The Selberg Trace Formula and the Ruelle Zeta Function
for Compact Hyperbolics

By A. Deitmar

We use a result of D. Fried [6, p. 529] to derive a trace formula of Selberg type
for differential forms on a compact oriented hyperbolic manifold. This yields the
analytic continuation of the corresponding Ruelle Zeta Function, its functional
equation and the location of its poles and zeroes. It turns out that the lengths of
the closed geodesics determine the Eigenvalues of the Laplacian on differential
forms and esp. the Betti numbers.

0. Notation

Let $X$ denote a compact oriented hyperbolic manifold of dimension $d \geq 2$,
$I$ its fundamental group. The universal covering manifold of $X$ is the hyperbolic
space $H_d$ of dimension $d$ which can be written as $G/K$ with $G = SO^+(d, 1)$ and
$K = SO(d)$, see [1].

We write the Iwasawa decomposition of $G$ as $G = NAK$. The centralizer $M$
of $A$ in $K$ is isomorphic to $SO(d - 1)$. For $\sigma \in \hat{M}$ and $x \in \mathbb{R}$ let $\pi_{\sigma, x}$ denote the
principal series representation of $G$ as in [2, p. 177]. The Plancherel measure on
$\{\sigma\} \times \mathbb{R}$ has a smooth density $P_\sigma$ with respect to the Lebesgue measure on $\mathbb{R}$.
The function $P_\sigma$ is a polynomial of degree $d - 1$ if $d$ is odd and a polynomial $\tilde{p}_\sigma$
of degree $d - 1$ times $\tanh px$ if $d$ is even, see [3]. Let $\sigma_j$ denote the representation
of $M \cong SO(d - 1)$ on $\wedge^j \mathbb{C}^{d-1}$. We write $P_j := P_{\sigma_j}$ and $\tilde{P}_j := \tilde{P}_{\sigma_j}$.

For an unitary representation $\chi$ of $I$ on $\mathbb{C}^m$, a $\chi$-twisted $j$-form is a $j$-form $\omega$
on $H_d$ with values in $\mathbb{C}^m$ that satisfies $\gamma^* \omega = \chi(\gamma) \omega$ for $\gamma \in I$. For trivial $\chi$
and $m = 1$ the $\chi$-twisted $j$-forms may be viewed as the $j$-forms on $X$.

For a closed geodesic $\gamma$ in $X$ we write $m_\gamma$ for the holonomy map of the parallel
displacement along $\gamma$.

We write $\Delta = d\delta + \delta d$ for the Laplacian on forms and $\Delta_j = d\delta_j + \delta d_j$
for its restriction to $\chi$-twisted $j$-forms. By $\alpha_n^j$ we denote the sequence of all nonvanishing
Eigenvalues of $d\delta_j$, counted with multiplicities and by $\beta_n^j$ the sequence of all
nonvanishing Eigenvalues of $d\delta_j$, also counted with multiplicities. These Eigen-
values are all positive and may be enumerated in ascending order. Further on let $b_j$
denote the multiplicity of the Eigenvalue zero of $\Delta_j$, i.e. the twisted Betti num-
ber. Looking how the outer differential $d$ and the Hodge star operator act on
Eigenforms gives
\[ \alpha_n^j = \beta_n^{j+1}, \quad j = 0, \ldots, d - 1, \]
\[ \alpha_n^{d-j} = \alpha_n^{j-1}, \quad j = 1, \ldots, d. \]
We also will need the alternating sum of Betti numbers
\[ \widetilde{b}_j := \sum_{\nu=0}^{j} (-1)^{\nu+j} b_{\nu}, \]
and the number \[ c_j := j - \frac{d - 1}{2}. \]

1. Results

Our first result is a trace formula of Selberg type [5, p. 74].

**Theorem 1.** Let \( \varepsilon > 0 \) and \( h \) be a holomorphic function in the strip \( \left\{ |\text{Im } z| < \frac{d - 1}{2} + \varepsilon \right\} \). Let \( h \) be even and satisfy \( h(z) = O(|z|^{-d-\varepsilon}) \) then we have with absolute convergence of all sums and integrals
\[
\begin{align*}
    h(ic_j) \widetilde{b}_j &+ \sum_n h \left( \sqrt{\alpha_n^{j} - c_j^2} \right) \\
    &= m \left( \frac{d - 1}{j} \right) \text{vol } (X) \int_{-\infty}^{\infty} h(x) P_j(x) \, dx + \sum_{\gamma} a_{\gamma} e^{-\frac{d - 1}{2} l(\gamma)} \hat{h}(l(\gamma))
\end{align*}
\]
where \( \hat{h}(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} h(s) s^{j+u} \, ds \), the sum runs over all closed geodesics in \( X \), \( l(\gamma) \) denotes the length of \( \gamma \) and
\[
a_{\gamma} = \text{tr} \chi(\gamma) \cdot \text{tr} (\sigma_{\gamma}(m_{\gamma})) \frac{l(\gamma_0)}{\det (1 - m_{\gamma} e^{-l(\gamma)})}. \]

The closed geodesics \( \gamma \) are in a \( 1-1 \) correspondence to the nontrivial conjugacy classes in \( \Gamma \). That is why one may write \( \text{tr} \chi(\gamma) \) for a geodesic \( \gamma \). The holonomy maps also may be considered as elements of \( M \) and \( \gamma_0 \) is the primitive closed geodesic underlying \( \gamma \).

The next result is the Weyl Asymptotic law.

**Theorem 2.** Define \( A_j(x) := \# \{ n \in \mathbb{N} : \alpha_n^j \leq x \} \) for \( x > 0 \). We have for \( j = 0, \ldots, d - 1 \)
\[ A_j(x) \sim C(d,j) m \left( \frac{d - 1}{j} \right) \text{vol } (X) \sqrt{x}^d, \quad x \to \infty \]
with
\[ c(d, j) = \frac{2\Gamma \left( \frac{d+1}{2} \right)^2}{\Gamma(d+1)(d-1)\pi^{(d-1)}} \quad \text{for } d \text{ odd} \]
and
\[ c(d, j) = \frac{2(d-1)^2}{d(d+1-2(j+1))2^d\pi^{d+2}\Gamma \left( \frac{d}{2} \right)} \quad \text{for } d \text{ even, } j < \frac{d}{2} \]
and
\[ d(c, j) = c(d, d-1-j) \quad \text{for } d \text{ even, } j \geq \frac{d}{2}. \]

To formulate our next result we need the Ruelle Zeta Function \( R \) defined for \( \Re s > d-1 \) by
\[ R(s) := \prod_\gamma \det (1_m - \chi(\gamma) e^{-s\ell(\gamma)}). \]
The product is taken over all prime geodesics of \( X \), where only one of \( \gamma \) and \( \gamma' \) is taken when they are equivalent.

**Theorem 3.** The function \( R \) extends meromorphically to the entire plane and satisfies the functional equation
\[ R(-z) = R(z)^{(-1)^d+1} f(z) \]
with
\[ f(z) = \exp \sum_{j=0}^{d-1} 4\pi m \left( \frac{d-1}{j} \right) \vol(X) \int_0^{\frac{z+\frac{d-1}{2}-j}{2}} P_j(iw) \, dw. \]
The poles and zeroes of \( R \) split into two classes: the topological poles and zeroes and the trivial poles and zeroes. The topological ones are situated as follows: The poles are at \( \pm i(\alpha_n^j - \tilde{c}_j)^k - c_j \) for \( j = 0, 2, 4, \ldots, <d \) with order equals the multiplicity of the eigenvalue \( \alpha_n^j \), if \( \alpha_n^j \neq \tilde{c}_j \) and twice this multiplicity if \( \alpha_n^j = \tilde{c}_j \). The zeroes are at \( \pm i(\alpha_n^j - \tilde{c}_j)^k - c_j \) for \( j = 1, 3, 5, \ldots <d \) with order equals the multiplicity \( \ldots \) (as above). At the real line it may happen that some of these points coincide. Since this requires \( \alpha_n^j \leq \tilde{c}_j \) this only happens finitely often. For \( d \) odd these are all poles and zeroes of \( R \).

For even \( d \) we have additional (trivial) zeroes or poles at the numbers \( z = k \in \mathbb{Z}, k < \frac{d-1}{2} \) of (vanishing-) order
\[ 4i \sum_{j=\frac{k}{2}-h-1}^{d-1} (-1)^j m \left( \frac{d-1}{j} \right) \vol(X) \tilde{P}_j(i(k - c_j)). \]

As consequence we conclude that the lengths of the closed geodesics determine all Eigenvalues \( \alpha_n^j \) and their multiplicites and viz. This statement is a so called Huber Theorem.
2. Proofs

In [6] it was shown, by computing the trace of the heat kernel, that for $t > 0$

$$b_j + \sum_n e^{-t \alpha_n^j} + e^{-t \beta_n^j} = T_t(\sigma_j) + T_t(\sigma_{j-1})$$

where

$$T_t(\sigma_j) := m \binom{d-1}{j} \vol(X) \int_\infty^{-\infty} e^{-t(x^2 + c_j^2)} P_j(x) \, dx$$

$$+ \sum_\nu a_\nu \frac{e^{-\frac{d-1}{2} t (\nu)}}{\sqrt{\pi t}} e^{-t(\nu^2/4t) - \nu^2 j^2}$$

which is the special case $h(s) = \exp(-s^2 + c_j^2)$ of Theorem 1. Using $\alpha_n^j = \beta_n^{j+1}$ we get

(*)

$$\tilde{b}_j + \sum_n e^{-t \alpha_n^j} = T_t(\sigma_j).$$

At first we derive Theorem 2. For $t > 0$ let $N(t)$ denote the number of closed geodesics on $X$ of length $\leq t$. Let also $P(t)$ denote the number of prime closed geodesics on $X$ of length $\leq 1$. From [8, p. 693] we take (Geodesic prime number theorem)

$$P(t) \sim N(t) \sim \frac{e^{(d-1)t}}{(d-1)t} \quad \text{for } t \to \infty.$$  

This makes the geodesic terms in $T_t(\sigma_j)$ stay bounded as $t \to 0$. Hence the growth of $T_t(\sigma_j)$ as $t \to 0$ only depends on the leading term of $P_j$. This leading term is got by means of [3]. The constant in [3] can be eliminated by means of the fact that in the case $j = 0$ the above formula must coincide with the classical Selberg Trace Formula. After these computations Theorem 2 is got by means of a Tauberian Theorem [7, p. 421]. □

Let $\varphi(t)$ denote the right hand side of (*) and define for complex $s$ the function $g_s(t) := \varphi(t) \exp(t(s^2 + c_j^2))$. For $\text{Im} \ s > \frac{d-1}{2}$ we have

$$f(s) := \int_\infty^t g_s(t) \, dt = -\tilde{b}_j \frac{e^{s^2 + c_j^2}}{s^2 + c_j^2} + \sum_n \frac{e^{-s_n^j + s^2 + c_j^2}}{\alpha_n^j - s^2 - c_j^2}$$

$$= m \binom{d-1}{j} \vol(\gamma) \int_\infty^{-\infty} \frac{e^{-x^2 + s^2}}{x^2 - s^2} P_j(x) \, dx$$

$$+ \sum_\nu a_\nu e^{-\frac{d-1}{2} t (\nu)} \frac{1}{2\pi} \int_\infty^{-\infty} \frac{e^{-x^2 + s^2}}{x^2 - s^2} e^{-i(\nu)x} \, dx.$$
Now we take $h$ as in the assumptions of Theorem 1 but with the stronger growth condition $h(z) = O(\exp(-\alpha |z|^4))$ for some $\alpha > 0$. For $T > 0$ let $\gamma_T$ denote the positive oriented boundary of the rectangle with vertices $\pm T \pm i\left(\frac{d-1}{2} + \frac{\varepsilon}{2}\right)$. Computing $\lim_{T \to \infty} \int_{\gamma_T} f(s) s h(s) \, ds$ gives the right hand side of the claim in Theorem 1 by the argument principle. For the other side leave out the vertical edges of $\gamma_T$ and interchange order of integration. This proves Theorem 1 for $h$ satisfying the stronger growth condition. For arbitrary $h$, $\alpha > 0$ let $h_\alpha(s) := h(s) \exp(-\alpha s^4)$. The function $h_\alpha$ fits into the Trace Formula and the limit $\alpha \to 0$ gives Theorem 1.

Now let $b_1, \ldots, b_d$ be pairwise distinct complex numbers with real part bigger than $\frac{1}{2}(d - 1)$. Applied to the function

$$h_b(s) := \prod_{k=1}^{d} \frac{1}{s^2 + b_k^2}$$

the Trace Formula gives

$$\prod_{k} \frac{b_k^2}{b_k^2 + b_k} \sum_{n} \prod_{k} (b_k^2 + \alpha_n^2 - c_j^2)^{-1}$$

$$= m \binom{d-1}{j} \text{vol}(X) \int_{-\infty}^{\infty} h_b(x) P_j(x) \, dx$$

$$+ \binom{d}{j} \frac{1}{2} \sum_{k=1}^{d} \prod_{l=1}^{d} (b_k^2 - b_l^2)^{-1} \frac{1}{b_k} \mathcal{E}_j \left( b_k + \frac{d-1}{2} \right).$$

Where $\mathcal{E}_j$ denotes Selbergs Xi-Function:

$$\mathcal{E}_j(z) := \sum_{\gamma} a_\gamma e^{-z(\gamma)}.$$

We conclude that the function $\mathcal{E}_j \left( z + \frac{d-1}{2} \right)$ has simple poles at $z = \pm c_j$, $z = \pm i(\alpha_n^j - c_j^2)^{\pm}$ of residues

$$\text{Res}_{z = \pm c_j} \mathcal{E}_j \left( z + \frac{d-1}{2} \right) = \begin{cases} \tilde{b} & \text{if } c_j \neq 0, \\ z\tilde{b} & \text{if } c_j = 0, \end{cases}$$

and

$$\text{Res}_{z = \pm \sqrt{\alpha_n^j - c_j^2}} \mathcal{E}_j \left( z + \frac{d-1}{2} \right) = \begin{cases} \text{mult } (\alpha_n^j) & \text{if } \alpha_n^j \neq c_j^2, \\ 2 \text{ mult } (\alpha_n^j) & \text{if } \alpha_n^j = c_j^2. \end{cases}$$
In case of even $d$ we have additional poles at the points $z = k + \frac{1}{2}$ for $k = -1, -2, \ldots$ of residue

$$4 \, \text{im} \left( \frac{d - 1}{j} \right) \, \text{vol} (X) \tilde{P}_j \left( i \left( k + \frac{1}{2} \right) \right).$$

We further get the functional equation

$$\Xi_j \left( -z + \frac{d - 1}{2} \right) = -\Xi_j \left( z + \frac{d - 1}{2} \right) - 4m \left( \frac{d - 1}{j} \right) \, \text{vol} (X) \pi P_j (iz).$$

We define Selberg's Zeta Function by

$$Z_j (z) := \exp \left( - \sum_{\gamma} a_{\gamma} \frac{e^{-z \ell(\gamma)}}{e(\gamma)} \right)$$

and get $\Xi_j = Z_j / Z_j$ hence the poles of $\Xi_j$ become zeroes and poles of $Z_j$ and

$$Z_j \left( -z + \frac{d - 1}{2} \right) = Z_j \left( z + \frac{d - 1}{2} \right) \exp \int_{0}^{z} 4 \text{im} \, \text{vol} (X) \left( \frac{d - 1}{j} \right) P_j (iw) \, dw.$$

A computation shows that $R(z) = \prod_{j=0}^{d-1} Z_j (Z + j)^{(-1)^j}$ and this gives Theorem 3. \qed

Remark. For odd $d$ we have $\tilde{b}_j = \tilde{b}_{d-1-j}$ and hence $Z_j = Z_{d-1-j}$. For even $d$ we get $Z_j = g Z_{d-j-1}$ with some rational $g$.

References


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Anschrift des Autors: ANTON DEITMAR, Mathematisches Institut der Westfälischen Wilhelms-Universität, Einsteinstr. 62, D-4400 Münster