Holomorphic torsion and closed geodesics

Anton Deitmar

Contents

Introduction

In his seminal paper [12], David Fried expressed the analytic torsion of a hyperbolic manifold as a special value of a zeta function. Later in [13] he did the same for the holomorphic torsion of a complex hyperbolic manifold. In the paper [26], based on [25], Moscovici and Stanton generalized Fried's work on analytic torsion to locally symmetric spaces of fundamental rank one.

In the current paper we extend Fried's result on holomorphic torsion to arbitrary hermitian locally symmetric spaces. In the papers [12, 13], the torsion numbers are described in the form $R(0)c$, where R is a geometrically defined zeta function and c only depends on the volume and the universal cover of the space. In this paper we interpret c as the reciprocal of the L^2-torsion. The main result of this paper can also be viewed as a geometric analogue of Lichtenbaum's conjecture [22]. See also [10].

Fried's first paper on analytic torsion [12], which only treated hyperbolic spaces, has been generalized to spaces of fundamental rank one by Moscovici and Stanton in [26] and by the author to all spaces of positive fundamental rank [8]. It then became clear, that the analogous treatment of holomorphic torsion requires a different approach. The basic problem is to find suitable test functions for the trace formula to read off the analytic continuation of Selberg-type zeta functions. To explain this, we fix a semisimple Lie group G together with a cocompact lattice $\Gamma \subset G$. By integration of the right regular representation, a function $f \in C_c^\infty(G)$ induces an operator $R(F)$ on the Hilbert space $L^2(\Gamma\backslash G)$. Having a smooth kernel, the operator $R(f)$ is trace class and its trace equals the integral over the diagonal of the kernel. Computing this integral, one gets an identity, known as the trace formula, expressing the (spectral) trace $\mathrm{tr} R(f)$ in terms of a (geometric) sum of orbital integrals. For suitable test functions f, the geometric side gives Selberg-type zeta functions and the comparison with the spectral side yields analytic continuation of the zeta function. To find such test functions, one can either look for test functions one knows the spectral side of, or for ones with given geometric trace. The first, "spectral" approach is the classical one. Usually one takes functions which are given by the functional calculus of a nice invariant operator like a Laplace operator. Fried and Moscovici/Stanton used heat kernels to this end. For higher rank groups, however, this approach cannot alway distinguish geometric contributions from different Cartan subgroups, which is necessary to achieve analytic continuation. At this point, Juhl's habilitation thesis [18] gave a new idea. It is the first treatment systematically to employ the second, geometric approach. It is detailed in the book [19]. This technique only works in rank one situations, but it inspired the author to *combine* the two approaches in the following way. First, the geometric technique of Juhl is used to reduce the rank by one, and then a spectral construction is used to give a kernel on a Levi-subgroup which eliminates all unwanted contributions. The result, which is given in [9], is a Lefschetz formula isolating the geometric contributions of a single Cartan subgroup.

This result is more general then what we need in this paper, as it is valid for all semisimple Lie groups. We will specialize it here to groups attached to hermitian symmetric spaces which requires a detailed knowledge of the latter.

1 Notation

This paper depends heavily on [9]. We take over all notation from that paper. For instance, G will denote a connected semisimple Lie group with finite center, and K will be a maximal compact subgroup. The attached locally symmetric space is denoted by $X = G/K$. For every discrete, torsion-free subgroup $\Gamma \subset G$ the space $\Gamma\backslash X$ is a locally symmetric space. We will only be interested in the compact case, so $\Gamma\backslash X$, or equivalently, $\Gamma\backslash G$ will be compact. In this case Γ is said to be a *uniform lattice* in G.

Let L be a Lie group. An element x of L is called *neat* if for every finite dimensional representation η of L the linear map $\eta(x)$ has no nontrivial root of unity as an eigenvalue. A subset A of L is called neat if each of its members are. Every neat subgroup is torsion free modulo the center of L. Every arithmetic group has a subgroup of finite index which is neat [4].

Lemma 1.1 *Let $x \in L$ be semisimple and neat. Let L_x denote the centralizer of x in L. Then for each $k \in \mathbb{N}$ the connected components of L_x and L_{x^k} coincide.*

Proof: It suffices to show that the Lie algebras coincide. The Lie algebra of L_x is just the fixed space of $\mathrm{Ad}(x)$ in $\mathrm{Lie}(L)$. Since $\mathrm{Ad}(x)$ is semisimple and does not have a root of unity for an eigenvalue this fixed space coincides with the fixed space of $\mathrm{Ad}(x^k)$. The claim follows. □

Let $\Gamma \subset G$ denote a cocompact discrete subgroup which is neat. Since Γ is torsion free it acts fixed point free on the contractible space X and hence Γ is the fundamental group of the Riemannian manifold

$$X_\Gamma = \Gamma \backslash X = \Gamma \backslash G/K$$

it follows that we have a canonical bijection of the homotopy classes of loops:

$$[S^1 : X_\Gamma] \to \Gamma/\text{conjugacy}.$$

For a given class $[\gamma]$ let X_γ denote the union of all closed geodesics in the corresponding class in $[S^1 : X_\Gamma]$. Then X_γ is a smooth submanifold of X_{Γ_H} [11], indeed, it follows that

$$X_\gamma \ \cong \ \Gamma_\gamma \backslash G_\gamma / K_\gamma,$$

where G_γ and Γ_γ are the centralizers of γ in G and Γ and K_γ is a maximal compact subgroup of G_γ. Further all closed geodesics in the class $[\gamma]$ have the same length l_γ.

Lemma 1.2 *For $\gamma \in \Gamma$ and $n \in \mathbb{N}$ we have $X_{\gamma^n} = X_\gamma$.*

Proof: By the last lemma we have for the connected components, $G_\gamma^0 = G_{\gamma^n}^0$. By definition one has that X_{γ^n} is a subset of X_γ. Since both are connected submanifolds of X_Γ they are equal if their dimensions are the same. We have

$$\begin{aligned} \dim X_\gamma &= \dim G_\gamma/K_\gamma \\ &= \dim G_\gamma^0/K_\gamma^0 \\ &= \dim G_{\gamma^n}^0/K_{\gamma^n}^0 \\ &= \dim G_{\gamma^n}/K_{\gamma^n} \\ &= \dim X_{\gamma^n}. \end{aligned}$$

□

2 Meromorphic continuation

Recall that G is a connected semisimple Lie group with finite center. Fix a maximal compact subgroup K with Cartan involution θ.

2.1 The zeta function

Fix a θ-stable Cartan subgroup H of split rank 1. Note that such a H doesn't always exist. It exists only if the absolute ranks of G and K satisfy the relation:

$$\text{rank } G - \text{rank } K \leq 1.$$

This certainly holds if G has a compact Cartan subgroup or if the real rank of G is one. In the case rank $G =$ rank K, i.e., if G has a compact Cartan there will in general be several G-conjugacy classes of split rank one Cartan subgroups. In the case rank $G -$ rank $K = 1$, however, there is only one. The number $FR(G) :=$ rank $G -$ rank K is called the *fundamental rank* of G.

Write $H = AB$ where A is the connected split component and $B \subset K$ is compact. Choose a parabolic subgroup P with Langlands decomposition $P = MAN$. Then $K_M = K \cap M$ is a maximal compact subgroup of M. Let $\bar{P} = MA\bar{N}$ be the opposite parabolic. As in the last section let $A^- = \exp(\mathfrak{a}_0^-) \subset A$ be the negative Weyl chamber of all $a \in A$ which act contractingly on $\mathfrak{n}$. Fix a finite dimensional representation (τ, V_τ) of K_M.

Let $H_1 \in \mathfrak{a}_0^-$ be the unique element with $B(H_1) = 1$.

Let $\Gamma \subset G$ be a cocompact discrete subgroup which is neat. An element $\gamma \neq 1$ will be called *primitive* if $\tau \in \Gamma$ and $\tau^n = \gamma$ with $n \in \mathbb{N}$ implies $n = 1$. Every $\gamma \neq 1$ is a power of a unique primitive element. Obviously primitivity is a property of conjugacy classes. Let $\mathcal{E}_P^p(\Gamma)$ denote the subset of $\mathcal{E}_P(\Gamma)$ consisting of all primitive classes. Recall the length l_γ of any geodesic in the class $[\gamma]$. If γ is conjugate to $am \in A^- M_{ell}$ then $l_\gamma = l_a$, where $l_a = |\log a| = \sqrt{B(\log a)}$. Let A_γ be the connected split component of G_γ, then A_γ is conjugate to A.

We say that M is *orientation preserving*, if M acts by orientation preserving maps on the manifold M/K_M, where $K_M = K \cap M$. For a complex vector space V, on which A acts linearly, and $\lambda \in \mathfrak{a}^*$, we write V^λ for the generalized λ-Eigenspace, i.e.,

$$V^\lambda = \{v \in V : (a-\lambda)^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

In this section we are going to prove the following

Theorem 2.1 *Suppose that τ is in the image of the restriction map $res^M_{K_M}$ or that M is orientation preserving. Let Γ be neat and (ω, V_ω) a finite dimensional unitary representation of Γ. For $\mathrm{Re}(s) >> 0$ define the generalized Selberg zeta function:*

$$Z_{P,\tau,\omega}(s) = \prod_{[\gamma]\in\mathcal{E}_P^p(\Gamma)} \prod_{N\geq 0} \det\left(1 - e^{-sl_\gamma}\gamma \,\middle|\, V_\omega \otimes V_\tau \otimes S^N(\mathfrak{n})\right)^{\chi(A_\gamma\backslash X_\gamma)},$$

where $S^N(\mathfrak{n})$ denotes the N-th symmetric power of the space $\mathfrak{n}$ and γ acts on $V_\omega \otimes V_\tau \otimes S^N(\mathfrak{n})$ via $\omega(\gamma) \otimes \tau(b_\gamma) \otimes Ad^N(a_\gamma b_\gamma)$, here $\gamma \in \Gamma$ is conjugate to $a_\gamma b_\gamma \in A^- B$. Finally, $\chi(A_\gamma\backslash X_\gamma)$ is the Euler-characteristic of $A_\gamma\backslash X_\gamma$.

Then $Z_{P,\tau,\omega}$ has a meromorphic continuation to the entire plane. The vanishing order of $Z_{P,\tau,\omega}(s)$ at a point $s = \lambda(H_1)$, $\lambda \in \mathfrak{a}^*$, is

$$m(s) = (-1)^{\dim N} \sum_{\pi\in\hat{G}} N_{\Gamma,\omega}(\pi) \sum_{p,q} (-1)^{p+q} \dim\left(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_M \otimes V_{\breve{\tau}} \right)^{K_M}.$$

Further, all poles and zeroes of the function $Z_{P,\tau,\omega}(s + |\rho_0|)$ lie in $\mathbb{R} \cup i\mathbb{R}$.

Proof: We apply the Lefschetz Theorem (Thm 4.2.1) of [9] with the test function $\varphi(a) = l_a^{j+1} e^{-sl_a}$ for $j \in \mathbb{N}$ and $s \in \mathbb{C}$ with $j, \mathrm{Re}(s)$ large enough. The geometric side of the Lefschetz formula is $\left(\frac{\partial}{\partial s}\right)^j \frac{Z'}{Z}(s_)$, and by the Lefschetz formula this equals $\left(\frac{\partial}{\partial s}\right)^j \sum_{s_0\in\mathbb{C}} \frac{m(s_0)}{s-s_0}$. The theorem follows. □

2.2 The functional equation

If the Weyl group $W(G, A)$ is nontrivial, then is has order two. To give the reader a feeling of this condition consider the case $G = SL_3(\mathbb{R})$. In that case the Weyl group $W(G, A)$ is trivial. On the other hand, consider the case when the fundamental rank of G is 0; this is the most interesting case to us since only then we have several conjugacy classes of splitrank-one Cartan subgroups. Here the *splitrank* of a Cartan subgroup $H = AB$ is the dimension of A, where B is compact and A is a split torus. In that case it follows that the dimension of all irreducible factors of the symmetric space $X = G/K$ is even, hence the point-reflection at the point eK is in the connected component of the group of isometries of X. This reflection can be thought of as an element of K which induces a nontrivial element of the Weyl group $W(G, A)$. So we see that in this important case we have $|W(G, A)| = 2$.

Let w be the nontrivial element. It has a representative in K which we also denote by w. Then $wK_Mw^{-1} = K_M$ and we let τ^w be the representation given by $\tau^w(k) = \tau(wkw^{-1})$. It is clear from the definitions that

$$Z_{P,\tau,\omega} = Z_{\bar{P},\tau^w,\omega}.$$

We will show a functional equation for $Z_{P,\tau,\omega}$. This needs some preparation. Assume G admits a compact Cartan $T \subset K$, then a representation $\pi \in \hat{G}$ is called *elliptic* if Θ_π is nonzero on the compact Cartan. Let $\hat{G}_{ell}$ be the set of elliptic elements in $\hat{G}$ and denote by $\hat{G}_{ds}$ the subset of discrete series representations. Further let $\hat{G}_{lds}$ denote the set of all discrete series and all limits of discrete series representations. In Theorem 2.1 we have shown that the vanishing order of $Z_{P,\tau,\omega}(s)$ at the point $s = \mu(H_1)$, $\mu \in \mathfrak{a}^*$ is

$$(-1)^{\dim N} \sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) m(\pi, \tau, \mu),$$

where

$$m(\pi, \tau, \mu) = \sum_{p,q} (-1)^{p+q} \dim \left(H^q(\mathfrak{n}, \pi_K)(\mu) \otimes \wedge^p \mathfrak{p}_M \otimes V_{\breve{\tau}} \right)^{K_M}.$$

A *standard representation* is a representation parabolically induced from a tempered representation, see [20], p.383. Any character Θ_π for $\pi \in \hat{G}$ is an integer linear combination of characters of standard representations (loc. cit.). From this it follows that for $\pi \in \hat{G}$ the character restricted to the compact Cartan T is

$$\Theta_\pi |_T = \sum_{\pi' \in \hat{G}_{lds}} k_{\pi,\pi'} \Theta_{\pi'} |_T,$$

with integer coefficients $k_{\pi,\pi'}$.

Lemma 2.2 *There is a* $C > 0$ *such that for* $\mathrm{Re}(\mu(H_1)) < -C$ *the order of* $Z_{P,\tau,\omega}(s)$ *at* $s = \mu(H_1)$ *is*

$$(-1)^{\dim N} \sum_{\pi \in \hat{G}_{ell}} N_{\Gamma,\omega}(\pi) \sum_{\pi' \in \hat{G}_{lds}} k_{\pi,\pi'} m(\pi', \tau, \mu).$$

Proof: For any $\pi \in \hat{G}$ we know that if $\Theta_\pi|_{AB} \neq 0$ then in the representation of Θ_π as linear combination of standard characters there must occur lds-characters and characters of representations $\pi_{\xi,\nu}$ induced from P. Since $\Theta_{\pi_{\xi,\nu}} = \Theta_{\pi_{w\xi,-\nu}}$ any contribution of $\pi_{\xi,\nu}$ for $\mathrm{Re}(s) << 0$ would also give a pole or zero of $Z_{P,\tau^w,\omega}$ for $\mathrm{Re}(s) >> 0$. In the latter region we do have an Euler product, hence there are no poles or zeroes. □

Now consider the case $FR(G) = 1$, so there is no compact Cartan, hence no discrete series.

Theorem 2.3 *Assume that the fundamental rank of* G *is* 1*, then there is a polynomial* P *of degree* $\leq \dim G + \dim N$ *such that*

$$Z_{P,\tau,\omega}(s) = e^{P(s)} Z_{P,\tau^w,\omega}(2|\rho_0| - s).$$

Proof: By Theorem 2.1 the functions $Z_{P,\tau,\omega}(s)$ and $Z_{P,\tau^w,\omega}(2|\rho_0| - s)$ have the same zeros and poles. Further, they are both of finite genus, hence the claim. □

Now assume $FR(G) = 0$ so there is a compact Cartan subgroup T. As Haar measure on G we take the Euler-Poincaré measure. The sum in the lemma can be rearranged to

$$(-1)^{\dim N} \sum_{\pi' \in \hat{G}_{lds}} m(\pi', \tau, \mu) \sum_{\pi \in \hat{G}_{ell}} N_{\Gamma,\omega}(\pi) k_{\pi,\pi'}.$$

We want to show that the summands with π' in the limit of the discrete series add up to zero. For this suppose π' and π'' are distinct and belong to the limit of the discrete series. Assume further that their Harish-Chandra parameters agree. By the Paley-Wiener theorem [7] there is a smooth compactly supported function $f_{\pi',\pi''}$ such that for any tempered $\pi \in \hat{G}$:

$$\mathrm{tr}\pi(f_{\pi',\pi''}) = \begin{cases} 1 & \text{if } \pi = \pi' \\ -1 & \text{if } \pi = \pi'' \\ 0 & \text{else.} \end{cases}$$

Plugging $f_{\pi',\pi''}$ into the trace formula one gets

$$\sum_{\pi \in \hat{G}_{ell}} N_{\Gamma,\omega}(\pi) k_{\pi,\pi'} = \sum_{\pi \in \hat{G}_{ell}} N_{\Gamma,\omega}(\pi) k_{\pi,\pi''},$$

so that in the above sum the summands to π' and π'' occur with the same coefficient. Let π_0 be the induced representation whose character is the sum of the characters of the π'', where π'' varies over all lds-representations with the same Harish-Chandra parameter as π'. Then for $\mathrm{Re}(\mu(H_1)) < -C$ we have $m(\pi_0, \tau, \mu) = 0$. Thus it follows that the contribution of the limit series vanishes.

Plugging the pseudo-coefficients [21] of the discrete series representations into the trace formula gives for $\pi \in \hat{G}_{ds}$:

$$\sum_{\pi' \in \hat{G}_{ell}} k_{\pi',\pi} N_{\Gamma,\omega}(\pi') = \dim \omega (-1)^{\frac{\dim X}{2}} \chi(X_\Gamma) d_\pi,$$

where d_π is the formal degree of π.

The infinitesimal character λ of π can be viewed as an element of the coset space $(\mathfrak{t}^*)^{reg}/W_K$. So let J denote the finite set of connected components of $(\mathfrak{t}^*)^{reg}/W_K$, then we get a decomposition $\hat{G}_{ds} = \coprod_{j \in J} \hat{G}_{ds,j}$.

In the proof of Theorem 2.1 we used the Hecht-Schmid character formula to deal with the global characters. On the other hand it is known that global characters are given on the regular set by sums of toric characters over the Weyl denominator. So on $H = AB$ the character Θ_π for $\pi \in \hat{G}$ is of the form $\mathcal{N}/D$, where D is the Weyl denominator and the numerator $\mathcal{N}$ is of the form

$$\mathcal{N}(h) = \sum_{w \in W(\mathfrak{t},\mathfrak{g})} c_w h^{w\lambda},$$

where $\lambda \in \mathfrak{h}^*$ is the infinitesimal character of π. Accordingly, the expression $m(\pi, \tau, \mu)$ expands as a sum

$$m(\pi, \tau, \mu) = \sum_{w \in W(\mathfrak{t}, \mathfrak{g})} m_w(\pi, \tau, \mu).$$

Lemma 2.4 *Let $\pi, \pi' \in \hat{G}_{ds,j}$ with infinitesimal characters λ, λ' which we now also view as elements of $(\mathfrak{h}^*)^+$, then*

$$m_w(\pi, \tau, \mu) = m_w(\pi', \tau, \mu + w(\lambda' - \lambda)|_{\mathfrak{a}}).$$

Proof: In light of the preceding it suffices to show the following: Let $\tau_\lambda, \tau_{\lambda'}$ denote the numerators of the global characters of π and π' on $\mathfrak{h}^+$. Write

$$\tau_\lambda(h) = \sum_{w \in W(\mathfrak{t}, \mathfrak{g})} c_w h^{w\lambda}$$

for some constants c_w. Then we have

$$\tau_{\lambda'}(h) = \sum_{w \in W(\mathfrak{t}, \mathfrak{g})} c_w h^{w\lambda'}.$$

To see this, choose a λ'' dominating both λ and λ', then apply the Zuckerman functors $\varphi_{\lambda''}^{\lambda}$ and $\varphi_{\lambda''}^{\lambda'}$. Proposition 10.44 of [20] gives the claim. □

Write π_λ for the discrete series representation with infinitesimal character λ. Let $d(\lambda) := d_{\pi_\lambda}$ be the formal degree then $d(\lambda)$ is a polynomial in λ, more precisely from [1] we take

$$d(\lambda) = \prod_{\alpha \in \Phi^+(\mathfrak{t}, \mathfrak{g})} \frac{(\alpha, \lambda + \rho)}{(\alpha, \rho)},$$

where the ordering Φ^+ is chosen to make λ positive.

Putting things together we see that for $\mathrm{Re}(s)$ small enough the order of $Z_{P,\tau,\omega}(s)$ at $s = \mu(H_1)$ is

$$\begin{aligned} \mathcal{O}(\mu) \quad &= \quad \dim \omega (-1)^{\frac{\dim X}{2}} \chi(X_\Gamma) \\ & \quad \times \sum_{j \in J} \sum_{w \in W(\mathfrak{t}, \mathfrak{g})} \sum_{\pi \in \hat{G}_{ds,j}} d(\lambda_\pi) m_w(\pi_j, \tau, \mu + w(\lambda_j - \lambda_\pi)|_{\mathfrak{a}}), \end{aligned}$$

where $\pi_j \in \hat{G}_{ds,j}$ is a fixed element. The function $\mu \mapsto m_w(\pi_j, \tau, \mu)$ takes nonzero values only for finitely many μ. Since further $\lambda \mapsto d(\lambda)$ is a polynomial it follows that the regularized product

$$D_{P,\tau,\omega}(s) := \widehat{\prod_{\mu,\mathcal{O}(\mu)\neq 0}} (s - \mu(H_1))^{\mathcal{O}(\mu)}$$

exists. We now have proven the following theorem.

Theorem 2.5 *With*

$$\hat{Z}_{P,\tau,\omega}(s) := Z_{P,\tau,\omega}(s) D_{H,\tau,\omega}(s)^{-1}$$

we have

$$\hat{Z}_{P,\tau,\omega}(2|\rho_0| - s) = e^{Q(s)} \hat{Z}_{P,\tau,\omega}(s),$$

where Q is a polynomial. □

Proposition 2.6 *Let τ be a finite dimensional representation of M then the order of $Z_{P,\tau,\omega}(s)$ at $s = \lambda(H_1)$ is*

$$(-1)^{\dim(N)} \sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(^\theta\pi) \sum_{q=0}^{\dim(\mathfrak{m}\oplus\mathfrak{n}/\mathfrak{k}_M)} (-1)^q \dim(H^q(\mathfrak{m}\oplus\mathfrak{n}, K_M, \pi_K \otimes V_{\breve{\tau}})^\lambda).$$

This can also be expressed as

$$(-1)^{\dim(N)} \sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(^\theta\pi) \sum_{q=0}^{\dim(\mathfrak{m}\oplus\mathfrak{n}/\mathfrak{k}_M)} (-1)^q \dim(\mathrm{Ext}^q_{(\mathfrak{m}\oplus\mathfrak{n},K_M)}(V_\tau, V_\pi)^\lambda).$$

Proof: Extend V_τ to a $\mathfrak{m}\oplus\mathfrak{n}$-module by letting $\mathfrak{n}$ act trivially. We then get

$$H^p(\mathfrak{n}, \pi_K) \otimes V_\tau \cong H^p(\mathfrak{n}, \pi_K \otimes V_{\breve{\tau}}).$$

The $(\mathfrak{m}, K_M)$-cohomology of the module $H^p(\mathfrak{n}, \pi_K \otimes V_\tau)$ is the cohomology of the complex (C^*) with

$$\begin{aligned} C^q &= \mathrm{Hom}_{K_M}(\wedge^q \mathfrak{p}_M, H^p(\mathfrak{n}, \pi_K) \otimes V_{\breve{\tau}}) \\ &= (\wedge^q \mathfrak{p}_M \otimes H^p(\mathfrak{n}, \pi_K) \otimes V_{\breve{\tau}})^{K_M}, \end{aligned}$$

since $\wedge^p\mathfrak{p}_M$ is a self-dual K_M-module. Therefore we have an isomorphism of virtual A-modules:

$$\sum_q(-1)^q(H^p(\mathfrak{n},\pi_K)\otimes\wedge^q\mathfrak{p}_M\otimes V_{\breve{\tau}})^{K_M}\cong\sum_q(-1)^qH^q(\mathfrak{m},K_M,H^p(\mathfrak{n},\pi_K\otimes V_{\breve{\tau}})).$$

Now one considers the Hochschild-Serre spectral sequence in the relative case for the exact sequence of Lie algebras

$$0\to\mathfrak{n}\to\mathfrak{m}\oplus\mathfrak{n}\to\mathfrak{m}\to 0$$

and the $(\mathfrak{m}\oplus\mathfrak{n},K_M)$-module $\pi\otimes V_{\breve{\tau}}$. We have

$$E_2^{p,q}=H^q(\mathfrak{m},K_M,H^p(\mathfrak{n},\pi_K\otimes V_{\breve{\tau}}))$$

and

$$E_\infty^{p,q}=\mathrm{Gr}^q(H^{p+q}(\mathfrak{m}\oplus\mathfrak{n},K_M,\pi_K\otimes V_{\breve{\tau}})).$$

Now the module in question is just

$$\chi(E_2)=\sum_{p,q}(-1)^{p+q}E_2^{p,q}.$$

Since the differentials in the spectral sequence are A-homomorphisms this equals $\chi(E_\infty)$. So we get an A-module isomorphism of virtual A-modules

$$\sum_{p,q}(-1)^{p+q}(H^p(\mathfrak{n},\pi_K)\otimes\wedge^q\mathfrak{p}_M\otimes V_{\breve{\tau}})^{K_M}\cong\sum_j(-1)^jH^j(\mathfrak{m}\oplus\mathfrak{n},K_M,\pi_K\otimes V_{\breve{\tau}}).$$

The second statement is clear by [5] p.16. □

2.3 The Ruelle zeta function

The generalized Ruelle zeta function can be described in terms of the Selberg zeta function as follows.

Theorem 2.7 *Let* Γ *be neat and choose a parabolic* P *of splitrank one. For* $\mathrm{Re}(s)>>0$ *define the zeta function*

$$Z_{P,\omega}^R(s)=\prod_{[\gamma]\in\mathcal{E}_H^p(\Gamma)}\det\left(\ 1-e^{-sl_\gamma}\omega(\gamma)\ \right)^{\chi_1(X_\gamma)},$$

then $Z^R_{P,\omega}(s)$ extends to a meromorphic function on $\mathbb{C}$. More precisely, let $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$ be the root space decomposition of $\mathfrak{n}$ with respect to the roots of $(\mathfrak{a}, \mathfrak{g})$ then

$$Z^R_{P,\omega}(s) = \prod_{q=0}^{\dim \mathfrak{n}_\alpha} \prod_{p=0}^{\dim \mathfrak{n}_{2\alpha}} Z_{P,(\wedge^q \mathfrak{n}_\alpha)\otimes(\wedge^p \mathfrak{n}_{2\alpha}),\omega}(s + (q+2p)|\alpha|)^{(-1)^{p+q}}.$$

In the case when $\text{rank}_{\mathbb{R}} G = 1$ this zeta function coincides with the Ruelle zeta function of the geodesic flow of X_Γ.

Proof: For any finite dimensional virtual representation τ of M we compute

$$\log Z_{P,\tau,\omega}(s) = \sum_{[\gamma]\in\mathcal{E}_P^p(\Gamma)} \chi_{_1}(X_\gamma) \sum_{N\geq 0} \text{tr}(\log(1 - e^{-sl_\gamma}\gamma)|\omega \otimes \tau \otimes S^N(\mathfrak{n}))$$

$$\begin{aligned}
&= \sum_{[\gamma]\in\mathcal{E}_P^p(\Gamma)} \chi_{_1}(X_\gamma) \sum_{N\geq 0}\sum_{n\geq 1} \frac{1}{n} e^{-sl_\gamma n}\text{tr}(\gamma|\omega \otimes \tau \otimes S^N(\mathfrak{n})) \\
&= \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \chi_{_1}(X_\gamma) \frac{e^{-sl_\gamma}}{\mu(\gamma)} \text{tr}\omega(\gamma) \frac{\text{tr}\tau(b_\gamma)}{\det(1-(a_\gamma b_\gamma)^{-1}|\mathfrak{n})} \\
&= \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \chi_{_1}(X_\gamma) \frac{e^{-sl_\gamma}}{\mu(\gamma)} \text{tr}\omega(\gamma) \frac{\text{tr}\tau(b_\gamma)}{\text{tr}((a_\gamma b_\gamma)^{-1}| \wedge^* \mathfrak{n})}.
\end{aligned}$$

Since $\mathfrak{n}$ is an M-module defined over the reals we conclude that the trace $\text{tr}((a_\gamma m_\gamma)^{-1}|\wedge^*\mathfrak{n})$ is a real number. Therefore it equals its complex conjugate which is $\text{tr}(a_\gamma^{-1} m_\gamma| \wedge^* \mathfrak{n})$. Now split into the contributions from $\mathfrak{n}_\alpha$ and $\mathfrak{n}_{2\alpha}$. The claim now becomes clear. □

3 The Patterson conjecture

The content of this section is not strictly needed in the rest of the paper, but it is a result of independent interest.

In this section G continues to be a connected semisimple Lie group with finite center. Fix a parabolic $P = MAN$ of splitrank one, ie, $\dim A = 1$. Let $\nu \in \mathfrak{a}^*$ and let (σ, V_σ) be a finite dimensional complex representation of M.

Consider the principal series representation $\pi_{\sigma,\nu}$ on the Hilbert space $\pi_{\sigma,\nu}$ of all functions f from G to V_σ such that $f(xman) = a^{-(\nu+\rho)}\sigma(m)^{-1}f(x)$ and such that the restriction of f to K is an L^2-function. Let $\pi_{\sigma,\nu}^\infty$ denote the Fréchet space of smooth vectors and $\pi_{\sigma,\nu}^{-\infty}$ for its continuous dual. For $\nu \in \mathfrak{a}^*$ let $\bar{\nu}$ denote its complex conjugate with respect to the real form $\mathfrak{a}_0^*$.

We will now formulate the Patterson conjecture [6].

Theorem 3.1 *If the order of the Weyl group $W(G,A)$ equals 2, then the cohomology group $H^p(\Gamma, \pi_{\sigma,\nu}^{-\infty} \otimes V_\omega)$ is finite dimensional for every $p \geq 0$ and the vanishing order of the Selberg zeta function $Z_{P,\tau,\omega}(s + |\rho_0|)$ at $s = \nu(H_1)$ equals*

$$\mathrm{ord}_{s=\nu(H_1)} Z_{P,\tau,\omega}(s + |\rho_0|) = \chi_1(\Gamma, \pi_{\breve{\sigma},-\nu}^{-\infty} \otimes V_\omega)$$

if $\nu \neq 0$ and

$$\mathrm{ord}_{s=0} Z_{P,\tau,\omega}(s + |\rho_0|) = \chi_1(\Gamma, \hat{H}_{\breve{\sigma},0}^{-\infty} \otimes V_\omega),$$

where $\hat{H}_{\sigma,0}^{-\infty}$ is a certain nontrivial extension of $\pi_{\sigma,0}^{-\infty}$ with itself.

Further $\chi(\Gamma, \pi_{\sigma,\nu}^{-\infty} \otimes V_{\breve{\omega}})$ vanishes.

Proof: Let $\mathfrak{a}_0$ be the real Lie algebra of A and $\mathfrak{a}_0^-$ the negative Weyl chamber with $\exp(\mathfrak{a}_0^-) = A^-$. Let $H_1 \in \mathfrak{a}_0^-$ be the unique element of norm 1. Recall that the vanishing order equals

$$\sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) \sum_{q=0}^{\dim N} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^{p+q} \dim(H_q(\mathfrak{n}, \pi_K)^{\nu-\rho_0} \otimes \wedge^p \mathfrak{p}_M \otimes V_{\breve{\sigma}})^{K_M}.$$

Set $\lambda = \nu - \rho_0$.

Lemma 3.2 *If $\lambda \neq -\rho_0$ then $\mathfrak{a}$ acts semisimply on $H_q(\mathfrak{n}, \pi_K)^\lambda$. The space $H_\rho(\mathfrak{n}, \pi_K)^{-\rho_0}$ is annihilated by $(H + \rho_0(H))^2$ for any $H \in \mathfrak{a}$.*

Since $W(G,A)$ is nontrivial, the proof in [6] Prop. 4.1. carries over to our situation. $\square$

To prove the theorem, assume first $\lambda \neq -\rho_0$. Let $\pi \in \hat{G}$, then

$$\sum_{q=0}^{\dim N} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^{p+q} \dim(H_q(\mathfrak{n}, \pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_M \otimes V_{\breve{\sigma}})^{K_M}$$

$$= \sum_{q=0}^{\dim N} \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^{p+q} \dim(H^0(\mathfrak{a}, H^p(\mathfrak{m}, K_M, H_q(\mathfrak{n}, \pi_K) \otimes V_{\breve{\sigma},-\lambda}))),$$

where $V_{\breve{\sigma},-\lambda}$ is the representation space of the representation $\breve{\sigma} \otimes (-\lambda)$. We want to show that this equals

$$\sum_{q=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{p+q+r} p \dim H^p(\mathfrak{a} \oplus \mathfrak{m}, K_M, H_q(\mathfrak{n}, \pi_K) \otimes V_{\breve{\sigma},-\lambda}).$$

Since the Hochschild-Serre spectral sequence degenerates for a one dimensional Lie algebra we get that

$$\dim H^p(\mathfrak{a} \oplus \mathfrak{m}, K_M, V)$$

equals

$$\dim H^0(\mathfrak{a}, H^{p-1}(\mathfrak{m}, K_M, V)) + \dim H^0(\mathfrak{a}, H^p(\mathfrak{m}, K_M, V)).$$

This implies

$$\sum_{p,q \geq 0} (-1)^{p+q+r} p \sum_{b=p-1}^{p} \dim H^0(\mathfrak{a}, H^b(\mathfrak{m}, K_M, H_q(\mathfrak{n}, \pi_K) \otimes V_{\breve{\sigma},-\lambda}))$$

$$= \sum_{b,q \geq 0} (-1)^q \sum_{p=b}^{b+1} p \; \dim H^0(\mathfrak{a}, H^b(\mathfrak{m}, K_M, H_q(\mathfrak{n}, \pi_K) \otimes V_{\breve{\sigma},-\lambda})).$$

It follows that the vanishing order equals

$$\sum_{q=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{p+q+1} p \dim H^p(\mathfrak{a} \oplus \mathfrak{m}, K_M, H_q(\mathfrak{n}, \pi_K) \otimes V_{\breve{\sigma},-\lambda}).$$

$$= \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{p+q+1} \dim \mathrm{Ext}^p_{\mathfrak{a} \oplus \mathfrak{m}, K_M}(H_q(\mathfrak{n}, \breve{\pi_K}), V_{\breve{\sigma},-\lambda}).$$

For a $(\mathfrak{g}, K)$-module V and a $(\mathfrak{a} \oplus \mathfrak{m}, M)$-module U we have [17]:

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_P^G(U)) \;\cong\; \mathrm{Hom}_{\mathfrak{a} \oplus \mathfrak{m}, K_M}(H_0(\mathfrak{n}, V), U \otimes \mathbb{C}_{\rho_0}),$$

where $\mathbb{C}_{\rho_0}$ is the one dimensional A-module given by ρ_0. Thus

$$\sum_{\pi\in\hat{G}} N_{\Gamma,\omega}(\pi) \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q+1} \dim \mathrm{Ext}^p_{\mathfrak{a}\oplus\mathfrak{m},K_M}(H_q(\mathfrak{n},\breve{\pi_K}),\pi^{\infty}_{\breve{\sigma},-\lambda+\rho_P,K})$$

$$= \sum_{p=0}^{\infty}(-1)^{p+1} \dim \mathrm{Ext}^p_{\mathfrak{g},K}(C^\infty(\Gamma\backslash G,\breve{\omega})_K,\pi^{\infty}_{\breve{\sigma},-\lambda,K})$$

Diualizing shows that this equals

$$\sum_{p\geq 0}(-1)^{p+1} \dim \mathrm{Ext}^p_{\mathfrak{g},K}(\pi^{\infty}_{\sigma,\lambda,K},C^\infty(\Gamma\backslash G,\omega)_K).$$

Next by [5], Chap. I,

$$\mathrm{Ext}^p_{\mathfrak{g},K}(\pi^{\infty}_{\sigma,\lambda,K},C^\infty(\Gamma\backslash G,\omega)_K) \;\cong\; H^p(\mathfrak{g},K,\mathrm{Hom}_{\mathbb{C}}(\pi^{\infty}_{\sigma,\lambda,K},C^\infty(\Gamma\backslash G,\omega)_K).$$

For any two smooth G-representations V,W the restriction map gives an isomorphism

$$\mathrm{Hom}_{ct}(V,W)_K \;\cong\; \mathrm{Hom}_{\mathbb{C}}(V_K,W_K)_K,$$

where Hom_{ct} means continuous homomorphisms. Therefore, using the classical identification of $(\mathfrak{g},K)$ with differentiable and continuous cohomology as in [5] we get

$$\begin{aligned}
\mathrm{Ext}^q_{\mathfrak{g},K}(\pi^{\infty}_{\sigma,\lambda,K},C^\infty(\Gamma\backslash G,\omega)_K) &\cong H^p(\mathfrak{g},K,\mathrm{Hom}_{ct}(\pi^{\infty}_{\sigma,\lambda},C^\infty(\Gamma\backslash G,\omega))_K)\\
&\cong H^p(\mathfrak{g},K,\mathrm{Hom}_{ct}(\pi^{\infty}_{\sigma,\lambda},C^\infty(\Gamma\backslash G,\omega)))\\
&\cong H^p_d(G,\mathrm{Hom}_{ct}(\pi^{\infty}_{\sigma,\lambda},C^\infty(\Gamma\backslash G,\omega)))\\
&\cong H^p_{ct}(G,\mathrm{Hom}_{ct}(\pi^{\infty}_{\sigma,\lambda},C^\infty(\Gamma\backslash G,\omega)))\\
&\cong \mathrm{Ext}^p_G(\pi^{\infty}_{\sigma,\lambda},C^\infty(\Gamma\backslash G,\omega))\\
&\cong \mathrm{Ext}^p_\Gamma(\pi^{\infty}_{\sigma,\lambda},\omega)\\
&\cong H^p(\Gamma,\pi^{-\infty}_{\breve{\sigma},-\lambda}\otimes\omega).
\end{aligned}$$

This gives the claim for $s\neq 0$. In the case $s=0$ we have to replace $H^0(\mathfrak{a},\cdot)$ by $H^0(\mathfrak{a}^2,\cdot)$, where $\mathfrak{a}^2$ means the subalgebra of $U(\mathfrak{a})$ generated by H^2, $H\in\mathfrak{a}$. Then the induced representation $\mathrm{Ind}_P^G(U)$ is replaced by a suitable self-extension. $\square$

4 Holomorphic torsion

4.1 Holomorphic torsion

Let D be a self-adjoint unbounded operator on a Hilbert space. Assume that D has eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots$ and that the zeta function

$$\zeta_D(s) = \sum_{j\ge 1:\lambda_j>0} \lambda_j^{-s}$$

converges in some half plane and extends to a meromorphic function on the plane which is regular at $s = 0$. In that case we call D *zeta-admissible* and define

$$\det{}'(D) = \exp\left(-\zeta_D'(0)\right).$$

Let $E = (E_0 \to E_1 \to \cdots \to E_n)$ be an elliptic complex over a compact smooth manifold M. That is, each E_j is a vector bundle over M and there is differential operators $d_j : \Gamma^\infty(E_j) \to \Gamma^\infty(E_{j+1})$ of order one such that the sequence of principal symbols

$$\sigma_E(x,\xi) \ : \ 0 \to E_{0,x} \to E_{1,x} \to \cdots \to E_{n,x} \to 0$$

is exact whenever $\xi \in T_xM^*$ is nonzero.

Assume each E_k is equipped with a Hermitian metric. Then we can form the Laplace operators $\Delta_j = d_j d_j^* + d_j^* d_j$ as second order differential operators. When considered as unbounded operators on the spaces $L^2(M, E_k)$ these are known to be zeta admissible. Now define the *torsion* of E as

$$\tau_1(E) = \prod_{k=0}^{n} \det{}'(\Delta_k)^{k(-1)^{k+1}}.$$

Note that this definition differs by an exponent 2 from the original one [27].

The corresponding notion in the combinatorial case, i.e. for a finite CW-complex and the combinatorial Laplacians, was introduced by Reidemeister in the 1930's who used it to distinguish homotopy equivalent spaces which are not homeomorphic. The torsion as defined above, also called *analytic torsion*, was defined by Ray and Singer in [27] where they conjectured that the combinatorial and the analytical torsion should coincide. This conjecture was later proven independently by J. Cheeger and W. Müller.

For a compact smooth Riemannian manifold M and $E \to M$ a flat Hermitian vector bundle the complex of E-valued forms on M satisfies the conditions above so that we can define the torsion $\tau(E)$ via the de Rham complex. Now assume further, M is Kählerian and E holomorphic then we may also consider the torsion $T(E)$ of the Dolbeault complex $\bar\partial \;:\; \Omega^{0,\cdot}(M,E) \to \Omega^{0,\cdot+1}(M,E)$. This then is called the *holomorphic torsion*. The holomorphic torsion is of significance in Arakelov theory [28] where it is used as normalization factor for families of Hermitian metrics.

We now define L^2*-torsion*. For the following see also [23]. Let M denote a compact oriented smooth manifold, Γ its fundamental group and $\tilde M$ its universal covering. Let $E = E_0 \to \cdots \to E_n$ denote an elliptic complex over M and $\tilde E = \tilde E_0 \to \cdots \to \tilde E_n$ its pullback to $\tilde M$. Assume all E_k are equipped with Hermitian metrics.

Let $\tilde\triangle_p$ and $\triangle_p$ denote the corresponding Laplacians. The ordinary torsion was defined via the trace of the complex powers $\triangle_p^s$. The L^2-torsion will instead be defined by considering the complex powers of $\tilde\triangle_p$ and applying a different trace functional. Write $\mathcal{F}$ for a fundamental domain of the Γ-action on $\tilde M$ then as a Γ-module we have

$$L^2(\tilde E_p) \;\cong\; l^2(\Gamma) \otimes L^2(\tilde E_p \mid_{\mathcal{F}}) \;\cong\; l^2(\Gamma) \otimes L^2(E_p).$$

The von Neumann algebra $VN(\Gamma)$ generated by the right action of Γ on $l^2(\Gamma)$has a canonical trace making it a type II_1 von Neumann algebra if Γ is infinite [14]. This trace and the canonical trace on the space $B(L^2(E))$ of bounded linear operators on $L^2(E)$ define a trace tr_Γ on $VN(\Gamma) \otimes B(L^2(E))$ which makes it a type II_∞ von Neumann algebra. The corresponding dimension function is denoted $\dim_\Gamma$. Assume for example, a Γ-invariant operator T on $L^2(E)$ is given as an integral operator with a smooth kernel k_T, then a computation shows

$$\mathrm{tr}_\Gamma(T) = \int_{\mathcal{F}} \mathrm{tr}(k_T(x,x))\; dx.$$

It follows for the heat operator $e^{-t\tilde\Delta_p}$ that

$$\mathrm{tr}_\Gamma e^{-t\tilde\Delta_p} = \int_{\mathcal{F}} \mathrm{tr} < x \mid e^{-t\tilde\Delta_p} \mid x > dx.$$

From this we read off that $\mathrm{tr}_\Gamma e^{-t\tilde{\Delta}_p}$ satisfies the same small time asymptotic as $\mathrm{tr} e^{-t\Delta_p}$.

Let $\tilde{\Delta}_p' = \tilde{\Delta}_p|_{\ker(\tilde{\Delta}_p)^\perp}$. Unfortunately very little is known about large time asymptotic of $\mathrm{tr}_\Gamma(e^{-t\tilde{\Delta}_p'})$ (see [24]). Let

$$NS(\Delta_p) = \sup\{\alpha \in \mathbb{R} \mid \mathrm{tr}_\Gamma e^{-t\tilde{\Delta}_p'} = O(t^{-\alpha/2}) \text{ as } t \to \infty\}$$

denote the *Novikov-Shubin invariant* of Δ_p ([15], [24]).

Then $NS(\Delta_p)$ is always ≥ 0; in this section we will *assume* that the Novikov-Shubin invariant of Δ_p is positive. This is in general an unproven conjecture. In the cases of our concern in later sections, however, the operators in question are homogeneous and it can be proven then that their Novikov-Shubin invariants are in fact positive. We will consider the integral

$$\zeta^1_{\Delta_p}(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \mathrm{tr}_\Gamma e^{-t\tilde{\Delta}_p'}\, dt,$$

which converges for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function on the entire plane which is holomorphic at $s = 0$, as is easily shown by using the small time asymptotic ([3],Thm 2.30).

Further the integral

$$\zeta^2_{\Delta_p(s)}(s) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \mathrm{tr}_\Gamma e^{-t\tilde{\Delta}_p'}\, dt$$

converges for $\mathrm{Re}(s) < \frac{1}{2} NS(\Delta_p)$, so in this region we define the *L^2-zeta function* of Δ_p as

$$\zeta^{(2)}_{\Delta_p}(s) = \zeta^1_{\Delta_p}(s) + \zeta^2_{\Delta_p}(s).$$

Assuming the Novikov-Shubin invariant of Δ_p to be positive we define the *L^2-determinant* of Δ_p as

$$\det{}^{(2)}(\Delta_p) = \exp\left(-\frac{d}{ds}\bigg|_{s=0} \zeta^{(2)}_{\Delta_p}(s) \right).$$

Now let the L^2-torsion be defined by

$$T^{(2)}(E) = \prod_{p=0}^n \det{}^{(2)}(\Delta_p)^{p(-1)^{p+1}}.$$

Again let M be a Kähler manifold and $E \to M$ a flat Hermitian holomorphic vector bundle then we will write $T_{hol}^{(2)}(E)$ for the L^2-torsion of the Dolbeault complex $\Omega^{0,*}(M, E)$.

4.2 Computation of Casimir eigenvalues

In this section we give formulas for Casimir eigenvalues of irreducible representations of reductive groups which hold under conditions on the K-types.

Let G be a connected semisimple Lie group with finite center. Fix a maximal compact subgroup K with Cartan involution θ. Let $P = MAN$ be the Langlands decomposition of a parabolic subgroup P of G. Modulo conjugation we can assume that AM is stable under θ and then $K_M = K \cap M$ is a maximal compact subgroup of M. Let $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{p}_M$ be the corresponding Cartan decomposition of the complexified Lie algebra of M. Let C_M denote the Casimir operator of M induced by the Killing form on G.

Lemma 4.1 *Let (σ, V_σ) be an irreducible finite dimensional representation of M. Let (ξ, V_ξ) be an irreducible unitary representation of M and assume*

$$\sum_{p=0}^{\dim(\mathfrak{p})} (-1)^p \dim(V_\xi \otimes \wedge^p \mathfrak{p}_M \otimes V_\sigma)^{K_M} \neq 0,$$

then the Casimir eigenvalues satisfy

$$\xi(C_M) = \sigma(C_M).$$

Proof: Recall that the Killing form of G defines a K_M-isomorphism between $\mathfrak{p}_M$ and its dual $\mathfrak{p}_M^*$, hence in the assumption of the lemma we may replace $\mathfrak{p}_M$ by $\mathfrak{p}_M^*$. Let ξ_K denote the $(\mathfrak{m}, K_M)$-module of K_M-finite vectors in V_ξ and let $C^q(\xi_K \otimes V_\sigma) = \mathrm{Hom}_{\mathfrak{k}_M}(\wedge^q \mathfrak{p}_M, \xi_K \otimes V_\sigma) = (\wedge^q \mathfrak{p}_M^* \otimes \xi_K \otimes V_\sigma)^{\mathfrak{k}_M}$ the standard complex for the relative Lie algebra cohomology $H^q(\mathfrak{m}, \mathfrak{k}_M, \xi_K \otimes V_\sigma)$. Further $(\wedge^q \mathfrak{p}_M^* \otimes \xi_K \otimes V_\sigma)^{K_M}$ forms the standard complex for the relative $(\mathfrak{m}, K_M)$-cohomology $H^q(\mathfrak{m}, K_M, \xi_K \otimes V_\sigma)$. In [5], p.28 it is shown that

$$H^q(\mathfrak{m}, K_M, \xi_K \otimes V_\sigma) = H^q(\mathfrak{m}, \mathfrak{k}_M, \xi_K \otimes V_\sigma)^{K_M/K_M^0}.$$

Our assumption implies $\sum_q (-1)^q \dim H^q(\mathfrak{m}, K_M, \xi_K \otimes V_\sigma) \neq 0$, therefore there is a q with $0 \neq H^q(\mathfrak{m}, K_M, \xi_K \otimes V_\sigma) = H^q(\mathfrak{m}, \mathfrak{k}_M, \xi_K \otimes V_\sigma)^{K_M/K_M^0}$, hence $H^q(\mathfrak{m}, \mathfrak{k}_M, \xi_K \otimes V_\sigma) \neq 0$. Now Proposition 3.1 on page 52 of [5] says that $\pi(C_M) \neq \sigma(C_M)$ implies that $H^q(\mathfrak{m}, \mathfrak{k}_M, \xi_K \otimes V_\sigma) = 0$ for all q. The claim follows. □

Let $X = G/K$ be the symmetric space to G and assume that X is Hermitian, i.e., X has a complex structure which is stable under G. Let θ denote the Cartan involution fixing K pointwise. Since X is hermitian it follows that G admits a compact Cartan subgroup $T \subset K$. We denote the real Lie algebras of G, K and T by $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{t}_0$ and their complexifications by $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{t}$. We will fix a scalar multiple B of the Killing form. As well, we will write B for its diagonal, so $B(X) = B(X, X)$. Denote by $\mathfrak{p}_0$ the orthocomplement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to B; then via the differential of exp the space $\mathfrak{p}_0$ is isomorphic to the real tangent space of $X = G/K$ at the point eK. Let $\Phi(\mathfrak{t}, \mathfrak{g})$ denote the system of roots of $(\mathfrak{t}, \mathfrak{g})$, let $\Phi_c(\mathfrak{t}, \mathfrak{g}) = \Phi(\mathfrak{t}, \mathfrak{k})$ denote the subset of compact roots and $\Phi_{nc} = \Phi - \Phi_c$ the set of noncompact roots. To any root α let $\mathfrak{g}_\alpha$ denote the corresponding root space. Fix an ordering Φ^+ on $\Phi = \Phi(\mathfrak{t}, \mathfrak{g})$ and let $\mathfrak{p}_\pm = \bigoplus_{\alpha \in \Phi_{nc}^+} \mathfrak{g}_{\pm\alpha}$. Then the complexification $\mathfrak{p}$ of $\mathfrak{p}_0$ splits as $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ and the ordering can be chosen such that this decomposition corresponds via exp to the decomposition of the complexified tangent space of X into holomorphic and antiholomorphic part. By Lemma 2.2.3 of [9] we can, replacing G by a double cover if necessary, assume that the adjoint homomorphism $K \to SO(\mathfrak{p})$ factors over $\mathrm{Spin}(\mathfrak{p})$.

Lemma 4.2 *Let (τ, V_τ) denote an irreducible representation of K. Assume X Hermitian and let (π, W_π) be an irreducible unitary representation of G and assume that*

$$\sum_{p=0}^{\dim \mathfrak{p}_-} (-1)^p \dim(W_\pi \otimes \wedge^p \mathfrak{p}_- \otimes V_\tau)^K \neq 0$$

then we have

$$\pi(C) = \tau \otimes \epsilon(C_K) - B(\rho) + B(\rho_K),$$

where ϵ is the one dimensional representation of K satisfying $\epsilon \otimes \epsilon \cong \wedge^{top} \mathfrak{p}_+$.

Proof: This follows from Lemma 2.1.1 of [9]. □

4.3 The local trace of the heat kernel

On a Hermitian globally symmetric space the heat operator is given by convolution with a function on the group of isometries. In this section we determine the trace of this function on any irreducible unitary representation.

Let H denote a θ-stable Cartan subgroup of G so $H = AB$ where A is the connected split component and B compact. The dimension of A is called the split rank of H. Let $\mathfrak{a}$ denote the complex Lie algebra of A. Then $\mathfrak{a}$ is an abelian subspace of $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Let $X \mapsto X^c$ denote the complex conjugation on $\mathfrak{g}$ according to the real form $\mathfrak{g}_0$. The next lemma shows that $\mathfrak{a}$ lies skew to the decomposition $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

Lemma 4.3 *Let $Pr_\pm$ denote the projections from $\mathfrak{p}$ to $\mathfrak{p}_\pm$. Then we have* $\dim Pr_+(\mathfrak{a}) = \dim Pr_-(\mathfrak{a}) = \dim \mathfrak{a}$*, or, what amounts to the same:* $\mathfrak{a} \cap \mathfrak{p}_\pm = 0$*.*

Proof: $\mathfrak{a}$ consists of semisimple elements whereas $\mathfrak{p}_\pm$ consists of nilpotent elements only. $\square$

Now let $\mathfrak{a}'$ denote the orthocomplement of $\mathfrak{a}$ in $Pr_+(\mathfrak{a}) \oplus Pr_-(\mathfrak{a})$. For later use we write $\mathcal{V} = \mathfrak{a} \oplus \mathfrak{a}' = \mathcal{V}_+ \oplus \mathcal{V}_-$, where $\mathcal{V}_\pm = \mathcal{V} \cap \mathfrak{p}_\pm = Pr_\pm(\mathfrak{a})$.

Let (τ, V_τ) be a finite dimensional representation of the compact group K and let $E_\tau = G \times_K V_\tau = (G \times V)/K$ be the corresponding G-homogeneous vector bundle over X. The sections of the bundle E_τ are as a G-module given by the space of K-invariants

$$\Gamma^\infty(E_\tau) = \left(C^\infty(G) \otimes V_\tau\right)^K ,$$

where K acts on $C^\infty(G) \otimes V_\tau$ by $k.(f(x) \otimes v) = f(xk^{-1}) \otimes \tau(k)v$. and G acts on $\left(C^\infty(G) \otimes V_\tau\right)^K$ by left translations on the first factor.

Consider the convolution algebra $C_c^\infty(G)$ of compactly supported smooth functions on G. Let $K \times K$ act on it by right and left translations and on $\mathrm{End}_{\mathbb{C}}(V_\tau)$ by $(k_1, k_2)T = \tau(k_1)T\tau(k_2^{-1})$. Then the space of invariants

$$\left(C_c^\infty(G) \otimes \mathrm{End}_{\mathbb{C}}(V_\tau)\right)^{K \times K}$$

is seen to be an algebra again and it acts on $\Gamma^\infty(E_\tau) = \left(C^\infty(G) \otimes V_\tau\right)^K$ by convolution on the first factor and in the obvious way on the second.

This describes all G-invariant smoothing operators on E_τ which extend to all sections. G-invariant smoothing operators which not extend to all sections will be given by Schwartz kernels in

$$\left(C^\infty(G) \otimes \mathrm{End}_{\mathbb{C}}(V_\tau)\right)^{K\times K}.$$

Let $n = 2m$ denote the real dimension of X and for $0 \le p, q \le m$ let $\Omega^{p,q}(X)$ denote the space of smooth (p,q)-forms on X. The above calculus holds for the space of sections $\Omega^{p,q}(X)$. D.Barbasch and H. Moscovici have shown in [2] that the heat operator $e^{-t\triangle_{p,q}}$ has a smooth kernel $h_t^{p,q}$ of rapid decay in

$$(C^\infty(G) \otimes \mathrm{End}(\wedge^p\mathfrak{p}_+ \otimes \wedge^q\mathfrak{p}_-))^{K\times K}.$$

Now fix p and set for $t > 0$

$$f_t^p = \sum_{q=0}^m q(-1)^{q+1} \ \mathrm{tr}\ h_t^{p,q},$$

where tr means the trace in $\mathrm{End}(\wedge^p\mathfrak{p}_+ \otimes \wedge^q\mathfrak{p}_-)$.

We want to compute the trace of f_t^p on the principal series representations. To this end let $H = AB$ as above and let P denote a parabolic subgroup of G with Langlands decomposition $P = MAN$. Let (ξ, W_ξ) denote an irreducible unitary representation of M, e^ν a quasicharacter of A and set $\pi_{\xi,\nu} = \mathrm{Ind}_P^G(\xi \otimes e^{\nu+\rho_P} \otimes 1)$, where ρ_P is the half of the sum of the P-positive roots.

Let C denote the Casimir operator of G attached to the form B.

Proposition 4.4 *The trace of f_t^p under $\pi_{\xi,\nu}$ vanishes if* $\dim\mathfrak{a} > 1$. *If* $\dim\ \mathfrak{a} = 1$ *it equals*

$$e^{t\pi_{\xi,\nu}(C)} \sum_{q=0}^{\dim(\mathfrak{p}_-)-1} (-1)^q \ \dim\left(W_\xi \otimes \wedge^p\mathfrak{p}_+ \otimes \wedge^q(\mathfrak{a}^\perp \cap \mathfrak{p}_-)\right)^{K\cap M},$$

where $\mathfrak{a}^\perp$ is the orthocomplement of $\mathfrak{a}$ in $\mathfrak{p}$.

Proof: As before let $\mathfrak{a}'$ be the span of all $X - X^c$, where $X + X^c \in \mathfrak{a}$, $X \in \mathfrak{p}_+$ then $\mathcal{V} = \mathfrak{a} \oplus \mathfrak{a}' = \mathcal{V}_+ \oplus \mathcal{V}_-$ where $\mathcal{V}_\pm = \mathcal{V} \cap \mathfrak{p}_\pm$. The group $K_M = K \cap M$

acts trivially on $\mathfrak{a}$ so for $x \in K_M$ we have $X + X^c = \mathrm{Ad}(x)(X + X^c) = \mathrm{Ad}(x)X + \mathrm{Ad}(x)X^c$. Since K respects the decomposition $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+$, we conclude that K_M acts trivially on $\mathcal{V}$, hence on $\mathcal{V}_-$. Let $r = \dim \mathfrak{a} = \dim \mathcal{V}_-$. As a K_M-module we have

$$\begin{aligned} \wedge^p \mathfrak{p}_- &= \sum_{a+b=q} \wedge^b \mathcal{V}_- \otimes \wedge^a \mathcal{V}_-^\perp \\ &= \sum_{a+b=q} \binom{r}{b} \wedge^a \mathcal{V}_-^\perp, \end{aligned}$$

where $\mathcal{V}_-^\perp = \mathfrak{a}^\perp \cap \mathfrak{p}_-$. By definition we get

$$\begin{aligned} \mathrm{tr}\pi_{\xi,\nu}(f_t^p) &= \mathrm{tr}\pi_{\xi,\nu}\left(\sum_{q=0}^m q(-1)^{q+1} h_t^{p,q}\right) \\ &= e^{t\pi_{\xi,\nu}(C)} \sum_{q=0}^m q(-1)^{q+1} \dim(V_{\pi_{\xi,\nu}} \otimes \wedge^p \mathfrak{p}_+ \otimes \wedge^q \mathfrak{p}_-)^K \end{aligned}$$

By Frobenius reciprocity this equals

$$\begin{aligned} & e^{t\pi_{\xi,\nu}(C)} \sum_{q=0}^m q(-1)^{q+1} \dim(W_{\pi_\xi} \otimes \wedge^p \mathfrak{p}_+ \otimes \wedge^q \mathfrak{p}_-)^{K\cap M} \\ = \;& e^{t\pi_{\xi,\nu}(C)} \sum_{q=0}^m \sum_{a=0}^q q(-1)^{q+1} \binom{r}{q-a} \dim(W_{\pi_\xi} \otimes \wedge^p \mathfrak{p}_+ \otimes \wedge^a \mathcal{V}_-^\perp)^{K\cap M} \\ = \;& e^{t\pi_{\xi,\nu}(C)} \sum_{a=0}^m \sum_{q=a}^m q(-1)^{q+1} \binom{r}{q-a} \dim(W_{\pi_\xi} \otimes \wedge^p \mathfrak{p}_+ \otimes \wedge^a \mathcal{V}_-^\perp)^{K\cap M} \end{aligned}$$

By taking into account $a \le m - r$ we get

$$\sum_{q=a}^m q(-1)^q \binom{r}{q-a} = \begin{cases} (-1)^{a+1} & \text{if } r = 1, \\ 0 & \text{if } r > 1, \end{cases}$$

and the claim follows. □

We will now combine this result with the computation of Casimir eigenvalues. This requires an analysis whether the homogeneous space M/K_M embedded into $X = G/K$ inherits the complex structure or not.

Fix a θ-stable Cartan subgroup $H = AB$ with $\dim(A) = 1$ and a parabolic $P = MAN$. Fix a system of positive roots $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ in $\Phi(\mathfrak{g}, \mathfrak{h})$ such that for $\alpha \in \Phi^+$ and α nonimaginary it follows $\alpha^c \in \Phi^+$. Further assume that Φ^+ is compatible with the choice of P, i.e., for any $\alpha \in \Phi^+$ the restriction $\alpha|_{\mathfrak{a}}$ is either zero or positive. Let ρ denote the half sum of positive roots. For $\xi \in \hat{M}$ let $\lambda_\xi \in \mathfrak{b}^*$ denote the infinitesimal character of ξ. Recall that we have

$$\pi_{\xi,\nu}(C) = B(\nu) + B(\lambda_\xi) - B(\rho).$$

Lemma 4.5 *There exists a unique real root $\alpha_r \in \Phi^+$.*

Proof: Recall that a root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ is real if and only if it annihilates $\mathfrak{b} = Lie_{\mathbb{C}}(B)$. Hence the real roots are elements of $\mathfrak{a}^*$ which is one dimensional, so, if there were two positive real roots, one would be positive multiple of the other which is absurd.

To the existence. We have that $\dim X = \dim \mathfrak{a} + \dim \mathfrak{n}$ is even, hence $\dim \mathfrak{n}$ is odd. Now $\mathfrak{n} = \oplus_\alpha \mathfrak{g}_\alpha$, where the sum runs over all $\alpha \in \Phi^+$ with $\alpha|_{\mathfrak{a}} \neq 0$. The complex conjugation permutes the $\mathfrak{g}_\alpha$ and for any nonreal α we have $(\mathfrak{g}_\alpha)^c \neq \mathfrak{g}_\alpha$, hence the nonreal roots pair up, thus $\mathfrak{n}$ can only be odd dimensional if there is a real root. □

Let $c = c(H)$ denote the number of positive restricted roots in $\Phi(\mathfrak{g}, \mathfrak{a})$. Let $\Phi^+(\mathfrak{g}, \mathfrak{a})$ denote the subset of positive restricted roots.

Lemma 4.6 *There are three possibilities:*

- $c = 1$ *and* $\Phi^+(\mathfrak{a}, \mathfrak{g}) = \{\alpha_r\}$,
- $c = 2$ *and* $\Phi^+(\mathfrak{a}, \mathfrak{g}) = \{\frac{1}{2}\alpha_r, \alpha_r\}$ *and*
- $c = 3$ *and* $\Phi^+(\mathfrak{a}, \mathfrak{g}) = \{\frac{1}{2}\alpha_r, \alpha_r, \frac{3}{2}\alpha_r\}$.

Proof: For any root $\alpha \in \Phi(\mathfrak{h}, \mathfrak{g})$ we have that $2B(\alpha, \alpha_r)/B(\alpha_r)$ can only take the values $0, 1, 2, 3$, so the only possible restricted roots in $\Phi^+(\mathfrak{a}, \mathfrak{g})$ are $\frac{1}{2}\alpha_r, \alpha_r, \frac{3}{2}\alpha_r$.

If $\frac{3}{2}\alpha_r \in \Phi^+(\mathfrak{a}, \mathfrak{g})$ then there is a root β in $\Phi(\mathfrak{h}, \mathfrak{g})$ with $\beta|_{\mathfrak{a}} = \frac{3}{2}\alpha_r|_{\mathfrak{a}}$. Then $B(\alpha_r, \beta) > 0$ hence $\eta = \beta - \alpha_r$ is a root. Since $\eta|_{\mathfrak{a}} = \frac{1}{2}\alpha_r$ we get that then $\frac{1}{2}\alpha_r \in \Phi(\mathfrak{a}, \mathfrak{g})$. From this the claim follows. □

Recall that a *central isogeny* $\varphi : L_1 \to L_2$ of Lie groups is a surjective homomorphism with finite kernel which lies in the center of L_1.

Lemma 4.7 *There is a central isogeny $M_1 \times M_2 \to M$ such that the inverse image of K_M is of the form $K_{M_1} \times K_{M_2}$ where K_{M_j} is a maximal compact subgroup of M_j for $j = 1, 2$ and such that $\mathfrak{p}_M = \mathfrak{p}_{M_1} \oplus \mathfrak{p}_{M_2}$ as $K_{M_1} \times K_{M_2}$-module and*

- *the map $X \mapsto [X, X_{\alpha_r}]$ induces a K_M-isomorphism*

$$\mathfrak{p}_{M_1} \ \cong \ [\mathfrak{p}_M, \mathfrak{g}_{\alpha_r}],$$

- *with $\mathfrak{p}_{M_2,\pm} := \mathfrak{p}_{M_2} \cap \mathfrak{p}_\pm$ we have*

$$\mathfrak{p}_{M_2} = \mathfrak{p}_{M_2,+} \oplus \mathfrak{p}_{M_2,-}.$$

The latter point implies that the symmetric space M_2/K_{M_2}, naturally embedded into G/K, inherits the complex structure. This in particular implies that M_2 is orientation preserving. Further we have $\mathfrak{p}_{M_1} \cap \mathfrak{p}_+ = \mathfrak{p}_{M_1} \cap \mathfrak{p}_- = 0$, which in turn implies that the symmetric space M_1/K_{M_1} does not inherit the complex structure of G/K.

Proof: At first we reduce the proof to the case that the center Z of G is trivial. So assume the proposition proved for G/Z then the covering $M_1 \times M_2 \to M$ is gotten by pullback from that of M/Z. Thus we may assume that G has trivial center.

Let H be a generator of $\mathfrak{a}_0$. Write $H = Y + Y^c$ for some $Y \in \mathfrak{p}_+$. According to the root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_M \oplus \mathfrak{p}_M \oplus \mathfrak{n} \oplus \theta(\mathfrak{n})$ we write $Y = Y_a + Y_k + Y_p + Y_n + Y_{\theta(n)}$. Because of $\theta(Y) = -Y$ it follows $Y_k = 0$ and $Y_{\theta(n)} = -\theta(Y_n)$. For arbitrary $k \in K_M$ we have $\mathrm{Ad}(k)Y = Y$ since $\mathrm{Ad}(k)H = H$ and the projection Pr_+ is K_M-equivariant. Since the root space decomposition is stable under K_M it follows $\mathrm{Ad}(K)Y_* = Y_*$ for $* = a, p, n$. The group M has a compact Cartan, hence K_M cannot act trivially on any nontrivial element of $\mathfrak{p}_M$, so $Y_p = 0$ and $Y_n \in \mathfrak{g}_{\alpha_r}$. Since $Y = Y_a + Y_n - \theta(Y_n)$ and $Y \notin \mathfrak{a}$ it follows that $Y_n \neq 0$, so Y_n generates $\mathfrak{g}_{\alpha_r}$ and so K_M acts trivially on $\mathfrak{g}_{\alpha_r}$.

Let $\mathfrak{p}_{M_2} \subset \mathfrak{p}_M$ by definition be the kernel of the map $X \mapsto [X, Y_n]$. Let $\mathfrak{p}_{M_1}$ be its orthocomplement in $\mathfrak{p}_M$. The group K_M stabilizes Y_n, so it follows that K_M leaves $\mathfrak{p}_{M_2}$ stable hence the orthogonal decomposition $\mathfrak{p}_M = \mathfrak{p}_{M_1} \oplus \mathfrak{p}_{M_2}$ is K_M-stable. Thus the symmetric space M/K_M decomposes into a product accordingly and so does the image of M in the group of isometries of M/K_M. It follows that the Lie algebra $\mathfrak{m}$ of M splits as a direct sum of ideals $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_0$ is the Lie algebra of the kernel M_0 of the map $M \to \mathrm{Iso}(M/K_M) = \overline{M_1} \times \overline{M_2}$. Let $L = AM$ be the Levi component and let L_c denote the centralizer of $\mathfrak{a}$ in the group $\mathrm{Int}(\mathfrak{g})$. Then L_c is a connected complex group and, since G has trivial center, L injects into L_c. The Lie algebra $\mathfrak{l}$ of L resp. L_c decomposes as $l = \mathfrak{a} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and so there is an isogeny $A_c \times M_{0,c} \times M_{1,c} \times M_{2,c} \to L_c$. The group A injects into A_c and we have $A_c = A \times U$ where U is a compact one dimensional torus. Let $\hat{M}_1 \subset M$ be the preimage of $\overline{M_1}$ and define $\hat{M}_2$ analogously. Then $\hat{M}_1$ and $\hat{M}_2$ are subgroups of L_c and we may define M_1 to be the preimage of $\hat{M}_1$ in $U \times M_{0,c} \times M_{1,c}$ and M_2 to be the preimage of $\hat{M}_2$ in $M_{2,c}$. Then the map $M_1 \times M_2 \to M$ is a central isogeny.

By definition we already have $\mathfrak{p}_{M_1} \cong [\mathfrak{p}_M, \mathfrak{g}_{\alpha_r}]$. Now let $X \in \mathfrak{p}_{M_2}$, then $[X, Y] = [X, Y_n] + \theta([X, Y_n]) = 0$ since $Y_n \in \mathfrak{g}_{\alpha_r}$. By $[X, H] = 0$ this implies $[X, Y^c] = 0$ and so $[Pr_\pm(X), H] = 0 = [Pr_\pm(X), Y] = [Pr_\pm(X), Y^c]$ for $Pr_\pm$ denoting the projection $\mathfrak{p} \to \mathfrak{p}_\pm$. Hence $Pr_\pm(X) \in \mathfrak{p}_{M_2}$ which implies

$$\mathfrak{p}_{M_2} = \mathfrak{p}_{M_2,+} \oplus \mathfrak{p}_{M_2,-}.$$

The lemma is proven. □

Let

$$\mathfrak{n}_{\alpha_r} := \bigoplus_{\alpha \in \Phi(\mathfrak{h},\mathfrak{g}) \ \alpha|_{\mathfrak{a}} = \alpha_r|_{\mathfrak{a}}} \mathfrak{g}_{\alpha_r}$$

and define $\mathfrak{n}_{\frac{1}{2}\alpha_r}$ and $\mathfrak{n}_{\frac{3}{2}\alpha_r}$ analogously. Let $\mathfrak{n}_r^\alpha := \mathfrak{n}_{\frac{1}{2}\alpha_r} \oplus \mathfrak{n}_{\frac{3}{2}\alpha_r}$, then $\mathfrak{n} = \mathfrak{n}_{\alpha_r} \oplus \mathfrak{n}_r^\alpha$.

Proposition 4.8 *There is a K_M-stable subspace $\mathfrak{n}_-$ of $\mathfrak{n}^{\alpha_r}$ such that as K_M-module $\mathfrak{n}_r^\alpha \cong \mathfrak{n}_- \oplus \tilde{\mathfrak{n}_-}$, where $\tilde{.}$ denotes the contragredient, and as K_M-module:*

$$\mathfrak{p}_- \ \cong \ \mathbb{C} \oplus \mathfrak{p}_{M_1} \oplus \mathfrak{p}_{M_2,-} \oplus \mathfrak{n}_-.$$

Proof: The map $a + p + n \mapsto a + p + n - \theta(n)$ induces a K_M-isomorphism

$$\mathfrak{a} \oplus \mathfrak{p}_M \oplus \mathfrak{n} \ \cong \ \mathfrak{p}.$$

We now prove that there is a K_M-isomorphism $\mathfrak{n}_{\alpha_r} \cong \mathfrak{g}_{\alpha_r} \oplus [\mathfrak{p}_M, \mathfrak{g}_{\alpha_r}]$. For this let $\alpha \neq \alpha_r$ be any root in $\Phi(\mathfrak{h}, \mathfrak{g})$ such that $\alpha|_{\mathfrak{a}} = \alpha_r|_{\mathfrak{a}}$. Then $B(\alpha, \alpha_r) > 0$ and hence $\beta := \alpha - \alpha_r$ is a root. The root β is imaginary. Assume β is compact, then $\mathfrak{g}_\beta \subset \mathfrak{k}_M$ and we have $[\mathfrak{g}_{\alpha_r}, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha_r}$ which contradicts $[\mathfrak{k}_M, \mathfrak{g}_{\alpha_r}] = 0$. It follows that β is noncompact and so $\mathfrak{g}_\beta \subset \mathfrak{p}_M$, which implies $\mathfrak{n}_{\alpha_r} = \mathfrak{g}_{\alpha_r} \oplus [\mathfrak{p}_M, \mathfrak{g}_{\alpha_r}]$.

Now the last lemma implies the assertion. □

We have $M \leftarrow M_1 \times M_2$ and so any irreducible representation ξ of M can be pulled back to $M_1 \times M_2$ and be written as a tensor product $\xi = \xi_1 \otimes \xi_2$. Let ξ be irreducible admissible then Proposition 4.4 and Proposition 4.8 imply

$$\mathrm{tr}\pi_{\xi,\nu}(f_t^0) = e^{t\pi_{\xi,\nu}(C)} \sum_{q \geq 0} (-1)^q \dim \left(W_\xi \otimes \wedge^q(\mathfrak{p}_{M_1} \oplus \mathfrak{p}_{M_2,-} \oplus \mathfrak{n}_-)\right)^{K_M}.$$

For $c \geq 0$ let

$$\wedge^c \mathfrak{n}_- = \bigoplus_{i \in I_c} \sigma_i^c \otimes \tau_i^c$$

be a decomposition as $K_M \leftarrow K_{M_1} \times K_{M_2}$-module of $\mathfrak{n}_-$ where τ_i^c is an irreducible K_{M_2}-module and σ_i^c is the image of the projection $\mathfrak{n}_r^\alpha = \mathfrak{n}_- \oplus \tilde{\mathfrak{n}_-} \rightarrow \mathfrak{n}_-$ of an irreducible M_1-submodule of $\mathfrak{n}_r^\alpha$. Then we conclude

$$\mathrm{tr}\pi_{\xi,\nu}(f_t^0) = e^{t\pi_{\xi,\nu}(C)} \sum_{a,b,c \geq 0} \sum_{i \in I_c} (-1)^{a+b+c} \dim \left(W_{\xi_1} \otimes \wedge^a \mathfrak{p}_{M_1} \otimes \sigma_i^c\right)^{K_{M_1}}$$

$$\times \dim \left(W_{\xi_2} \otimes \wedge^b \mathfrak{p}_{M_2,-} \otimes \tau_i^c\right)^{K_{M_2}}.$$

Fix an irreducible representation $\xi = \xi_1 \otimes \xi_2$ of M. Let $\xi' := \tilde{\xi_1} \otimes \xi_2$, where $\tilde{\xi_1}$ is the contragredient representation to ξ_1.

We will use the following notation: For an irreducible representation π we denote its infinitesimal character by $\wedge_\pi$. We will identify $\wedge_\pi$ to a corresponding element in the dual of a Cartan subalgebra (modulo the Weyl group), so that it makes sense to write an expression like $B(\wedge_\pi)$. In the case of σ_i^c we have the following situation: either σ_i^c already extends to a representation of M_1 or $\sigma_i^c \oplus \tilde{\sigma_i^c}$ does. In either case we write $\wedge_{\sigma_i^c}$ for the corresponding infinitesimal character with respect to the group M_1.

Lemma 4.9 *If* $\xi \cong \xi'$ *and* $\mathrm{tr}\pi_{\xi,\nu}(f_t^0) \neq 0$ *for some* $t > 0$ *then*

$$\pi_{\xi,\nu}(C) = B(\wedge_{\sigma_i^c}) + B(\wedge_{\tau_i^c \otimes \epsilon}) + B(\nu) - B(\rho)$$

for some $c \geq 0$ and some $i \in I_c$. Here ϵ is the one dimensional representation of K_{M_2} such that $\epsilon \otimes \epsilon \cong \wedge^{top}\mathfrak{p}_{M_2}$.

Proof: Generally we have

$$\begin{aligned} \pi_{\xi,\nu}(C) &= B(\wedge_{\pi_{\xi,\nu}}) - B(\rho) \\ &= B(\wedge_\xi) + B(\nu) - B(\rho) \\ &= B(\wedge_{\xi_1}) + B(\wedge_{\xi_2}) + B(\nu) - B(\rho). \end{aligned}$$

Now $\mathrm{tr}\pi_{\xi,\nu}(f_t^0) \neq 0$ implies firstly

$$\sum_{a\geq 0}(-1)^a \dim\left(W_{\xi_1} \otimes \wedge^a \mathfrak{p}_{M_1} \otimes \sigma_i^c\right)^{K_{M_1}} \neq 0$$

for some c, i. Now either σ_i^c already extends to an irreducible representation of M_1 or $\sigma_i^c \oplus \tilde{\sigma_i^c}$ does. In the second case we get

$$\sum_{a\geq 0}(-1)^a \dim\left(W_{\xi_1} \otimes \wedge^a \mathfrak{p}_{M_1} \otimes (\sigma_i^c \oplus \tilde{\sigma_i^c})\right)^{K_{M_1}}$$

$$= 2\sum_{a\geq 0}(-1)^a \dim\left(W_{\xi_1} \otimes \wedge^a \mathfrak{p}_{M_1} \otimes \sigma_i^c\right)^{K_{M_1}} \neq 0.$$

So in either case Lemma 4.1 implies

$$B(\wedge_{\xi_1}) - B(\rho_{M_1}) = \xi_1(C_M) = \sigma_i^c(C_M) = B(\wedge_{\sigma_i^c}) - B(\rho_M),$$

hence $B(\wedge_{\xi_1}) = B(\wedge_{\sigma_1^c})$.

Next $\mathrm{tr}\pi_{\xi,\nu}(f_t^0) \neq 0$ implies

$$\sum_{b\geq 0}(-1)^b \dim\left(W_{\xi_2} \otimes \wedge^b \mathfrak{p}_{M_2,-} \otimes \tau_i^c\right)^{K_{M_2}} \neq 0$$

for the same c, i. In that case with $\tau = \tau_i^c$ Lemma 4.2 implies

$$\xi_2(C_{M_2}) = \tau \otimes \epsilon(C_{K_{M_2}}) - B(\rho_{M_2}) + B(\rho_{K_{M_2}}).$$

Since $\xi_2(C_{M_2}) = B(\wedge_{\xi_2}) - B(\rho_{M_2})$ we conclude

$$B(\wedge_{\xi_2}) = \tau \otimes \epsilon(C_{K_{M_2}}) + B(\rho_{K_{M_2}}) = B(\wedge_{\tau\otimes\epsilon}).$$

□

Along the same lines we get

Lemma 4.10 *If ξ is not isomorphic to ξ' and* $\mathrm{tr}\pi_{\xi,\nu}(f_t^0)+\mathrm{tr}\pi_{\xi',\nu}(f_t^0)\neq 0$ *for some* $t>0$ *then*

$$\pi_{\xi,\nu}(C) = B(\wedge_{\sigma_i^c}) + B(\wedge_{\tau_i^c\otimes\epsilon}) + B(\nu) - B(\rho)$$

for some $c\geq 0$ *and some* $i\in I_c$.

We abbreviate $s_i^c := B(\wedge_{\sigma_i^c}) + B(\wedge_{\tau_i^c\otimes\epsilon})$. We have shown that if $\xi\cong\xi'$ then

$$\mathrm{tr}\pi_{\xi,\nu}(f_t^0)$$

equals

$$e^{t(B(\nu)-B(\rho))}\sum_{a,b,c\geq 0}\sum_{i\in I_c} e^{t(s_i^c)}(-1)^{a+b+c}\dim\left(W_{\xi_1}\otimes\wedge^a\mathfrak{p}_{M_1}\otimes\sigma_i^c\right)^{K_{M_1}}$$
$$\times\dim\left(W_{\xi_2}\otimes\wedge^b\mathfrak{p}_{M_2,-}\otimes\tau_i^c\right)^{K_{M_2}}.$$

Define the Fourier transform of f_t^0 by

$$\hat{f_t^0}_H(\nu,b^*) = \mathrm{tr}\pi_{\xi_{b^*},\nu}(f_t^0).$$

Let B^* be the character group of B. According to $M=M_1\times M_2$ we can write $B=B_1\times B_2$ and so we see that any character b^* of B decomposes as $b_1^*\times b_2^*$. Let $b^{*\prime} := \overline{b_1^*}\times b_2^*$. We get

Lemma 4.11 *The sum of Fourier transforms*

$$\hat{f_t^0}_H(\nu,b^*) + \hat{f_t^0}_H(\nu,b^{*\prime})$$

equals

$$e^{t(B(\nu)-B(\rho))}\sum_{a,b,c\geq 0}\sum_{i\in I_c} e^{ts_i^c}(-1)^{a+b+c}\dim\left((V_{b_1^*}\oplus V_{\overline{b_1^*}})\otimes\wedge^a\mathfrak{p}_{M_1}\otimes\sigma_i^c\right)^{K_{M_1}}$$
$$\times\dim\left(V_{b_2^*}\otimes\wedge^b\mathfrak{p}_{M_2,-}\otimes\tau_i^c\right)^{K_{M_2}}.$$

4.4 The global trace of the heat kernel

Fix a discrete cocompact torsion-free subgroup Γ of G then the quotient $X_\Gamma := \Gamma\backslash X = \Gamma\backslash G/K$ is a non-euclidean Hermitian locally symmetric space.

Since X is contractible and Γ acts freely on X the group Γ equals the fundamental group of X_Γ. Let (ω, V_ω) denote a finite dimensional unitary representation of Γ.

D. Barbasch and H. Moscovici have shown in [2] that the function f_t^0 satisfies the conditions to be plugged into the trace formula. So in order to compute $\mathrm{tr} R_{\Gamma,\omega}(f_t^0)$ we have to compute the orbital integrals of f_t^0. At first let $h \in G$ be a nonelliptic semisimple element. Since the trace of f_t^0 vanishes on principal series representations which do not come from splitrank one Cartan subgroups, we see that $\mathcal{O}_h(f_t^0) = 0$ unless $h \in H$, where H is a splitrank one Cartan. Write $H = AB$ as before. We have $h = a_h b_h$ and since h is nonelliptic it follows that a_h is regular in A. We say that h is *split regular* in H. We can choose a parabolic $P = MAN$ such that a_h lies in the negative Weyl chamber A^-.

Let V denote a finite dimensional complex vector space and let A be an endomorphism of V. Let $\det(A)$ denote the determinant of A, which is the product over all eigenvalues of A with algebraic multiplicities. Let $\det'(A)$ be the product of all nonzero eigenvalues with algebraic multiplicities.

In [16], sec 17 Harish-Chandra has shown that for $h_0 \in H$

$$\mathcal{O}_{h_0}(f_t^0) = \frac{\varpi_{h_0}(\ {}'F^H_{f_t^0}(h))|_{h=h_0}}{c_{h_0} h_0^{\rho_P} \det'(1 - h_0^{-1}|(\mathfrak{g}/\mathfrak{h})^+)},$$

where $(\mathfrak{g}/\mathfrak{h})^+$ is the positive part of the root space decomposition of a compatible ordering. Further ϖ_h is the differential operator attached to h as follows. Let $\mathfrak{g}_h$ denote the centralizer of h in $\mathfrak{g}$ and let $\Phi^+(\mathfrak{g}_h, \mathfrak{h})$ the positive roots then

$$\varpi_h = \prod_{\alpha \in \Phi^+(\mathfrak{g}_h, \mathfrak{h})} H_\alpha,$$

where H_α is the element in $\mathfrak{h}$ dual to α via the bilinear form B. In comparison to other sources the formula above for the orbital integral lacks a factor $[G_h : G_h^0]$ which doesn't occur because of the choices of Haar measures made. We assume the ordering to come from an ordering of $\Phi(\mathfrak{b}, \mathfrak{m})$ which is such that for the root space decomposition $\mathfrak{p}_M = \mathfrak{p}_M^+ \oplus \mathfrak{p}_M^-$ it holds $\mathfrak{p}_{M_2} \cap \mathfrak{p}_M^+ = \mathfrak{p}_{M_2,-}$. For short we will henceforth write $\varpi_{h_0} F(h_0)$ instead of $\varpi_{h_0} F(h)|_{h=h_0}$ for any function F.

Our results on the Fourier transform of f_t^0 together with a computation as in the proof of Lemma 2.3.6 of [9] imply that ${}'F^H_{f_t^0}(h)$ equals

$$\frac{e^{-l_h^2/4t}}{\sqrt{4\pi t}} e^{-tB(\rho)} \det(1-h^{-1}|\mathfrak{k}_M^+ \oplus \mathfrak{p}_{M_1}^+) \sum_{c\geq 0,\ i\in I_c} (-1)^c e^{ts_i^c} \mathrm{tr}(b_h^{-1}|\sigma_i^c \oplus \tau_i^c).$$

Note that this computation involves a summation over B^* and thus we may replace σ_i^c by its dual $\breve{\sigma}_i^c$ without changing the result.

Now let (τ, V_τ) be a finite dimensional unitary representation of K_M and define for $b \in B$ the monodromy factor:

$$L^M(b,\tau) = \frac{\varpi_b(\det(1-b|(\mathfrak{k}_M/|\mathfrak{b})^+)\mathrm{tr}(\tau(b)))}{\varpi_b(\det(1-b|(\mathfrak{m}/\mathfrak{b})^+))}.$$

Note that the expression $\varpi_b(\det(1-b|(\mathfrak{m}/\mathfrak{b})^+))$ equals

$$|W(\mathfrak{m}_b,\mathfrak{b})| \prod_{\alpha\in\Phi_b^+(\mathfrak{m},\mathfrak{b})} (\rho_b,\alpha) \det{}'(1-b|(\mathfrak{m}/\mathfrak{b})^+),$$

and that for $\gamma = a_\gamma b_\gamma$ split-regular we have

$$|W(\mathfrak{m}_b,\mathfrak{b})| \prod_{\alpha\in\Phi_b^+(\mathfrak{m},\mathfrak{b})} (\rho_b,\alpha) = |W(\mathfrak{g}_\alpha,\mathfrak{h})| \prod_{\alpha\in\Phi_\gamma^+} (\rho_\gamma,\alpha),$$

so that writing $L^M(\gamma,\tau) = L^M(b_\gamma,\tau)$ we get

$$L^M(\gamma,\tau) = \frac{\varpi_\gamma(\det(1-\gamma|(\mathfrak{k}_M/\mathfrak{b})^+)\mathrm{tr}(\tau(b_\gamma)))}{|W(\mathfrak{g}_\gamma,\mathfrak{h})| \prod_{\alpha\in\Phi_\gamma^+}(\rho_\gamma,\alpha) \det{}'(1-\gamma|(\mathfrak{m}/\mathfrak{b})^+)}.$$

From the above it follows that $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f_t^0)$ equals:

$$\frac{\chi_1(X_\gamma) l_{\gamma_0}}{\det(1-\gamma|\mathfrak{n})} \frac{e^{-l_\gamma^2/4t}}{\sqrt{4\pi t}} a_\gamma^{\rho_P} \sum_{c\geq 0,\ i\in I_c} (-1)^c\ e^{ts_i^c}\ \mathrm{tr}(b_\gamma|\sigma_i^c) L^{M_2}(\gamma,\tau_i^c).$$

Note that here we can replace σ_i^c by $\tilde{\sigma_i^c}$.

For a splitrank one Cartan $H = AB$ and a parabolic $P = MAN$ let $\mathcal{E}_P(\Gamma)$ denote the set of Γ-conjugacy classes $[\gamma] \subset \Gamma$ such that γ is G-conjugate to an element of A^-B, where A^- is the negative Weyl chamber given by P. We have proven:

Theorem 4.12 *Let X_Γ be a compact locally Hermitian space with fundamental group Γ and such that the universal covering is globally symmetric without compact factors. Assume Γ is neat and write $\triangle_{p,q,\omega}$ for the Hodge Laplacian on (p,q)-forms with values in a the flat Hermitian bundle E_ω, then it theta series defined by*

$$\Theta(t) = \sum_{q=0}^{\dim_{\mathbb{C}} X_\Gamma} q(-1)^{q+1} \mathrm{tr}\; e^{-t\triangle_{0,q,\omega}}$$

equals

$$\sum_{P/conj.} \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \chi_1(X_\gamma) \frac{l_{\gamma_0} a_\gamma^{\rho_P}}{\det(1-\gamma|\mathfrak{n})} \frac{e^{-l_\gamma^2/4t}}{\sqrt{4\pi t}} \times \sum_{c\geq 0,\; i\in I_c} (-1)^c\; e^{ts_i^c}\; \mathrm{tr}(b_\gamma|\sigma_i^c) L^{M_2}(\gamma, \tau_i^c).$$
$$+ f_t^0(e)\; \dim\omega\; vol(X_\Gamma).$$

The reader should keep in mind that by its definition we have for the term of the identity:

$$f_t^0(e)\; \dim\omega\; vol(X_\Gamma) = \sum_{q=0}^{\dim_{\mathbb{C}} X} q(-1)^{q+1} \mathrm{tr}_\Gamma(e^{-t\Delta_{0,q,\omega}}),$$

where tr_Γ is the Γ-trace. Further note that by the Plancherel theorem the Novikov-Shubin invariants of all operators $\Delta_{0,q}$ are positive.

4.5 The holomorphic torsion zeta function

Now let X be Hermitian again and let $\tau = \tau_1 \otimes \tau_2$ be an irreducible representation of $K_M \to K_{M_1} \times K_{M_2}$. Further assume that τ_1 lies in the image of the restriction map $res : \mathrm{Rep}(M_1) \to \mathrm{Rep}(K_{M_1})$.

Theorem 4.13 *Let* Γ *be neat and* (ω, V_ω) *a finite dimensional unitary representation of* Γ. *Choose a* θ*-stable Cartan* H *of splitrank one. For* $\mathrm{Re}(s) >> 0$ *define the zeta function* $Z^0_{P,\tau,\omega}(s)$ *to be*

$$\exp\left(- \sum_{[\gamma]\in\mathcal{E}_H(\gamma)} \frac{\chi_1(X_\gamma)\mathrm{tr}(\omega(\gamma))\mathrm{tr}\tau_1(b_\gamma)L^{M_2}(\gamma,\tau_2)}{\det(1-\gamma|\mathfrak{n})} \; \frac{e^{-sl_\gamma}}{\mu_\gamma}\right).$$

Then $Z^0_{P,\tau,\omega}$ *has a meromorphic continuation to the entire plane. The vanishing order of* $Z^0_{P,\tau,\omega}(s)$ *at* $s = \lambda(H_1)$, $\lambda \in \mathfrak{a}^*$ *is* $(-1)^{\dim \mathfrak{n}}$ *times*

$$\sum_{\pi\in\hat{G}} N_{\Gamma,\omega}(\pi) \sum_{p,q,r} (-1)^{p+q+r} \dim \left(H^q(\mathfrak{n},\pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_{M_1} \otimes \wedge^r \mathfrak{p}_{M_2,-} \otimes V_{\check{\tau}}\right)^{K_M}.$$

Note that in the special case $M_2 = 1$ *this function equals the Selberg zeta function.*

Proof: By Lemma 2.2.1 of [9] the group M_2 is orientation preserving. Since τ_1 lies in the image of the restriction, the Euler-Poincaré function $f^{M_1}_{\tau_1}$ for the representation τ_1 exists. Further for M_2 the function $g^{M_2}_{\tau_2}$ of Theorem 2.2.2 in [9] exists. We set $h_\tau(m_1, m_2) := f^{M_1}_{\tau_1}(m_1) g^{M_2}_{\tau_2}(m_2)$ and this function factors over M. Then for any function η on M which is a product $\eta = \eta_1 \otimes \eta_2$ on $M_1 \times M_2$ we have for the orbital integrals:

$$\mathcal{O}^M_m(\eta) = \mathcal{O}^{M_1}_{m_1}(\eta_1)\mathcal{O}^{M_2}_{m_2}(\eta_2).$$

With this in mind it is straightforward to see that the proof of Theorem 4.13 proceeds as the proof of Theorem 2.1 with the Euler-Poincaré function f_τ replaced by the function h_τ. □

Extend the definition of $Z^0_{P,\tau,\omega}(s)$ to arbitrary virtual representations in the following way. Consider a finite dimensional virtual representation $\xi = \oplus_i a_i \tau_i$ with $a_i \in \mathbb{Z}$ and $\tau_i \in \hat{K_M}$. Then let $Z^0_{P,\xi,\omega}(s) = \prod_i Z^0_{P,\tau_i,\omega}(s)^{a_i}$.

Theorem 4.14 *Assume* Γ *is neat, then for* $\lambda >> 0$ *we have the identity*

$$\prod_{q=0}^{\dim_{\mathbb{C}} X} \left(\frac{\det(\Delta_{0,q,\omega}+\lambda)}{\det^{(2)}(\Delta_{0,q,\omega}+\lambda)}\right)^{q(-1)^{q+1}} =$$

$$\prod_{P/\mathrm{conj.}} \prod_{c\geq 0} \prod_{i\in I_c} Z_{P,\sigma_i^c\otimes\tau_i^c,\omega}(|\rho_P| + \sqrt{\lambda + s_i^c})^{(-1)^c}$$

Proof: Consider Theorem 4.12. For any semipositive elliptic differential operator D_Γ the heat trace $\mathrm{tr} e^{-tD_\Gamma}$ has the same asymptotic as $t \to 0$ as the L^2-heat trace $\mathrm{tr}_\Gamma e^{-tD}$. Thus it follows that the function

$$h(t) := \sum_{q=0}^{\dim_{\mathbb{C}} X_\Gamma} q(-1)^{q+1} (\mathrm{tr} e^{-t\Delta_{0,q,\omega}} - \mathrm{tr}_\Gamma e^{-t\Delta_{0,q,\omega}})$$

is rapidly decreasing at $t = 0$. Therefore, for $\lambda > 0$ the Mellin transform of $h(t)e^{-t\lambda}$ converges for any value of s and gives an entire function. Let

$$\zeta_\lambda(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} h(t) e^{-t\lambda} dt.$$

We get that

$$\exp(-\zeta_\lambda'(0)) = \prod_{q=0}^{\dim_{\mathbb{C}} X} \left(\frac{\det(\Delta_{0,q,\omega} + \lambda)}{\det^{(2)}(\Delta_{0,q,\omega} + \lambda)} \right)^{q(-1)^{q+1}}.$$

On the other hand, Theorem 4.12 gives a second expression for $\zeta_\lambda(s)$. In this second expression we are allowed to interchange integration and summation for $\lambda >> 0$ since we already know the convergence of the Euler products giving the right hand side of our claim. □

Let n_0 be the order at $\lambda = 0$ of the left hand side of the last proposition. Then

$$n_0 = \sum_{q=0}^{\dim_{\mathbb{C}}(X)} q(-1)^q (h_{0,q,\omega} - h_{0,q,\omega}^{(2)}),$$

where $h_{0,q,\omega}$ is the $(0,q)$-th Hodge number of X_Γ with respect to ω and $h_{0,q,\omega}^{(2)}$ is the L^2-analogue. Conjecturally we have $h_{0,q,\omega}^{(2)} = h_{0,q,\omega}$, so $n_0 = 0$. For a splitrank one Cartan H, for $c \geq 0$ and $i \in I_c$ we let

$$n_{P,c,i,\omega} \ := \ \mathrm{ord}_{s=|\rho_P|+\sqrt{s_i^c}} Z_{P,\sigma_i^c \otimes \tau_i^c,\omega}(s)$$

so $n_{P,c,i,\omega}$ equals $(-1)^{\dim \mathfrak{n}}$ times

$$\sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) \sum_{p,q,r} (-1)^{p+q+r} \dim \left(H^q(\mathfrak{n}, \pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_{M_1} \otimes \wedge^r \mathfrak{p}_{M_2,-} \otimes V_{\breve{\sigma} \otimes \breve{\tau}} \right)^{K_M},$$

for $\lambda(H) = |\rho_P| + \sqrt{s_i^c}$. We then consider

$$c(X_\Gamma, \omega) = \prod_P \prod_{c \geq 0,\ i \in I_c} \left(2\sqrt{s_i^c}\right)^{(-1)^c n_{P,c,i,\omega}}.$$

We assemble the results of this section to

Theorem 4.15 *Let*

$$Z_\omega(s) = \prod_{P/\text{conj.}} \prod_{c \geq 0} \prod_{i \in I_c} Z_{P,\sigma_i^c \otimes \tau_i^c,\omega}\left(s + |\rho_P| + \sqrt{s_i^c}\right),$$

then Z_ω extends to a meromorphic function on the plane. Let n_0 be the order of Z_ω at zero then

$$n_0 = \sum_{q=0}^{\dim_{\mathbb{C}}(X)} q(-1)^q \left(h_{0,q}(X_\Gamma) - h_{0,q}^{(2)}(X_\Gamma)\right),$$

where $h_{p,q}(X_\Gamma)$ is the (p,q)-th Hodge number of X_Γ and $h_{p,q}^{(2)}(X_\Gamma)$ is the (p,q)-th L^2-Hodge number of X_Γ. Let $R_\omega(s) = Z_\omega(s)s^{-n_0}/c(X_\Gamma, \omega)$ then

$$R_\omega(0) = \frac{T_{hol}(X_\Gamma, \omega)}{T_{hol}^{(2)}(X_\Gamma)^{\dim \omega}}.$$

Proof: This follows from Theorem 4.14. □

References

[1] **Atiyah, M.; Schmid, W.:** *A Geometric Construction of the Discrete Series for Semisimple Lie Groups.* Invent. math. 42, 1-62 (1977).

[2] **Barbasch, D.; Moscovici, H.:** *L^2-Index and the Selberg Trace Formula.* J. Func. An. 53, 151-201 (1983).

[3] **Berline, N.; Getzler, E.; Vergne, M.:** *Heat Kernels and Dirac Operators.* Grundlehren 298. Springer 1992.

[4] **Borel, A.:** *Introduction aux groupes arithmétiques.* Hermann, Paris 1969.

[5] **Borel, A.; Wallach, N.:** *Continuous Cohomology, Discrete Groups, and Representations of Reductive Groups.* Ann. Math. Stud. 94, Princeton 1980.

[6] **Bunke, U.; Olbrich, M.:** Γ*-Cohomology and the Selberg Zeta Function* J. reine u. angew. Math. 467, 199-219 (1995).

[7] **Clozel, L.; Delorme, P.:** *Le théorème de Paley-Wiener invariant pour les groupes de Lie reductifs II.* Ann. sci. Éc. Norm. Sup. (4) 23, 193-228 (1990).

[8] **Deitmar, A.:** *Higher torsion zeta functions.* Adv. Math. 110, 109-128 (1995).

[9] **Deitmar, A.:** *A higher rank Lefschetz formula.* J. Fixed Point Theory Appl. 2, 1-40 (2007).

[10] **Deninger, C.:** *A dynamical systems analogue of Lichtenbaum's conjectures on special values of Hasse-Weil zeta functions.* http://arxiv.org/abs/math/0605724

[11] **Duistermaat, J.J.; Kolk, J.A.C.; Varadarajan, V.S.:** *Spectra of locally symmetric manifolds of negative curvature.* Invent. math. 52 (1979) 27-93.

[12] **Fried, D.:** *Analytic torsion and closed geodesics on hyperbolic manifolds.* Invent. math. 84, 523-540 (1986).

[13] **Fried, D.:** *Torsion and closed geodesics on complex hyperbolic manifolds.* Invent. Math. 91, 31-51 (1988).

[14] **Goodman, F.M.; de la Harpe, P.; Jones, V.F.R.:** *Coxeter Graphs and Towers of Algebras.* Springer 1989.

[15] **Gromov, M.; Shubin, M.A.:** *Von Neumann Spectra near Zero.* GAFA 1, 375-404 (1991).

[16] **Harish-Chandra:** *Harmonic analysis on real reductive groups I. The theory of the constant term.* J. Func. Anal. 19 (1975) 104-204.

[17] **Hecht, H.; Schmid, W.:** *Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules.* Acta Math. 151, 49-151 (1983).

[18] **Juhl, A.:** *Zeta-Funktionen, Index-Theorie und hyperbolische Dynamik.*
Habilitationsschrift. Humboldt-Universität zu Berlin 1993.

[19] **Juhl, A.:** *Cohomological theory of dynamical zeta functions.* Progress in Mathematics, 194. Birkhäuser Verlag, Basel, 2001.

[20] **Knapp, A.:** *Representation Theory of Semisimple Lie Groups.* Princeton University Press 1986.

[21] **Labesse, J.P.:** *Pseudo-coefficients très cuspidaux et K-théorie.* Math. Ann. 291, 607-616 (1991).

[22] **Lichtenbaum, S.:** *The Weil-etale topology on schemes over finite fields.* Compos. Math. 141, 689–702, (2005).

[23] **Lott, J.:** *Heat kernels on covering spaces and topological invariants.* J.Diff. Geom. 35, 471-510 (1992)

[24] **Lott, J.; Lück, W.:** *L^2-Topologicla Invariants of 3-Manifolds.* Invent. Math. 120, 15-60 (1995).

[25] **Moscovici, H.; Stanton, R.:** *Eta invariants of Dirac operators on locally symmetric manifolds.* Invent. math. 95, 629-666 (1989).

[26] **Moscovici, H.; Stanton, R.:** *R-torsion and zeta functions for locally symmetric manifolds.* Invent. math. 105, 185-216 (1991).

[27] **Ray, D.; Singer, I.:** *R-torsion and the Laplacian on Riemannian manifolds.* Adv. Math. 7, 145-220 (1971).

[28] **Soulé, C.; Abramovich, D.; Burnol, J.-F.; Kramer, J.:** *Lectures on Arakelov Geometry.* Cambridge University Press 1992.

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen

Germany
deitmar@uni-tuebingen.de

7.1.2010