Gamma'}{Z_\Gamma}(s) = \sum_{[\gamma] \neq 1} \frac{l(\gamma_0)}{\det(1 - a_\gamma m_\gamma | \mathfrak{n})} l(\gamma)^{j+1} e^{-sl(\gamma)}$$

Here the sum runs over the set of all conjugacy classes $[\gamma] \neq 1$ in Γ.

Now for the higher rank case, i.e. $\mathrm{rank}(G) \geq 1$. Then there can be several conjugacy classes of non-trivial parabolics. For the Selberg zeta function one only considers *cuspidal* parabolics, i.e., parabolics $P = MAN$ such that M admits a compact Cartan subgroup. Assume first that A is one-dimensional. Then one defines the Selberg zeta function exactly as before, except that the first product is extended over the set $\mathcal{E}_P(\Gamma)$ of all conjugacy classes $[\gamma]$ in Γ with $\gamma \sim_G a_\gamma m_\gamma$ where $a_\gamma \in A^-$ and $m_\gamma \in M$ is elliptic, and that the Euler factors come with an exponent which is an Euler number given below. One can prove that the product converges in a half-plane and $Z_\Gamma(s)$ extends to a meromorphic function on $\mathbb{C}$ [**9**].

For higher dimensional A it is natural to expect a Dirichlet series in several variables for a generalised Selberg zeta function. The reason is this: if $\dim A = 1$ then $a_\gamma \in A^-$ is determined by a single value, the length $l(\gamma)$. If $\dim A > 1$, then $a_\gamma \in A^-$ lies in a higher-dimensional Weyl chamber and the length alone does not pin it down. Therefore, one rather expects a zeta function in $r = \dim A$ variables.

If $[\gamma] \in \mathcal{E}_P(\Gamma)$, then γ lies in a conjugate $A_\gamma M_\gamma$ of AM. Let G_γ and Γ_γ denote the centralisers of γ in G and Γ respectively and let K_γ be a maximal compact subgroup of G_γ. Consider the locally symmetric space $X_\gamma = \Gamma_\gamma \backslash G_\gamma / K_\gamma$. Since $G_\gamma \subset A_\gamma M_\gamma$, the torus A_γ lies central in G_γ and therefore acts on X_γ. The Euler-number $\chi(A_\gamma \backslash X_\gamma)$ is non-zero. The group Γ_γ is a discrete subgroup of $A_\gamma M_\gamma$ and projects down to a lattice $\Gamma_{\gamma,A}$ in A_γ. Let

$$\lambda_\gamma = \mathrm{vol}\left(A_\gamma / \Gamma_{\gamma,A}\right).$$

Set

$$\mathrm{ind}(\gamma) = \frac{\lambda_\gamma\, \chi(A_\gamma \backslash X_\gamma)}{\det(1 - a_\gamma m_\gamma | \mathfrak{n})}.$$

If $\dim A = 1$, then $\lambda_\gamma = l(\gamma_0)$ and if the rank of G is one, then $\chi(A_\gamma \backslash X_\gamma) = 1$.

Let $\alpha_1, \dots, \alpha_r$ be positive multiples of the simple roots of (A, G) such that

$$\alpha_1 + \dots + \alpha_r = 2\rho \ \in \ \mathfrak{a}^*,$$

where $\mathfrak{a}^*$ is the Lie algebra of A and ρ is the modular shift of P. For $a \in A^-$ let $l(a) = |\alpha_1(\log a) \cdots \alpha_r(\log a)|$. For $s \in \mathbb{C}^r$ let

$$s \cdot \alpha \ \stackrel{\mathrm{def}}{=} \ s_1\alpha_1 + \dots + s_r\alpha_r \ \in \ \mathfrak{a}^*.$$

Each $\lambda \in \mathfrak{a}^*$ gives a continuous group homomorphism $A \to \mathbb{C}^\times$ written as $a \mapsto a^\lambda$. For $j \in \mathbb{N}$ let

$$L^j(s) \ \stackrel{\mathrm{def}}{=} \ \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \mathrm{ind}(\gamma)\, l(a)^{j+1}\, a_\gamma^{s \cdot \alpha}.$$

This is a multi-variable Dirichlet series which converges for $\mathrm{Re}(s_k) > 1$ for $k = 1, \dots, r$. We consider it a replacement for the lacking Selberg zeta function in several variables. Indeed in the rank one case the proof of the meromorphic continuation proceeds via the logarithmic derivative, but it is impossible to deduce meromorphicity in several variables in this way.

Let D be the differential operator

$$D = (-1)^r \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_r}.$$

For a complex vector space V on which A acts linearly, and $\lambda \in \mathfrak{a}^*$, let

$$V^\lambda = \{v \in V : \exists n \in \mathbb{N},\ (a - a^\lambda)^n v = 0\ \forall a \in A\}$$

be the generalised λ-weight space. For (π, V_π) in the unitary dual $\hat{G}$ of G, let $H^q(\mathfrak{n}, \pi_K)$ denote the Lie-algebra cohomology of the Harish-Chandra-module

$$\pi_K \ \stackrel{\mathrm{def}}{=} \ \{v \in V_\pi : v \text{ is } K-\text{finite}\}.$$

Then $H^q(\mathfrak{n}, \pi_K)$ is a Harish-Chandra module for $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ [**22**].

Since Γ is cocompact, the right regular representation of G on $L^2(\Gamma \backslash G)$ decomposes discretely,

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi)\pi$$

with finite multiplicities $N_\Gamma(\pi)$.

For $\lambda \in \mathfrak{a}^*$, $\pi \in \hat{G}$, let

$$m_\lambda(\pi) = \sum_{p,q \geq 0} (-1)^{p+q+\dim N} \dim \left(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \bigwedge^p \mathfrak{p}_M \right)^{K_M},$$

where $K_M = K \cap M$ and $\mathfrak{p}_M$ is the positive part in the Cartan decomposition $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{p}_M$ of $\mathfrak{m} = \mathrm{Lie}(M)$. Let

$$q_M = \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^p \dim \left(\bigwedge^p \mathfrak{p}_M \right)^{K_M}.$$

Then it turns out that $q_M \in \mathbb{N}$; see [**12**].

THEOREM 1.1. *For j large enough the Dirichlet series $L^j(s)$ converges locally uniformly on $\{\mathrm{Re}(s_k) > 1\}$. It can be written as a Mittag-Leffler series*

$$L(s) = D^{j+1} \frac{q_M}{(s_1 - 1) \cdots (s_r - 1)}$$

$$+ \sum_{\pi \in \hat{G} - \{1\}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) D^{j+1} \frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)},$$

where the co-ordinates λ_k are defined by $\lambda = \lambda_1 \alpha_1 + \cdots + \lambda_r \alpha_r$. For $\pi \neq 1$ and $\lambda \in \mathfrak{a}^*$ with $m_\lambda(\pi) \neq 0$ we have $\mathrm{Re}(\lambda_k) \geq -1$ for $k = 1, \ldots, r$ and there is at least one k with $\mathrm{Re}(\lambda_k) < -1$. The Mittag-Leffler series converges locally uniformly on $\{\mathrm{Re}(s_k) > 1\}$.

This theorem is sufficient to prove asymptotic assertions about geodesics and class numbers [**10, 12**], but it will not grant meromorphic continuation of L^j to all of $\mathbb{C}^r$.

QUESTION 1.2. *Does $L^j(s)$ extend to a meromorphic function on $\mathbb{C}^r$?*

In one variable, a convergent Mittag-Leffler series guarantees meromorphicity. In several variables, however, poles can accumulate even though the Mittag-Leffler series converges. The question of meromorphicity of $L^j(s)$ thus amounts to subtle questions of the distribution of automorphic representations which are beyond the scope of our present methods.

The theorem is derived from a Lefschetz formula which we will present next.

THEOREM 1.3. *(Lefschetz formula)*
For $\varphi \in C_c^\infty(A^-)$ we have

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{\lambda \in \mathfrak{a}^*} m_\lambda(\pi) \int_A \varphi(a) a^{\lambda+\rho}\, da = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \mathrm{ind}(\gamma)\, \varphi(a_\gamma).$$

The proof of this formula [**13**] uses the trace formula and the Osborne conjecture [**22**].

2. A conjectural Lefschetz formula

In this section we will formulate a Lefschetz formula for an arithmetic subgroup Γ of G which is not necessarily cocompact. Fix a cuspidal parabolic P. Let $\hat{G}_{\mathrm{adm}} \supset \hat{G}$ be the admissible dual, i.e. the set of classes of admissible irreducible representations under infinitesimal equivalence. Harish-Chandra proved that two unitary irreducible representations are unitarily equivalent iff they are infinitesimally equivalent. Therefore $\hat{G}$ can be considered as a subset of $\hat{G}_{\mathrm{adm}}$. For $\pi \in \hat{G}_{\mathrm{adm}}$ let $\Lambda_\pi \in \mathfrak{h}^*$ be a representative of the infinitesmal character of π. The unitary G-representation on $L^2(\Gamma\backslash G)$ decomposes as

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}},$$

where

$$L^2_{\mathrm{disc}} = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi)\, \pi$$

is a direct sum of irreducibles with finite multiplicities and L^2_{cont} is a sum of continuous Hilbert integrals. In particular, L^2_{cont} does not contain any irreducible subrepresentation.

Let $\mathfrak{a}_{\mathbb{R}}^{*,+} = \{t_1\alpha_1 + \cdots + t_r\alpha_r : t_1, \dots, t_r > 0\}$ be the positive dual cone and let $\overline{\mathfrak{a}_{\mathbb{R}}^{*,+}}$ be its closure in $\mathfrak{a}_{\mathbb{R}}^*$.

For $\mu \in \mathfrak{a}^*$ and $j \in \mathbb{N}$ let $C^{\mu,j}(A^-)$ denote the space of all functions on A which

- are j-times continuously differentiable on A,
- are zero outside A^-,
- satisfy $|D\varphi| \le C|a^\mu|$ for every invariant diffferential operator D on A of degree $\le j$, where $C > 0$ is a constant, which depends on D.

This space can be topologized with the seminorms

$$N_D(\varphi) = \sup_{a\in A} |a^{-\mu} D\varphi(a)|,$$

$D \in U(\mathfrak{a})$, $\deg(D) \le j$. Since the space of operators D as above is finite dimensional, one can choose a basis $D_1, \dots, D_n$ and set

$$\|\varphi\| = N_{D_1}(\varphi) + \cdots + N_{D_n}(\varphi).$$

The topology of $C^{\mu,j}(A^-)$ is given by this norm and thus $C^{\mu,j}(A^-)$ is a Banach space.

Conjecture 2.1. *(Lefschetz Formula)*
For $\lambda \in \mathfrak{a}*$ and $\pi \in \hat{G}_{\mathrm{adm}}$ there is an integer $N_{\Gamma,\mathrm{cont}}(\pi, \lambda)$ which vanishes if $\mathrm{Re}(\lambda) \notin \overline{\mathfrak{a}_{\mathbb{R}}^{*,+}}$ and there are $\mu \in \mathfrak{a}^*$ and $j \in \mathbb{N}$ such that for each $\varphi \in C^{\mu,j}(A^-)$ we have

$$\sum_{\substack{\pi\in\hat{G}_{\mathrm{adm}}\\ \lambda\in\mathfrak{a}^*}} m_\lambda(\pi)\left(N_\Gamma(\pi) + N_{\Gamma,\mathrm{cont}}(\pi,\lambda)\right) \int_A \varphi(a) a^\lambda \, da = \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \mathrm{ind}(\gamma)\, \varphi(a_\gamma).$$

Either side of this identity represents a continuous functional on $C^{\mu,j}(A^-)$.

The numbers $N_{\Gamma,\mathrm{cont}}(\pi)$ represent vanishing orders of the automorphic scattering matrix. The conjecture can be proven for SL_2.

3. A possible proof for $\mathbb{Q}$-rank one

In this section we give a rough sketch of a possible proof for $\mathbb{Q}$-rank one congruence groups. We will point out the technical problems, each of which requires further study. The main problem consists in a growth assertion for the logarithmic derivative of the scattering matrix. The "proof" uses Arthur's trace formula.

3.1. The spectral side. We will now recall Arthur's trace formula. This formula is the equality of two distributions on $\mathcal{G}(\mathbb{A})$,

$$J_{\mathrm{geom}} = J_{\mathrm{spec}}.$$

The geometric distribution J_{geom} can be described in terms of weighted orbital integrals. For the moment our interest however is focused on the spectral distribution J_{spec}.

From now on we will assume that

$$\mathrm{rank}_{\mathbb{Q}}(\mathcal{G}) = 1.$$

Then, up to conjugation, there is only one $\mathbb{Q}$-parabolic $\mathcal{P}_0$ different from $\mathcal{G}$. Let $\mathcal{P}_0 = \mathcal{L}_0\mathcal{N}_0$ be a Levi decomposition and let $P_0 = M_0A_0N_0$ be the Langlands decomposition of $P_0 = \mathcal{P}_0(\mathbb{R})$ with $M_0A_0 = L_0 = \mathcal{L}_0(\mathbb{R})$. Let Λ be an infinitesimal character of M_0 and let $\mathcal{H}_0(\Lambda)$ be the space of all functions ϕ on $\mathcal{N}_0(\mathbb{A})\mathcal{L}_0(\mathbb{Q})A_0\backslash\mathcal{G}(\mathbb{A})$ whose pullback to $\mathcal{L}_0(\mathbb{Q})\backslash\mathcal{L}_0(\mathbb{A})^1 \times K_\mathbb{A}$ is square integrable and which satisfy $\phi(Zx) = \Lambda(Z)\phi(x)$ in the distributional sense for every $Z \in \mathfrak{z}_{M_0}$. The Weyl group $W = W(G, A_0)$ acts on the set of all infinitesimal characters Λ of M_0. If one writes $\mathcal{O}$ for an orbit under that action, then $\mathcal{O}$ has one or two elements. Let $\mathcal{H}_0(\mathcal{O})$ denote the sum of the $\mathcal{H}_0(\Lambda)$, where Λ ranges over $\mathcal{O}$.

Let $\mathfrak{a}_0, \mathfrak{n}_0$ be the complex Lie algebras of A_0 and N_0 respectively. For $\phi \in \mathcal{H}_0(\mathcal{O})$ and $\lambda \in \mathfrak{a}_0^*$ we put $\phi_\lambda(x) = e^{\langle\lambda+\rho_0,H(x)\rangle}\phi(x)$, where for $X \in \mathfrak{a}_0$ we set $\rho_0(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|\mathfrak{n}_0)$ and $H\colon \mathcal{G}(\mathbb{A}) \to \mathfrak{a}_{0,\mathbb{R}}$ is defined by $e^{\langle\psi,H(nlk)\rangle} = \psi(l)$ for every $\psi \in X_\mathbb{Q}(\mathcal{L}_0)$, $n \in \mathcal{N}_0(\mathbb{A})$, and $k \in K_\mathbb{A}$. We get a representation $I_{\mathcal{O},\lambda}$ of $\mathcal{G}(\mathbb{A})$ on $\mathcal{H}_0(\mathcal{O})$ by

$$(I_{\mathcal{O},\lambda}(y)\phi)(x) = \phi_\lambda(xy), \qquad x, y \in \mathcal{G}(\mathbb{A}).$$

Let w_0 denote the non-trivial element of the Weyl group $W(G, A_0)$ as well as any representative in $\mathcal{G}(\mathbb{Q})$. Let $\lambda \in \mathfrak{a}_0^*$. In the theory of Eisenstein series one considers the operator $M(\mathcal{O}, \lambda)$ on the subspace of $K_\mathbb{A}$-finite vectors in $\mathcal{H}_0(\mathcal{O})$, which is defined for $\mathrm{Re}(\lambda - \rho_0)$ positive with respect to $\mathcal{P}_0$ by

$$(M(\mathcal{O}, \lambda)\phi)_{-\lambda}(x) = \int_{\mathcal{N}_0(\mathbb{A})} \phi_\lambda(w_0nx)\, dn$$

and has a meromorphic continuation to $\mathfrak{a}_0^*$. This operator satisfies

$$M(\mathcal{O}, \lambda)M(\mathcal{O}, w_0\lambda) = \mathrm{Id} \;\text{ and }\; M(\mathcal{O}, \lambda)^* = M(\mathcal{O}, \bar{\lambda})$$

and

$$M(\mathcal{O}, \lambda)\, I_{\mathcal{O},\lambda} = I_{\mathcal{O},w_0\lambda}\, M(\mathcal{O}, \lambda),$$

so it intertwines $I_{\mathcal{O},\lambda}$ and $I_{\mathcal{O},w_0\lambda} = I_{\mathcal{O},-\lambda}$. For an irreducible unitary representation π of $\mathcal{G}(\mathbb{A})$ we write $N(\pi)$ for the multiplicity of π in $L^2(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$. If T is a truncation parameter, the spectral side of the trace formula is given by

$$\sum_{\pi\in\Pi(\mathcal{G}(\mathbb{A}))} N(\pi)\,\mathrm{tr}\pi(f)$$

$$\begin{aligned} &+ \quad \frac{\rho(T)}{\pi}\sum_{\mathcal{O}}\int_{\mathfrak{a}_{0,\mathbb{R}}^*} \mathrm{tr} I_{\mathcal{O},\lambda}(f)\, d\lambda \\ &- \quad \frac{1}{4\pi}\sum_{\mathcal{O}}\int_{\mathfrak{a}_{0,\mathbb{R}}^*} \mathrm{tr}\left(M(\mathcal{O}, -i\lambda)M'(\mathcal{O}, i\lambda)\, I_{\mathcal{O},\lambda}(f)\right) d\lambda \\ &+ \quad \frac{1}{4}\sum_{\mathcal{O}:\#\mathcal{O}=1} \mathrm{tr}\left(M(\mathcal{O}, 0)\, I_{\mathcal{O},0}(f)\right). \end{aligned}$$

Here we have written $\Pi(\mathcal{G}(\mathbb{A}))$ for the unitary dual of the locally compact group $\mathcal{G}(\mathbb{A})$. We further have identified $\mathfrak{a}_{0,\mathbb{R}}^*$ with $\mathbb{R}$ by $t \mapsto 2t\rho$, which explains the derivative $M'(\mathcal{O}, \lambda) = \frac{d}{d\lambda}M(\mathcal{O}, \lambda)$. This formula is a special case of Theorem 8.2 in [**4**].

3.2. A simple trace formula. In this section we recall the "simple trace formula" from [**11**].

Let $\mathcal{H}$ be a linear algebraic $\mathbb{Q}$-group. If E is a $\mathbb{Q}$-algebra, any rational character χ of $\mathcal{H}$ defined over $\mathbb{Q}$ defines a homomorphism $\mathcal{H}(E) \to \mathrm{GL}_1(E)$. If E comes with an absolute value $|\,.\,|$, we define $\mathcal{H}(E)^1$ to be the subgroup of all elements g such that $|\chi(g)| = 1$ for all rational characters χ defined over $\mathbb{Q}$. We will use this notation in the cases when E is $\mathbb{R}$ or the ring $\mathbb{A}$ of adeles of $\mathbb{Q}$. One should be aware that $\mathcal{H}(\mathbb{R})^1$ could also be defined with respect to characters defined over the field $\mathbb{R}$, but this is not the point of view in the present paper.

Let $\mathcal{G}$ be a semisimple, simply connected reductive linear algebraic group over $\mathbb{Q}$. If $\mathcal{P}$ is a parabolic $\mathbb{Q}$-subgroup of $\mathcal{G}$ with unipotent radical $\mathcal{N}$, we have a Levi decomposition $\mathcal{P} = \mathcal{L}\mathcal{N}$. Generally, we denote the group of real points of a linear algebraic $\mathbb{Q}$-group by the corresponding roman letter, so that $P = LN$. However, if $\mathcal{A}$ is a maximal $\mathbb{Q}$-split torus of $\mathcal{L}$, we denote by A the connected component of the identity $\mathcal{A}(\mathbb{R})^0$. One has decompositions $\mathcal{L}(\mathbb{A}) = \mathcal{L}(\mathbb{A})^1 A$, $L = MA$ (direct products) and $P^1 = MN$, where $M = L^1$.

An element x of $\mathcal{G}(\mathbb{A})$ is called *parabolically singular* or *p-singular*, if there are $y \in \mathcal{G}(\mathbb{A})$ and a parabolic $\mathbb{Q}$-group $\mathcal{P} \neq \mathcal{G}$ such that $yxy^{-1} \in \mathcal{P}(\mathbb{A})^1$. A function f on $\mathcal{G}(\mathbb{A})$ is called *p-admissible* if f vanishes on all p-singular elements.

EXAMPLE 3.1. Suppose that the function f on $\mathcal{G}(\mathbb{A})$ is supported on $K_{\text{fin}} \times G$ and vanishes on all G-conjugates of $K_{\text{fin}} \times P^1$ for every parabolic $\mathbb{Q}$-subgroup $\mathcal{P} \neq \mathcal{G}$. Then f is p-admissible.

Proof: Let $q \in \mathcal{P}(\mathbb{A})^1$ for some proper Q-parabolic $\mathcal{P}$ and let $x \in \mathcal{G}(\mathbb{A})$. We have to show that $f(x^{-1}qx) = 0$. By the assumption on the support of f we have only to consider $q = q_{\text{fin}} q_\infty$ with $x^{-1} q_{\text{fin}} x \in K_{\text{fin}}$, i.e., $q_{\text{fin}} \in x K_{\text{fin}} x^{-1} \cap \mathcal{P}(\mathbb{A})$, a compact subgroup of $\mathcal{P}(\mathbb{A})$. Any continuous quasicharacter with values in $]0, \infty[$ will be trivial on that subgroup, hence $q_{\text{fin}} \in \mathcal{P}(\mathbb{A})^1$. Since q was already in $\mathcal{P}(\mathbb{A})^1$, it follows that $q_\infty \in \mathcal{P}(\mathbb{A})^1 \cap P = P^1$, and so $f(x^{-1}qx) = 0$ due to the assumption on f applied to the parabolic $\mathbb{Q}$-subgroup $\mathcal{P}$. □

An element $\gamma \in \mathcal{G}(\mathbb{Q})$ is called $\mathbb{Q}$-*elliptic* if it is not contained in any parabolic $\mathbb{Q}$-subgroup other than $\mathcal{G}$ itself. This notion is clearly invariant under conjugation, and we say that a class o is $\mathbb{Q}$-elliptic if some (hence any) of its elements is so. It is known that $\mathbb{Q}$-elliptic elements are semisimple, so $\mathbb{Q}$-elliptic classes o are just conjugacy classes in $\mathcal{G}(\mathbb{Q})$.

Let $\Gamma \subset \mathcal{G}(\mathbb{Q}) \subset G$ be a congruence subgroup, i.e., there exists a compact open subgroup K_Γ of $\mathcal{G}(\mathbb{A}_{\text{fin}})$ such that $\Gamma = K_\Gamma \cap \mathcal{G}(\mathbb{Q})$. Suppose the test function f is of the form $f = \frac{1}{\mathrm{vol}(K_\Gamma)} \otimes f_\infty$, where f_∞ is p-regular on G. Then f is p-regular [**11**].

PROPOSITION 3.2. *Let f be as above. Then we have*

$$J_{\text{geom}}(f) = \sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)\ \mathcal{O}_\gamma(f_\infty),$$

where the sum on the right-hand side runs over the set of all conjugacy classes $[\gamma]$ in the group Γ which consist of $\mathbb{Q}$-elliptic elements.

Proof: see [**11**]. □

PROPOSITION 3.3. *If the $\mathbb{Q}$-rank of $\mathcal{G}$ is one and f is p-admissible, then the spectral side of the trace formula reduces to*

$$\begin{aligned} &\sum_{\pi\in\Pi(\mathcal{G}(\mathbb{A}))} N(\pi)\,\mathrm{tr}\pi(f) \\ &- \quad \frac{1}{4\pi}\sum_{\mathcal{O}}\int_{\mathfrak{a}^*_{0,\mathbb{R}}} \mathrm{tr}\left(M(\mathcal{O},-i\lambda)M'(\mathcal{O},i\lambda)\,I_{\mathcal{O},\lambda}(f)\right)\,d\lambda \\ &+ \quad \frac{1}{4}\sum_{\mathcal{O}:\#\mathcal{O}=1} \mathrm{tr}\left(M(\mathcal{O},0)\,I_{\mathcal{O},0}(f)\right). \end{aligned}$$

Proof: If f is p-admissible, then the geometric side of the trace formula, as we have seen, is independent of T. Therefore the spectral side also is constant as a function in the truncation parameter T. This means that the summand involving T must be zero. □

3.3. The continuous contribution. In this section we will treat the continuous spectral contribution which is

$$\frac{1}{4\pi}\int_{\mathfrak{a}^*_{0,\mathbb{R}}} \mathrm{tr}\left(M(\mathcal{O},-i\lambda)M'(\mathcal{O},i\lambda)\,I_{\mathcal{O},\lambda}(f)\right)\,d\lambda.$$

We abbreviate this as

$$\frac{1}{4\pi}\int_{\mathfrak{a}^*_{0,\mathbb{R}}} \mathrm{tr}\left(\mathcal{M}(\lambda)\,I_{\mathcal{O},\lambda}(f)\right)\,d\lambda,$$

or, if no confusion is possible, write it as

$$\frac{1}{4\pi i}\int_{i\mathfrak{a}^*_{0,\mathbb{R}}} \mathrm{tr}\ M(\lambda)^{-1}M'(\lambda)I_\lambda(f)\,d\lambda.$$

3.3.1. *Integral kernel.* Fix an infinitesimal character Λ of M_0. Let

$$V(\Lambda) \stackrel{\text{def}}{=} L^2\left(\mathcal{L}_0(\mathbb{Q})\backslash\mathcal{L}_0(\mathbb{A})^1\right)(\Lambda)$$

be the space of all square integrable functions ϕ on $\mathcal{L}_0(\mathbb{Q})\backslash\mathcal{L}_0(\mathbb{A})^1$ satisfying $T\phi = \Lambda(T)\phi$ in the distributional sense for every $T\in\mathfrak{z}_{M_0}$. Let $V(\mathcal{O})$ be the sum of the $V(\Lambda)$ where Λ ranges over $\mathcal{O}$. Let R_0 denote the right regular representation of $\mathcal{L}_0(\mathbb{A})^1$ on this space. Note that R_0 is a direct sum of irreducible representations. The representation $I_{\mathcal{O},\lambda}$ can be viewed as the unitarily induced representation

$$\mathrm{Ind}_{\mathcal{P}(\mathbb{A})}^{\mathcal{G}(\mathbb{A})}\left(V(\mathcal{O})\otimes\lambda\otimes 1\right),$$

where we have used $\mathcal{P}(\mathbb{A}) = \mathcal{L}_0(\mathbb{A})^1 A_0 \mathcal{N}_0(\mathbb{A})$. In other words, $I_{\mathcal{O},\lambda}$ can be viewed as the representation on the space $H_{\mathcal{O},\lambda}$ of functions $\phi\colon \mathcal{G}(\mathbb{A})\to V(\mathcal{O})$ satisfying $\phi(manx) = a^{\lambda+\rho_0}R_0(m)\phi(x)$ for $m\in\mathcal{L}_0(\mathbb{A})^1$, $a\in A_0$, $n\in\mathcal{N}_0(\mathbb{A})$, and $x\in\mathcal{G}(\mathbb{A})$, as well as $\int_{k_{\mathbb{A}}}|\phi(k)|^2\,dk < \infty$. On this space, the representation $I_{\mathcal{O},\lambda}$ is given by $I_{\mathcal{O},\lambda}(y)\phi(x) = \phi(xy)$.

Since $\mathcal{G}(\mathbb{A}) = \mathcal{P}(\mathbb{A})K_{\mathbb{A}}$, any function $\phi\in H_{\mathcal{O},\lambda}$ is uniquely determined by its restriction to $K_{\mathbb{A}}$. In this way, $H_{\mathcal{O},\lambda}$ can be identified with the space $L^2(K_{\mathbb{A}}, R_0)$ of all $\phi\in L^2(K_{\mathbb{A}}, V(\mathcal{O}))$ such that $\phi(mk) = R_0(m)\phi(k)$ for all $m\in K_{\mathbb{A}}\cap\mathcal{L}_0(\mathbb{A})$ and all $k\in K_{\mathbb{A}}$.

Now let f be as above, and let $\phi \in H_{\mathcal{O},\lambda}$. Then for $k_1 \in K_{\mathbb{A}}$,

$$\begin{aligned} I_{\mathcal{O},\lambda}(f)\phi(k_1) &= \int_{\mathcal{G}(\mathbb{A})} f(y)\phi(k_1 y)\, dy \\ &= \int_{\mathcal{G}(\mathbb{A})} f(k_1^{-1} y)\phi(y)\, dy \end{aligned}$$

and this equals

$$\int_{\mathcal{N}_0(\mathbb{A})} \int_{\mathcal{L}_0(\mathbb{A})^1} \int_{A_0} \int_{K_{\mathbb{A}}} f(k_1^{-1} nmak_2)\, a^{\lambda+\rho_0} R_0(m)\phi(k_2)\, dndmdadk_2.$$

We thus interpret $I_{\mathcal{O},\lambda}(f)$ as an integral operator on $L^2(K_{\mathbb{A}}, R_0)$ with kernel

$$\begin{aligned} k_{f,\lambda}(k_1,k_2) &= \int_{\mathcal{N}(\mathbb{A})\mathcal{L}_0(\mathbb{A})^1 A_0} f(k_1^{-1} nmak_2) a^{\lambda+\rho_0} R_0(m) dnam \\ &= \int_{A_0} \tilde{f}(k_1,k_2,a) a^{\lambda+\rho_0}\, da, \end{aligned}$$

where

$$\tilde{f}(k_1,k_2,a) = \int_{\mathcal{N}(\mathbb{A})\mathcal{L}_0(\mathbb{A})^1} f(k_1^{-1} nmak_2) R_0(m)\, dmda.$$

Thus we may view the kernel $k_{f,\lambda}$ pointwise as a Fourier transform in λ of the function $\tilde{f}$.

3.3.2. *Moderate growth.* An open set $U \subset \mathbb{C}$ is called an *admissible set* if each connected component of U is bounded.

Let g be a meromorphic function on $\mathbb{C}$. We say that g is *essentially of moderate growth* if there is a natural number N, a constant $C > 0$ and an admissible set U such that on $\mathbb{C} \setminus U$ one has $|g(z)| \le C|z|^N$. The minimal number N for which there exists such a set U is called the *growth exponent* of g.

LEMMA 3.4. *Let f be an entire function of finite order p and let $g = f'/f$ be its logarithmic derivative. Then g is essentially of moderate growth with growth exponent $\le 2p+3$.*

Proof: Let p be the order of f and let $a_1, a_2, \dots$ be the non-zero zeros of f, each repeated with multiplicity. By Hadamard's factorization theorem we can write

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_p(z/a_n),$$

where P is a polynomial of degree $\le p$ and

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

This implies

$$\frac{f'}{f}(z) = \frac{m}{z} + P'(z) + \sum_{n=1}^{\infty} \frac{1}{a_n}\frac{E_p'}{E_p}(z/a_n).$$

It suffices to show the claim for the sum over n. So we will assume that $m = 0 = P(z)$. As a consequence of Hadamard's factorization theorem one has

$$D \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |a_n|^{-p-1} < \infty,$$

and this implies that there is $C > 0$ such that for $x > 0$,

$$\#\{n : |a_n| \le x\} \ \le \ Cx^{p+2}.$$

For $a \in \mathbb{C}$ and $r > 0$ let $B_r(a)$ denote the open disk of radius r around a. For $n \in \mathbb{N}$ we define

$$r_n \ \stackrel{\text{def}}{=} \ |a_n|^{-p-1}.$$

We will first show that the set $U = \bigcup_n B_{r_n}(a_n)$ has only bounded components. Assume the contrary. By replacing the sequence (a_n) with a subsequence if necessary it suffices to assume that B consists of a single component and that the balls $B_{r_n}(a_n)$ and $B_{r_{n+1}}(a_{n+1})$ have a nonempty intersection for every $n \in \mathbb{N}$. This implies that

$$|a_n - a_{n+1}| \ < \ r_n + r_{n+1} = |a_n|^{-p-1} + |a_{n+1}|^{-p-1}.$$

The finiteness of D then implies that

$$\sum_n |a_n - a_{n+1}| \ < \ \infty.$$

This, however, implies that the sequence (a_n) converges, a contradiction. So the set U has only bounded components.

We have to estimate the absolute value of

$$\frac{1}{a_n}\frac{E_p'}{E_p}(z/a_n) = \frac{1}{a_n}\left(1 + \frac{z}{a_n} + \cdots + \left(\frac{z}{a_n}\right)^{p-1} - \frac{1}{1 - z/a_n}\right)$$

in $\mathbb{C} \smallsetminus U$. We first consider the case $|z| < |a_n|$. Then

$$\begin{aligned} \left|\frac{1}{a_n}\frac{E_p'}{E_p}(z/a_n)\right| &= \frac{1}{|a_n|}\left|1 + \frac{z}{a_n} + \cdots + \left(\frac{z}{a_n}\right)^{p-1} - \frac{1}{1 - \frac{z}{a_n}}\right| \\ &= \frac{1}{|a_n|}\left|\sum_{j=p}^{\infty}\left(\frac{z}{a_n}\right)^j\right| \\ &= \frac{1}{|a_n|}\left|\left(\frac{z}{a_n}\right)^p \frac{1}{1 - \frac{z}{a_n}}\right| \\ &= \frac{|z|^p}{|a_n|^{p+1}}\frac{1}{|1 - z/a_n|}. \end{aligned}$$

For $z \in \mathbb{C} \smallsetminus B$ and $|z| < |a_n|$ we have $|a_n - z| \ge r_n = |a_n|^{-p-1}$, hence

$$\frac{1}{|1 - z/a_n|} \ \le \ |a_n|^{p+1} \ < \ |z|^{p+2},$$

so

$$\left|\frac{1}{a_n}\frac{E_p'}{E_p}(z/a_n)\right| \ \le \ \frac{|z|^{2p+2}}{|a_n|^{p+1}}$$

and thus for $z \in \mathbb{C} \smallsetminus U$,

$$\left|\sum_{n:|a_n|>|z|}\frac{1}{a_n}\frac{E_p'}{E_p}(z/a_n)\right| \ \le \ D|z|^{2p+2}.$$

Next we consider the case $|z| \geq |a_n|$ for $z \in \mathbb{C} \smallsetminus U$. Then $|a_n - z| \geq |a_n|^{-p-1}$ and so $\frac{1}{|a_n - z|} \leq |a_n|^{p+1}$ so that

$$\left| \frac{1}{a_n} \frac{E_p'}{E_p}(z/a_n) \right| \leq |a_n|^{p+1} + \frac{p|z/a_n|^p}{|a_n|}.$$

Summing over n we get

$$\begin{aligned} \sum_{n:|a_n|\leq|z|} \left| \frac{1}{a_n} \frac{E_p'}{E_p}(z/a_n) \right| &\leq \#\{n : |a_n| \leq |z|\} \, |z|^{p+1} + Dp|z|^p \\ &\leq C|z|^{2p+3} + D|z|^p. \end{aligned}$$

The lemma follows. $\square$

3.3.3. *A conjecture.* We consider test functions of the form $f = \frac{1}{\mathrm{vol}(K_\Gamma)} \mathbb{1}_{K_\Gamma} \otimes f_\infty$. Note that for a unitary representation η of $\mathcal{G}(\mathbb{A}_{\mathrm{fin}})$ one has $\eta\left(\frac{1}{\mathrm{vol}(K_\Gamma)} \mathbb{1}_{K_\Gamma}\right) = \mathrm{Pr}_\Gamma$, the orthogonal projection onto the space η^{K_Γ} of K_Γ-invariants.

Comparing the version of the trace formula used in this paper to the non-adelic trace formula in Theorem 4.2 of [**24**] one sees that for f of the above form, the expression

$$\mathrm{tr}\left(\mathcal{M}(\lambda) I_{\mathcal{O},\lambda}(f)\right)$$

equals

$$\mathrm{tr}\left(c(\lambda,\mathcal{O})^{-1} c'(\lambda,\mathcal{O}) \pi_{\Gamma,\lambda}(f_\infty)\right),$$

where we write $c(\lambda,\mathcal{O})$ for the scattering operator of [**24**] restricted to the space attached to the orbit $\mathcal{O}$ and $\pi_{\Gamma,\lambda}$ equals $I_{\mathcal{O},\lambda}$ restricted to the space of K_Γ-invariants and considered as a G-representation.

Under the compact group K one has the isotypical decomposition

$$\pi_{\Gamma,\lambda} = \bigoplus_{\sigma\in\hat{K}} \pi_{\Gamma,\lambda}^\sigma$$

which is preserved by $c(\lambda\mathcal{O})$ and $\pi_{\Gamma,\lambda}(f_\infty)$. For each $\sigma \in \hat{K}$ the space $\pi_{\Gamma,\lambda}^\sigma$ is finite dimensional. Let $c(\lambda,\sigma,\mathcal{O})$ denote the restriction of $c(\lambda,\mathcal{O})$ to the isotype $\pi_{\Gamma,\lambda}^\sigma$.

Conjecture 3.5. *The map $\lambda \mapsto c(\lambda,\sigma,\mathcal{O})^{-1} c'(\lambda,\sigma,\mathcal{O})$ is a meromorphic matrix valued function of essentially moderate growth. The growth exponent is $\leq 2(\dim(G/K)+2)+3$. The exceptional set U can be chosen independent of σ.*

We want to give the integral over $i\mathfrak{a}_\mathbb{R}^*$ in the continuous contribution of the trace formula a different shape. To this end recall that the kernel $k_{f,\lambda}$ is a Paley-Wiener function in the argument λ.

We will formulate a general remark on Paley-Wiener functions. For a natural number n let $C_c^n(\mathbb{R})$ denote the space of n-times continuously differentiable compactly supported functions on $\mathbb{R}$. By a *Paley-Wiener function of order n* we mean a function h which is the Fourier transform of some $g \in C_c^n(\mathbb{R})$. Since it better fits into our applications we will change coordinates from z to iz. So a Paley-Wiener function h will be of the form

$$h(z) = \int_{-\infty}^{\infty} g(t) e^{zt} dt$$

for some $g \in C_c^n(\mathbb{R})$.

PROPOSITION 3.6. *Let h be a Paley-Wiener function of order n and fix $a \in \mathbb{C}$. There is a unique decomposition*

$$h = h_a^{+,n} + h_a^{-,n}$$

such that the functions $h_a^{\pm n}$ are holomorphic in $\mathbb{C} - \{a\}$, both have at most a pole of order $< n$ at a. Further for some $C > 0$ the following estimates hold:

$$\begin{aligned} |h_a^{+,n}(z)| &\leq \frac{C}{|z-a|^n} \quad \text{for } \mathrm{Re}(z) \leq 0,\ z \neq a, \\ |h_a^{-,n}(z)| &\leq \frac{C}{|z-a|^n} \quad \text{for } \mathrm{Re}(z) \geq 0,\ z \neq a. \end{aligned}$$

Proof: Let us show uniqueness first. Suppose we are given two decompositions $h = h^+ + h^- = h_1^+ + h_1^-$ of the above type then $\tilde{h} = h^+ - h_1^+ = h_1^- - h^-$ satisfies $|\tilde{h}(z)| \leq \frac{2C}{|z-a|^n}$ for all $z \neq a$. Therefore the entire function $(z-a)^n \tilde{h}(z)$ is bounded, hence constant. But this function vanishes at a by the pole order condition, whence the claim.

For the existence assume

$$h(z) = \int_{-\infty}^{\infty} g(t) e^{zt} dt$$

for some $g \in C_c^n(\mathbb{R})$. Now define

$$h_a^{+,n}(z) := (\frac{1}{z-a})^n \int_0^{\infty} (g(t)e^{at})^{(n)} e^{(z-a)t} dt - \frac{c(g)}{(z-a)^n}$$

and

$$h_a^{-,n}(z) := (\frac{1}{z-a})^n \int_{-\infty}^{0} (g(t)e^{at})^{(n)} e^{(z-a)t} dt + \frac{c(g)}{(z-a)^n},$$

where $c(g) = \int_0^\infty (g(t)e^{at})^{(n)} dt$. Partial integration shows that $h = h_a^{+,n} + h_a^{-,n}$, the rest is clear. $\square$

Note that if g vanishes at $t = 0$ to order $j+1$ and $n \leq j$, then

$$h_a^{\pm,n} = h_a^{\pm,n-1} = \ldots = h_a^{\pm,1}$$

and this further equals

$$h^{\pm}(z) := \int_0^\infty g(\pm t) e^{\pm tz} dt.$$

In this case we say that h is *orthogonal to polynomials of degree* $\leq j$.

If $a = 0$, we will generally drop the index, so $h_0^{\pm,n} = h^{\pm,n}$.

Finally note that, by the formula given above, one sees that if g depends differentiably or holomorphically on some parameter then the same holds for $h^\pm$.

Fix some $n \leq j$, but still large and denote by $k_{f,\lambda,a}^{\pm,n}$ the kernels we get by applying this construction to $k_{f,\lambda}$ as a function in λ. Write $T_{f,\lambda,a}^{\pm,n}$ for the corresponding operator at infinity and $I_{\lambda,a}^{\pm,n}(f)$ for the global operator $Pr_\Gamma \otimes T_{f,\lambda,a}^{\pm,n}$. Suppose $a \in \mathfrak{a}^*$ has negative real part and does not coincide with a pole of $M(\lambda)^{-1}M'(\lambda)$. We get that

$$\frac{1}{4\pi i} \int_{i\mathfrak{a}_\mathbb{R}^*} \mathrm{tr} M(\lambda)^{-1} M'(\lambda) I_\lambda(f) d\lambda$$

equals

$$\begin{aligned} & \frac{1}{4\pi i}\int_{i\mathfrak{a}_{\mathbb{R}}^*} \mathrm{tr} M(\lambda)^{-1}M'(\lambda)I_{\lambda,a}^{+;n}(f)d\lambda \\ + \quad & \frac{1}{4\pi i}\int_{i\mathfrak{a}_{\mathbb{R}}^*} \mathrm{tr} M(\lambda)^{-1}M'(\lambda)I_{\lambda,a}^{-;n}(f)d\lambda. \end{aligned}$$

We move the integration paths to the left and the right resp. to get the residues plus a term which tends to zero according to the conjecture. The above becomes

$$\begin{aligned} & \frac{1}{2}\sum_{\mathrm{Re}\lambda<0} \mathrm{tr} R_\lambda I_{\lambda,a}^{+;n}(f) \\ + \quad & \frac{1}{2}\mathrm{res}_{\lambda=a}\mathrm{tr} M(\lambda)^{-1}M'(\lambda)I_{\lambda,a}^{+;n}(f) \\ - \quad & \frac{1}{2}\sum_{\mathrm{Re}\lambda>0} \mathrm{tr} R_\lambda I_{\lambda,a}^{-;n}(f), \end{aligned}$$

where $R_{\lambda_0} := \mathrm{res}_{\lambda=\lambda_0} M(\lambda)^{-1}M'(\lambda)$.

We say that a function $f \in C_c^j(G)$ is *orthogonal to polynomials of degree* $\leq j$ if the operator valued function $\lambda \mapsto \pi_{\xi,\lambda}(f)$ satisfies this condition for any $\xi \in \hat{M}$. In that case it immediately gives that the above equals

$$\frac{1}{2}\sum_{\mathrm{Re}\lambda<0} \mathrm{tr} R_\lambda I_\lambda^+(f) - \frac{1}{2}\sum_{\mathrm{Re}\lambda>0} \mathrm{tr} R_\lambda I_\lambda^-(f).$$

Furthermore f is called *positive* if $I_\lambda^-(f) = 0$. In that case we end up with the simple expression

$$\frac{1}{2}\sum_{\mathrm{Re}\lambda<0} \mathrm{tr} R_\lambda I_\lambda(f).$$

From here the proof of the Lefschetz formula should proceed in a similar way to the compact case [**13**]. There is, however, a difference and a further difficulty in that the test functions chosen in [**13**] are not orthogonal to polynomials. They can, however, be chosen to be approximately orthogonal to polynomials, meaning that one can let them run through a sequence, giving the same geometric contribution, such that the polar contributions above vanish in the limit. This is done by shrinking the support of these functions so that in the limit it shrinks to a subset of $K_M A^-$. It remains to be shown that the necessary interchange of integral and limit is justified, but I believe this can be done. The major problem, in my view, of this approach lies in Conjecture 3.5, which I have at the moment no idea how to prove in general.

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