Corrections for “Principles of Harmonic Analysis”

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We thank the following people for pointing out errors in the book: *Robert Burckel, Cody Gunton, Yi Li, Michael Mueger, Kenneth Ross, Christian Schmidt, Frank Valckenborgh.*

1 Lemma 2.6.2

(pointed out by Frank Valckenborgh)

At the end of the proof it is stated that one gets strict inequality of the integrands for $x = 1$, however, this is not the case. For $x = 1$ the integrands are indeed equal. The proof has to be replaced with the proof below.

Lemma 2.6.2 *Let G be a locally compact group, and let $\mathcal{A}$ be the Banach algebra $L^1(G)$. With the involution $f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}$ the algebra $\mathcal{A}$ is a Banach-*-algebra but not a C^*-algebra unless G is trivial, in which case $\mathcal{A} = \mathbb{C}$.*

Proof: The axioms of an involution are easily verified, and so is the fact that $\|f^*\| = \|f\|$ for $f \in \mathcal{A}$. For the last assertion, we need a sublemma.

Sublemma.

(a) Let X be a locally compact Hausdorff space and let $x_1, \dots, x_n \in X$ pairwise different points. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ any given numbers. Then there exists $f \in C_c(G)$ with $f(x_j) = \lambda_j$ for each $j = 1, \dots, n$.

(b) Let G be a locally compact group and $g \in C_c(G)$ with the property that $\left|\int_G g(y)\,dy\right| = \int_G |g(y)|\,dy$. Then there is $\theta \in \mathbb{T}$ such that $g(x) \in \theta[0,\infty)$ for every $x \in G$.

Proof: (a) By the Hausdorff property there are open sets $U_{i,j}$ and $V_{i,j}$ for $i \neq j$ with $x_i \in U_{i,j}$, $x_j \in V_{i,j}$ and $U_{i,j} \cap V_{i,j} = \emptyset$. Set $W_j = \bigcap_{j \neq i} U_{j,i} \cap V_{i,j}$. Then W_j is an open neighborhood of x_j and the sets $W_1, \dots, W_n$ are pairwise disjoint. By the Lemma of Urysohn there is $f_j \in C_c(X)$ with support in W_j and $f(x_j) = 1$. Set $f = \lambda_1 f_1 + \dots + \lambda_n f_n$. Then f satisfies the assertion of the lemma.

(b) Let g be given as in the assumption. If $\int_G g(x)\,dx = 0$, then $\int_G |g(x)|\,dx = 0$, so $g = 0$. So we assume $\int_G g(x)\,dx \neq 0$. Replacing g by λg for some $\lambda \in \mathbb{T}$, we can assume $\int_G g(x)\,dx > 0$. Then

$$\int_G |g| = \int_G g = \mathrm{Re}\left(\int_G g\right) = \int_G \mathrm{Re}(g).$$

So $\int_G (|g| - \mathrm{Re}(G)) = 0$. Since the continuous function $|g| - \mathrm{Re}(G)$ is positive, it vanishes, hence $|g| = \mathrm{Re}(g)$, which means $g \geq 0$. $\square$

To show the lemma assume that $L^1(G)$ is a C^*-algebra. We show that $G = \{1\}$. By the C^*-property one has for every $f \in C_c(G)$ that

$$\begin{aligned}
\int_G \left| \int_G \Delta(x^{-1}y) f(y) \overline{f(x^{-1}y)}\,dy \right| dx &= \|f * f^*\| = \|f\|^2 \\
&= \int_G \int_G \Delta(x^{-1}y) |f(y)\overline{f(x^{-1}y)}|\,dy\,dx.
\end{aligned}$$

The outer integrals on both sides have continuous integrands ≥ 0. The integrands satisfy the inequality $\leq$. As the integrals are equal, the integrands are equal, too. So for every $x \in G$ we have

$$\left| \int_G \Delta(x^{-1}y) f(y) \overline{f(x^{-1}y)}\,dy \right| = \int_G \Delta(x^{-1}y) |f(y)\overline{f(x^{-1}y)}|\,dy.$$

By the Sublemma there is, for given $x \in G$, a $\theta \in \mathbb{T}$ such that $f(y)\overline{f(x^{-1}y)} \in \theta[0,\infty)$ for every $y \in G$.

Assume now that G is non-trivial. Thene there is $x_0 \neq 1$ in G. By the sublemma there is a function $f \in C_c(G)$ with $f(x_0) = f(x_0^{-1}) = i$ and

$f(1) = 1$. For $x = x_0$ we deduce

$$y = 1 \Rightarrow f(y)\overline{f(x^{-1}y)} = f(1)\overline{f(x_0^{-1})} = -i,$$
$$y = x_0 \Rightarrow f(y)\overline{f(x^{-1}y)} = f(x_0)\overline{f(1)} = i.$$

This is a contradiction! Therefore G is the trivial group. □

2 Lemma 3.2.2

(pointed out by Christian Schmidt)

The proof of the Lemma is a bit garbled at the end. Replace it with the proof below.

Lemma 3.2.2. *Let $\chi_0 \in \widehat{A}$. Let K be a compact subset of A, and let $\varepsilon > 0$. Then there exist $l \in \mathbb{N}$, functions $f_0, f_1, \dots, f_l \in L^1(A)$, and $\delta > 0$ such that for $\chi \in \widehat{A}$ the condition*

$$|\hat{f}_j(\chi) - \hat{f}_j(\chi_0)| \ < \ \delta \quad \textit{for every } j = 0, \dots, l$$

implies

$$|\chi(x) - \chi_0(x)| \ < \ \varepsilon \quad \textit{for every } x \in K.$$

Proof: For $f \in L^1(A)$ we have

$$\begin{aligned} \hat{f}(\chi) - \hat{f}(\chi_0) &= \int_A f(x)\overline{(\chi(x) - \chi_0(x))}\, dx \\ &= \int_A f(x)\overline{\chi_0(x)}\,(\overline{\chi(x)}\chi_0(x) - 1)\, dx \\ &= \widehat{f\bar{\chi}_0}(\chi\bar{\chi}_0) - \widehat{f\bar{\chi}_0}(1). \end{aligned}$$

So without loss of generality we can assume $\chi_0 = 1$.

Let $f \in L^1(A)$ with $\hat{f}(1) = \int_A f(x)\, dx = 1$. Then there is a unit-neighborhood U in A with $\|L_u f - f\|_1 < \varepsilon/3$ for every $u \in U$. As K is compact, there are finitely many $x_1, \dots, x_l \in A$ such that K is a subset of $x_1 U \cup \dots \cup x_l U$. Set $f_j = L_{x_j} f$ as well as $f_0 = f$ and let $\delta = \varepsilon/3$. Let $\chi \in \widehat{A}$ with

$$|\hat{f}_j(\chi) - 1| \ < \ \varepsilon/3 \quad \text{for every } j = 0, \dots, l.$$

Now let $x \in K$. Then there exists $1 \leq j \leq l$ and $u \in U$ such that $x = x_j u$. One gets

$$\begin{aligned} |\chi(x) - 1| &= |\overline{\chi(x)} - 1| \\ &\leq |\overline{\chi(x)} - \overline{\chi(x)}\hat{f}(\chi)| + |\hat{f}(\chi)\overline{\chi(x)} - \hat{f}_j(\chi)| + |\hat{f}_j(\chi) - 1| \\ &= |1 - \hat{f}(\chi)| + |\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| + |\hat{f}_j(\chi) - 1| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

where the last inequality uses

$$\begin{aligned} |\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| &\leq \|L_x f - L_{x_j} f\|_1 \\ &= \|L_{x_j}(L_u f - f)\|_1 \\ &= \|L_u f - f\|_1 < \varepsilon/3. \end{aligned}$$

The lemma and the theorem are proven. □

3 Corollary 6.1.9

In the proof of that corollary in the displayed formula the η has to change to π from the second line on, so the displayed formular should be this

$$\begin{aligned} \langle v, T^*\eta(g)w \rangle &= \langle Tv, \eta(g)w \rangle = \left\langle \eta(g^{-1})Tv, w \right\rangle \\ &= \left\langle T\pi(g^{-1})v, w \right\rangle = \left\langle \pi(g^{-1})v, T^*w \right\rangle \\ &= \langle v, \pi(g)T^*w \rangle . \end{aligned}$$

4 Proposition 6.2.3

(pointed out by Robert Burckel and Kenneth Ross)

In the proof of Proposition 6.2.3 the representation $\tilde{\pi}$ is defined by

$$\tilde{\pi}(x)\pi(f)v = \pi(L_x f)v.$$

However, it is not clear why the right hand side is unambiguously determined by the left hand side. So we replace the proof by the following refined proof.

Proof of Proposition 6.2.3. Note first that π is continuous by Lemma 2.7.1. We want define an operator $\tilde{\pi}(x)$ on the dense subspace $\pi(L^1(G))V$ of V. This space is made up of sums of the form $\sum_{i=1}^n \pi(f_i)v_i$ for $f_i \in L^1(G)$ and $v_i \in V$. We propose to define

$$\tilde{\pi}(x)\sum_{i=1}^n \pi(f_i)v_i \stackrel{\text{def}}{=} \sum_{i=1}^n \pi(L_x f_i)v_i.$$

We have to show well-definedness, which amounts to show that if $\sum_{i=1}^n \pi(f_i)v_i = 0$, then $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$ for every $x \in G$. For $x \in G$ and $f, g \in L^1(G)$ a short computation shows that $g^* * (L_x f) = (L_{x^{-1}}g)^* * f$. Based on this, we compute for $v, w \in V$ and $f_1, \dots, f_n \in L^1(G)$,

$$\begin{aligned}\left\langle \sum_{i=1}^n \pi(L_x f_i)v, \pi(g)w \right\rangle &= \sum_{i=1}^n \left\langle \pi\big(g^* * (L_x f_i)\big)v, w\right\rangle \\ &= \sum_{i=1}^n \langle \pi((L_{x^{-1}}g)^* * f_i)v, w\rangle \\ &= \left\langle \sum_{i=1}^n \pi(f_i)v, \pi(L_{x^{-1}}g)w \right\rangle.\end{aligned}$$

Now for the well-definedness of $\tilde{\pi}(x)$ assume $\sum_{i=1}^n \pi(f_i)v_i = 0$, then the above computation shows that the vector $\sum_{i=1}^n \pi(L_x f_i)v_i$ is orthogonal to all vectors of the form $\pi(g)w$, which span the dense subspace $\pi(L^1(G))V$, hence $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$ follows. The computation also shows that this, now well-defined operator $\tilde{\pi}(x)$ is unitary on the space $\pi(L^1(G))V$ and since the latter is dense in V, the operator $\tilde{\pi}(x)$ extends to a unique unitary operator on V with inverse $\tilde{\pi}(x^{-1})$, and we clearly have $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ for all $x, y \in G$. Since for each $f \in L^1(G)$ the map $G \to L^1(G)$ sending x to $L_x f$ is continuous by Lemma 1.4.2, it follows that $x \mapsto \tilde{\pi}(x)v$ is continuous for every $v \in V$. Thus $(\tilde{\pi}, V)$ is a unitary representation of G.

It remains to show that $\pi(f)$ equals $\tilde{\pi}(f)$ for every $f \in L^1(G)$. By continuity it is enough to show that $\langle \tilde{\pi}(f)\pi(g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle$ for all $f, g \in C_c(G)$ and $v, w \in V$. Since $g \mapsto \langle \pi(g)v, w\rangle$ is a continuous linear functional on

$L^1(G)$ we can use Lemma B.6.5 to get

$$\begin{aligned}
\langle \tilde{\pi}(f)\pi(g)v, w\rangle &= \int_G f(x) \langle \tilde{\pi}(x)(\pi(g)v), w\rangle \, dx \\
&= \int_G \langle \pi(f(x)L_x g)v, w\rangle \, dx \\
&= \left\langle \pi \left(\int_G f(x)L_x g \, dx \right) v, w \right\rangle \\
&= \langle \pi(f * g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle \, ,
\end{aligned}$$

which completes the proof. □

5 Theorem B.1.5

The monotone convergence Theorem is formulated mistakenly with the conclusion that the function be integrable. This need not be the case. The following is the correct formulation.

Theorem 5.1 (Monotone Convergence Theorem)
Let $(f_n)_{n\in\mathbb{N}}$ be a pointwise monotonically increasing sequence of positive integrable functions. For $x \in X$ let $f(x) = \lim_n f_n(x)$ (with possible value ∞). Then and one has

$$\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu,$$

where ∞ is a possible value on both sides.

6 Typos and smaller errors

(page,line)

7,1 The sentence should be: *Note that in a locally compact Husdorff space every point has a neighborhood consisting of compact sets.* The next sentence can be deleted.

(8,6) Write $(X, \mathcal{A}, \mu)$ instead of $(X, \mathcal{A}, \sigma)$.

(11,5) Write $|f(x)| > 1/n$ instead of $f(x) > 1/n$.

(11,-5) The text says: 'Let $x, y \in G$.' Instead it should read: 'Let $x, y \in G$ with $x^{-1}y \in U$.'

(13,-12) Write $J(g, \varphi)$ instead of $J(h, \varphi)$.

(14,-1) Write $(J(f, \varphi))_f$ instead of $J(f, \varphi)$.

(15,5) Write 'any real function' instead of 'any function'.

(15,15) Write 'neighborhood' instead of 'unit-neighborhood'.

(22,3) It should be f^H and not f^h.

(27,3) There are double integrals on both sides of the equality.

(39,1) Write $|\lambda| > 2\|a\|$ instead of $\lambda \neq 0$.

(56,10) The domain of the map Ψ is $C(\sigma_B(a))$.

(71,7) The "=" is missing in $L(f^*) = L(f)^*$.

(74,-40 Write: $g \in C_0(\hat{A}) \cong C^*(A)$ with $g \geq 0$ and $g^2 = \hat{f}$.

(83,9) Write $\|f\|_2^2$ instead of $\|f\|_2^*$.

(86,10) Remove spurious "$z =$".

(86,-2) Write "generates the topology" instead of "is a basis of the topology".

(94,-15) Replace "locally compact space" by "locally compact Hausdorff space".

(130,3) Replace "dense in G" by "dense in V_π".

(131,14) The dual $\hat{G}$ is defined as the set of all isomorphism classes of irreducible *unitary* representations.

(144,-8) Instead of $\mathrm{tr}(ST^*)$ it should be

$$\dim(\tau)\mathrm{tr}(ST^*).$$

(147,7) isometrically

(148,4) It should be

$$P = \dim(\tau) \int_K \overline{\chi_\tau(x)} \pi(x)\, dx.$$

(154,6) Replace 10.2.1 by 10.2.2.

(244,15) The first V_π should be D_π.

(244,17) The second η is η', so it should be $\langle W_\xi(\eta), W_{\xi'}(\eta')\rangle$.

(244,20) Same.

(253,-8) From this line on, some of the x's have to be replaced with t's. I give you the rest of the example as it should be:

$$W_\varphi(\psi)(t,y) = \int_{\mathbb{R}} \psi(x)\overline{\varphi(x-y)} e^{-2\pi i t x}\, dx.$$

... The group may be realized as the semi-direct product $(\mathbb{T}\times\mathbb{R}) \rtimes \mathbb{R}$ with action of $y \in \mathbb{R}$ on a pair $(z,t) \in \mathbb{T}\times\mathbb{R}$ given by

$$x(z,t) = (ze^{-2\pi i y t}, t).$$

The dual group $\widehat{\mathbb{T}\times\mathbb{R}}$ of $\mathbb{T}\times\mathbb{R}$ is given by the characters $\chi_{(n,s)}$ indexed by $n \in \mathbb{Z}$ and $s \in \mathbb{R}$ such that

$$\chi_{(n,s)}(z,t) = z^n e^{-2\pi i s t}.$$

Let $\chi = \chi_{(1,0)}$, i.e., $\chi(z,t) = z$. Then the action of $\mathbb{R}$ on χ is given by

$$\begin{aligned} y\cdot\chi(z,t) &= \chi(-y\cdot(z,t)) = \chi(ze^{2\pi i t y}, t) \\ &= ze^{2\pi i t y} = \chi_{(1,-y)}(z,t). \end{aligned}$$

... by the formula

$$\big(\sigma(z,t,y)\varphi\big)(x) = ze^{2\pi i t x}\varphi(x-y).$$

The corresponding wavelet transform for a function $\psi \in L^2(\mathbb{R})$ is then given by

$$\big(W_\varphi(\psi)\big)(z,t,y) = \overline{z}\int_{\mathbb{R}} \psi(x) e^{-2\pi i t x}\overline{\varphi(x-y)}\, dx.$$

... and applying Fubini several times we get

$$\begin{aligned}
\|W_\varphi(\psi)\|_2^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{T}} |W_\varphi(\psi)(z,t,y)|^2 \, dz \, dt \, dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \psi(x)\overline{\varphi(x-y)} e^{-2\pi i t x} \, dx \right|^2 \, dt \, dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{F_y}(t)|^2 \, dt \, dy = \int_{\mathbb{R}}\int_{\mathbb{R}} |F_y(x)|^2 \, dx \, dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\psi(x)\overline{\varphi(x-y)}|^2 \, dx \, dy = \|\psi\|_2^2 \|\varphi\|_2^2,
\end{aligned}$$

(299,13-23) The integration domain should be X, not Ω.

(301,11) It is $f^{-1}(f(X) \setminus B_\delta(0))$.

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