Corrections for “Principles of Harmonic Analysis” 2nd Ed.

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We thank the following people for pointing out errors in the book: *Jose Getino, Finn Harring, Alexandros Kazantzidhs, Ehssan Khanmohammadi, Linus Kramer, Sven Raum, Niklas Rodenbücher, N.S. Seyedi, Arjun Sudan.*

Lemma 1.3.10 on page 11 (pointed out by Linus Kramer)
In the last line of the proof, it is not possible to let δ tend to zero, as the neighborhood V and thus the choice of possible ϕ's depend on δ. To fix this, one has to rearrange the Lemma and its proof as follows:

Lemma 1.3.10. *Let $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$. Then there is a unit-neighborhood V in G such that*

$$J(f_1, \phi) + J(f_2, \phi) \ < \ J(f_1 + f_2, \phi) + \varepsilon$$

holds for every $\phi \in C_c^+(G) \smallsetminus \{0\}$ with support in V.

Proof. Choose a function $f' \in C_c^+(G)$ such that $f' \equiv 1$ on the support of $f_1 + f_2$. Choose $\delta > 0$ such that

$$\delta(f_1 + f_2 : f_0) + (\delta + \delta^2)(f' : f_0) < \varepsilon.$$

Set

$$f = f_1 + f_2 + \delta f', \quad h_1 = \frac{f_1}{f}, \quad h_2 = \frac{f_2}{f},$$

where we set $h_j(x) = 0$ if $f(x) = 0$. Then $h_j \in C_c^+(G)$ for $j = 1, 2$.

According to Lemma 1.3.7, every function in $C_c(G)$ is left uniformly continuous, so in particular, for h_j this means that there is a unit-neighborhood V such that for $x, y \in G$ with $x^{-1}y \in V$ and $j = 1, 2$ one has $|h_j(x) - h_j(y)| < \delta/2$. Let $\phi \in C_c^+(G) \smallsetminus \{0\}$ with support in V. Choose finitely many $s_k \in G, c_k > 0$ with $f \le \sum_k c_k L_{s_k}\phi$. Then $\phi(s_k^{-1}x) \neq 0$ implies $|h_j(x) - h_j(s_k)| < \delta/2$, and for all x one has

$$\begin{aligned} f_j(x) = f(x)h_j(x) &\le \sum_k c_k\phi(s_k^{-1}x)h_j(x) \\ &\le \sum_k c_k\phi(s_k^{-1}x)\left(h_j(s_k) + \frac{\delta}{2}\right), \end{aligned}$$

so that $(f_j : \phi) \le \sum_k c_k \left(h_j(s_k) + \frac{\delta}{2}\right)$, implying that $(f_1 : \phi) + (f_2 : \phi)$ is less than or equal to $\sum_k c_k(1+\delta)$. By Lemma 1.3.9 we have $J(f', \phi) \le (f' : f_0)$ and so

$$\begin{aligned} J(f_1, \phi) + J(f_2, \phi) &\le J(f, \phi)(1+\delta) \le \left(J(f_1 + f_2, \phi) + \delta J(f', \phi)\right)(1+\delta) \\ &= J(f_1 + f_2, \phi) + \delta J(f_1 + f_2, \phi) + \delta J(f', \phi) + \delta^2 J(f', \phi) \\ &\le J(f_1 + f_2, \phi) + \delta(f_1 + f_2 : f_0) + (\delta + \delta^2)(f' : f_0) \\ &< J(f_1 + f_2, \phi) + \varepsilon. \end{aligned}$$

□

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Lemma 2.7.2. on page 55 (pointed out by Finn Harring):
The proof missed the argument, why the restriction homomorphism res is surjective, if res* is injective. Now if res* is injective, then res has dense image. The image being compact, it must be the complete image space, so res is surjective.

However, the proof doesn't actually need this argument at all, it even becomes simpler without, as the following formulation shows.

Let $\mathcal{A} \subset \mathcal{B}$ be unital C^*-algebras and let $\mathcal{A}^\times \subset \mathcal{B}^\times$ be their groups of invertible elements.

(a) *One has $\mathcal{A}^\times = \mathcal{A} \cap \mathcal{B}^\times$.*

(b) *For $a \in \mathcal{A}$ one has $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.*

Proof. Part (b) immediately follows from part (a) by the definition of the spectrum. Therefore it suffices to prove (a). The inclusion "$\subset$" is clear, so suppose $a \in \mathcal{A} \cap \mathcal{B}^\times$. We have to show that $a \in \mathcal{A}^\times$. In a first step assume that $\mathcal{A}$ and $\mathcal{B}$ are commutative. Restriction of multiplicative functionals defines a continuous map $\mathrm{res} : \Delta_\mathcal{B} \to \Delta_\mathcal{A}$. Define $\mathrm{res}^* : C(\Delta_\mathcal{A}) \to C(\Delta_\mathcal{B})$ by $\mathrm{res}^* f(m) = f(\mathrm{res}(m))$. We get a commutative diagram

$$\begin{tikzcd} C(\Delta_\mathcal{A}) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}^*"'] & \mathcal{A} \arrow[d, hook] \\ C(\Delta_\mathcal{B}). \arrow[r, "\cong"] & \mathcal{B}, \end{tikzcd}$$

whose horizontal arrows are isomorphisms by the Gelfand-Naimark theorem. As a is invertible in $\mathcal{B}$, its image in $C(\Delta_\mathcal{B})$ has no zeros, so its image in $C(\Delta_\mathcal{A})$ has no zeros, hence is invertible, so a is invertible in $\mathcal{A}$. Next assume $\mathcal{A}$ commutative, but $\mathcal{B}$ possibly not. Then for given $a \in \mathcal{A} \cap \mathcal{B}^\times$ there exists $b \in \mathcal{B}$ with $ab = ba = 1$. Then b commutes with a^* as $a^*b = baa^*b = ba^*ab = ba^*$. Similarly, $b = a^{-1}$ commutes with b^*, so that the C^*-algebra $\mathcal{C} = C^*(1, a, b)$ generated by 1, a and b is commutative. As $a \in \mathcal{C}^\times$, we get $a \in \mathcal{A}^\times$ by the first step. Finally, if both $\mathcal{A}$ is non-commutative, then we first consider the case when $a \in \mathcal{A} \cap \mathcal{B}^\times$ is normal. Then the C^*-algebra $C^*(1, a)$ is commutative and hence by the second step we have $a \in C^*(1, a)^\times \subset \mathcal{A}^\times$. Finally, for $a \in \mathcal{A}$ arbitrary, we use the simple fact that a is invertible if and only if aa^* and a^*a are invertible to deduce that $a \in \mathcal{B}^\times$ implies $a \in \mathcal{A}^\times$. $\square$

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Lemma 3.4.6. As Arjun Sudan pointed out, the proof of the lemma only works as give, if $\eta \geq 0$. Fortunately, the conclusions can be derived by decomposing η. So the proof needs the following preamble:

Writing $\eta = \eta_+ - \eta_-$ with non-negative functions $\eta_\pm \in C_c(\widehat{A})$ and $\mathrm{supp}(\eta_\pm) \subset \mathrm{supp}(\eta)$, one can replace η with $\eta_\pm$ and assume that $\eta \geq 0$.

* * *

Proposition 5.2.1 on page 112 (pointed out by Sven Raum):

The proof (b) $\Rightarrow$ (c) actually proves (a) $\Rightarrow$ (c). The proof of (b) $\Rightarrow$ (c) (or (b) $\Rightarrow$ (a)) is missing.

Here we give a correct proof of (b) $\Rightarrow$ (c): Let T be a bounded operator on the Hilbert space H. Then conditions (b) and (c) of Proposition 5.2.1 state the following:

(b) For every orthonormal sequence e_j the sequence Te_j has a convergent subsequence.

(c) There exists a sequence F_n of finite rank operators such that $\|T-F_n\|_{\mathrm{op}}$ tends to zero, as $n \to \infty$.

In the first step, we show that (b) implies the following

(b') If e_j is an orthonormal sequence in H, then Te_j converges to 0.

For this let e_j be any orthonormal sequence in H. By assumption we see that every subsequence of e_j has a subsequence e_k such that $Te_k \to v$ for some $v \in H$. It suffices to show $v = 0$. First, since every orthonormal sequence converges weakly to 0, we have $\langle T^*v, e_k \rangle \to 0$. Given $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $\|T^*v - T^*Te_k\| < \varepsilon$ and $|\langle T^*v, e_k \rangle| < \varepsilon$ for all $k \geq K$. It then follows that

$$\|Te_k\|^2 = \langle T^*Te_k, e_k \rangle = \langle T^*Te_k - T^*v, e_k \rangle + \langle T^*v, e_k \rangle \leq 2\varepsilon$$

for all $k \geq K$.

For (b') $\Rightarrow$ (c) we construct an orthonormal sequence e_j as follows. Choose $e_1 \in H$ with $\|e_1\| = 1$ and $\|Te_1\| \geq \frac{1}{2}\|T\|$. Next assume $e_1, \dots, e_n$ already constructed and let U_n be their span. Then choose $e_{n+1} \in U_n^\perp$ with $\|e_{n+1}\| = 1$ and $\|Te_{n+1}\| \geq \frac{1}{2}\|T(I - P_n)\|$, where P_n is the orthogonal projection onto U_n. Then

$$\|T - TP_n\| = \|T(I - P_n)\| \ \leq\ 2\|Te_{n+1}\| \ \to\ 0,$$

as $n \to \infty$. So T is indeed the limit of a sequence of finite rank operators $F_n = TP_n$. $\square$

In the proof of Theorem C.4.5 it is not clear why the space $\tilde{D}$ constructed there, should be dense. It is, however, not needed, as the proof below shows.

Theorem C.4.5. *Suppose that D is a dense linear subspace of the Hilbert space V and that*

$$B : D \times D \to \mathbb{C}$$

is a positive semi-definite closed hermitian form on D. Then there exists a closed operator $C : D \subset V \to V$ such that

$$B(v, w) = \langle Cv, Cw \rangle$$

for all $v, w \in D$. If B is positive definite, then C is injective.

Proof. We equip D with the inner product

$$\langle\langle v, w \rangle\rangle = \langle v, w \rangle + B(v, w)$$

and claim that D is complete with respect to this inner product. Indeed, if $(v_n)_{n\in\mathbb{N}}$ is a sequence in D, which is Cauchy with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, then it follows that $(v_n)_{\in\mathbb{N}}$ is Cauchy in V, hence converges to some $v \in V$, and that $B(v_n - v_m, v_n - v_m) \to 0$ for $n, m \to \infty$. Since B is closed we get $v \in D$ and $v_n \to v$ with respect to $\langle \cdot, \cdot \rangle$.

We first regard the restriction of the inner product on V to D as a positive definite hermitian form on $(D, \langle\langle \cdot, \cdot \rangle\rangle)$. This is clearly bounded, and we obtain a positive definite operator $C_1 : D \to D$ such that

$$\langle v, w \rangle = \langle C_1 v, C_1 w \rangle + B(C_1 v, C_1 w)$$

for all $v, w \in D$. This means that C_1 is an isometry $(D, \langle ., . \rangle) \to (D, \langle\langle ., . \rangle\rangle)$. As D is dense in V, the operator C_1 extends uniquely to an isometry $\overline{C_1} : V \to (D, \langle\langle ., . \rangle\rangle)$. Since C_1 is positive definite, it has dense image in D with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. The extension $\overline{C_1}$ has complete image, as it is an isometry, on the other hand the image is dense, therefore $\overline{C_1}$ is also surjective, hence a unitary map $\overline{C_1} : V \xrightarrow{\cong} (D, \langle\langle ., . \rangle\rangle)$. Let $\overline{C_1}^{-1} : D \to V$ denote the inverse of $\overline{C_1}$. It then satisfies

$$\left\langle \overline{C_1}^{-1} v, \overline{C_1}^{-1} w \right\rangle = \langle v, w \rangle + B(v, w)$$

for all $v, w \in D$.

Consider now the hermitian form $B : D \times D \to \mathbb{C}$. Since it is continuous with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ there exists a positive operator $C_2 : D \to D$ with

$$B(v, w) = \langle C_2 v, C_2 w \rangle + B(C_2 v, C_2 w)$$

for all $v, w \in D$.

We want define $C : D \to V$ as the composition $\overline{C_1}^{-1} \circ C_2$. We observe that we have the equation

$$B(v, w) = \langle Cv, Cw \rangle$$

for all $v, w \in D$, as claimed. The operator C is closed since D is complete with respect to the inner product

$$\langle\langle v, w \rangle\rangle = \langle v, w \rangle + \langle Cv, Cw \rangle . \qquad \square$$

(page,line)

- p. v, line 10, Change "Weil asymptotic law" to "Weyl asymptotic law".
- p. 6, line 17 Lemma 1.1.3 should be Lemma 1.1.5.
- p. 6, line 21 $x \in H \rightsquigarrow x \in G$.
- p. 10, line 16, Change $KV \subset x_1U_1 \cap \cdots \cap x_nU_n$ to $KV \subset x_1U_1 \cup \cdots \cup x_nU_n$.
- p. 11, line 5, $(f, g) \rightsquigarrow (f : g)$.
- p. 22, Change "locally-compact" to "locally compact" in the Definition.
- p. 23, line -2, $x, y \in g \rightsquigarrow x, y \in G$.
- p. 24, symmetry should not be part of the definition of Dirac functions, but an extra property.
- p. 24, line 8, Put a period at the end of the sentence.
- p. 26, Definitions of 'partial order' and 'directed set' can be removed since they are included in Appendix A. Moreover, the notion of a 'directed set' appears earlier in the book as well, say on page 24.
- p. 34, Delete Exercise 1.7.
- p. 34, Exercise 1.9, Change "locally-compact" to "locally compact".
- p. 35, Exercise 1.11 (c) and (d): replace $\delta(g)$ with $\delta(g^{-1})$.

- p. 39, line 8, Change “for every $n \in \mathbb{N}$” to “for every unit-neighborhood U”.
- p. 42, Definition, Change “*sup*” to “sup”.
- p. 47, line -5, Change “$\|m\|\|a\| = \|a\|$” to “$\|m\|\|a\| \leq \|a\|$”.
- p. 48, line -4, Replace “$\mathcal{A}$” by “a”.
- p. 49, line -3 of the proof should read “$\mathrm{Im}(\hat{a}) = \sigma(a)$” and the proof should end here.
- p. 52, line -11, Change $\|\lambda|$” to “$|\lambda|$”.
- p. 55, line -7 should read “$f\colon \sigma(a) \to \mathbb{C}$”.
- p. 56, The displayed equation should read “$\Psi(f) = f \circ \hat{a}$”.
- p. 56, line 10 should read “$\Phi_a\colon C(\sigma_{\mathcal{B}}(a)) \to \mathcal{B}$”.
- p. 56, last line of the Example, Replace the second colon with “,”.
- p. 56, line 9, Place a blank space between “Gelfand transform” and $\widehat{.}$
- p. 57, line 2, Change “$\Phi(a)$” to “$\Psi(a)$”.
- p. 57, line 4 of the proof of Corollary 2.7.6, Change “$\Phi(a)$” to “$\Psi(a)$”.
- p. 57, Exercise 2.3, “Suppose that” should be “Show that”.
- p. 63, line -8, Change the comma at the end of the displayed formula to a period.
- p. 65, line -3, Change “$\hat{f}(\chi_j$” to “$\hat{f}(\chi_j)$”.
- p. 65, line -11, Change “$|m(L_x\phi_U)|$” to “$|m(L_x\phi_U)| + \varepsilon$” and replace “$\lim_U \|L_x\phi_U\|_1$” by “$\|L_x\phi_U\|_1$”.
- p. 68, Change “$(L^1(A))$” to “$L(L^1(A))$”.
- p. 69, line 2 of section 3.4, “LCA group” should be “LCA-group”.
- p. 70, line 9, “$f^*(\psi)$” should be “$f_*(\psi)$”.

- p. 70, line 11, $C_c(G) \rightsquigarrow C_c(A)$.
- p. 70, Proof of Lemma 3.4.3, The assumption "$g_n = g_n^*$ for every $n \in \mathbb{N}$" is unnecessary. Without this assumption, the computational part of the proof goes as follows:

$$\begin{aligned} &= \lim_n g_n * \phi * \phi * g_n^* = \lim_n g_n * g_n^* * \phi * \phi \\ &= \lim_n L(g_n * g_n^*)(\phi * \phi) = \hat{f}(\phi * \phi) = f * \phi * \phi \end{aligned}$$

  where we have used the facts that L is a $*$-homomorphism and that $g^2 = \hat{f}$.
- p. 71, Lemma 3.4.6, Change "considered as subspace" to "considered as a subspace".
- p. 72, line 11, Change "$C_0(A) \cap L^2(A)$" to "$C_0^*(A) \cap L^2(A)$".
- p. 76, line -8, Change "$\|\hat{f}\|_{\widehat{\hat{A}}}$" to "$\|\hat{\hat{f}}\|_{\widehat{\hat{A}}}$".
- p. 77, line 3 of the proof of Theorem 3.5.8, $\hat{f}(\delta_{x^{-1}}) \rightsquigarrow \hat{\hat{f}}(\delta_{x^{-1}})$.
- p. 77, line -3 of the proof of Theorem 3.4.8, Remove the dash in "Fourier-transform".
- p. 79, line -3 of the statement of Theorem 3.6.3, "$\widehat{B}^\perp \cong \widehat{A/B}$" should be "$B^\perp \cong \widehat{A/B}$".
- p. 79, line -5 of the proof of Theorem 3.6.3, The reference should be given to Theorem 3.5.8 rather than Theorem 1.5.3.
- p. 79, line -3 of the proof of Theorem 3.6.3, Change "$\overline{\chi(x)}$" to "$\chi(x)$".
- p. 83, Insert "commutes." at the end of Exercise 3.21.
- p. 83, line -3, Change "goes" to "go".
- p. 108, Part (c) of Proposition 5.1.1, Change "$f(T|_V)$" to "$T|_V$".
- p. 109, line 13, $e^{2\pi i T} \rightsquigarrow e^{2\pi i y T}$.
- p. 125, line1, $L_x, \psi \rightsquigarrow L_x\psi$.

- p. 127, line -1, "$\|w\|v\|$" should be "$\|w\|\|v\|$".
- p. 131, line -1, Change William's to Williams'.
- p. 139, line -2, Change Hom_K to $\mathrm{Hom}_{\mathbb{C}}$.
- p. 140, line 5 of the proof, Change "$\tau(\sigma_{kl})$" to "$\tau(\overline{\sigma_{kl}})$".
- p. 140, line 5 of the proof, Change "$\dim(\tau)$" in the displayed formula to "$\frac{1}{\dim(\tau)}$".
- p. 140, line -3 of the proof, Change "$\sqrt{\dim(\tau)}E_{kl}^{\tau}$" to "$\sqrt{\dim(\tau)}^{-1}E_{kl}^{\tau}$".
- p. 141, lines $-3, -4$ of the proof of Lemma 7.2.5, Change "$\psi(w \otimes \alpha)$" to "$\psi(v \otimes \alpha)$" in two places.
- p. 143, line -3, Change "$(v_1 + w_1)$" to "(v_1, w_1)".
- p. 144, lines 3 and 5 under Induced Representations, Change "Hilbert-space" to "Hilbert space."
- p. 145, line -2 of the proof, Change "$\mathrm{Hom}_{\mathbb{C}}$" to "$\mathrm{Hom}_K$" and replace "$\beta\,(\tau(k^{-1})u)$" by "$\beta(\tau(k)u)$".
- p. 149, Change "p" to "η" in two places in the proof of Lemma 7.5.6.
- p. 150, Exercise 7.4, Change "Plancherell" to "Plancherel".
- . 159, line 10, the half-sentence starting with "and there is a countable..." should be deleted. This condition is unnecessary and only leads to problems later.
- p. 168, line 11, Instaed of "unimodular closed cocompact subgroup" it should be "uniform lattice".
- p. 170, line 3, Change "$\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\,dh$" to "$\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$".
- p. 170, line -3, Put a period at the end of the sentence.
- p. 171, line 5, Change "Dirac-net" to "Dirac net".
- p. 172, The third line of Proposition 9.3.1 should read "with a continuous L^2-kernel k on X."

- p. 173, line 11, Change "$\|S\|_{\mathrm{HS}}$" to "$\|S\|_{\mathrm{HS}}^2$".
- p. 174, lines -1 and -2, Change "supp(f)" to "supp$(\tilde{f})$".
- p. 175, line 2 should read "$F \in C_c(G)^2$ and $F \geq \tilde{f}$."
- p. 176, line -2, Change "$v_{\alpha\mu}$" to "$v_{\alpha,\mu}$".
- p. 177, line 1, Change "right hand side" to "right-hand side".
- p. 177, line -6, Change "Lie-group" to "Lie group".
- p. 178, line 6, Change "proposition" to "lemma".
- p. 179, line 2 should read "$\nu(A) = \mu(\phi^{-1}(A))$".
- p. 181, line 3, Change "$d(\psi_j)_*\mu(x)$" to "$= d(\psi_j)_*\mu(x)$".
- p. 182, line 1, Change "$i \neq i$" to "$i \neq j$".
- p. 198, 3rd paragraph, "for $y \in G$" should be "for $x \in G$".
- p. 200, line 2, "$p(t) = ity$" should be "$p(t) = i + it(y-1)$".
- p. 201, line -6, Put a period at the end of the sentence.
- p. 202, line 3, Change "$m = \pm 1$" to "$m = -1$".
- p. 202, line -2 should read "let $U \subset V_\pi^K$ be a closed".
- p. 203, line 4 should read "$\tilde{f}(x) = \int_K \int_K f(kxl)\, dk\, dl$ so that $\tilde{f} \in \mathcal{H}$."
- p. 203, line 6, Change "$\pi(\bar{f})$" to "$\pi(\tilde{f})$."
- p. 204, line -14, Change "there exists for every $f \in \mathcal{H}$ a unique function" to "for every $f \in \mathcal{H}$ there exists a unique function".
- p. 204, line -9, Change "$\mathrm{i}\mathbb{R}$" to "$i\mathbb{R}$".
- p. 205, line 4, "h_f" should be "$h_f(r)$".
- p. 205, line 10, "a consequences" should be "a consequence".
- p. 205, Remove "and g is even" from line -3 of the Proof of Lemma 11.2.6.

- p. 206, line -6, Change "be definition" to "by definition".
- p. 207, lines 1 and 2 of Proposition 11.2.9, Change "$\mathcal{HS}$" to "$\mathcal{H}_{\rm sym}$". (This is the notation used in the rest of the chapter.)
- p. 208, line 3, Change "$\|\pi(g)\|_{\rm HS}$" to "$\|\pi(g)\|_{\rm HS}^2$".
- p. 208, line 5, Change "f" to "g".
- p. 211, Theorem 11.4.3, Change "$\mathrm{SL}(2,\mathbb{R})$" to "$\mathrm{SL}_2(\mathbb{R})$".
- p. 249, line -2 of the first proof, Change "enumerator" to "numerator".
- p. 275, line 3 of Examples A.5.3, $q\circ f \ \rightsquigarrow\ f\circ q$.
- p. 275, line -4, Change "Theorem" to "theorem".
- p. 281, line 5 of proof to Theorem A.8.3, $C_c(U) \ \rightsquigarrow\ C_c(X)$.
- p. 283, proof of Lemma A.10.3, line 2, $y\in K \ \rightsquigarrow\ y\in X$.
- p. 301, Theorem B.5.1, line 5, it should be $\int_X \phi\, d\mu = \int_X \phi h\, d\lambda$. ($\lambda$ and μ are swapped).
- p. 315, proof of Proposition C.3.1, line -2, Cange $\lambda(v)$ to $\alpha(v)$, and change $\alpha(v_0)$ to $|\alpha(v_0)|$.
- p. 327, Change "$C_{unif}(G)$" to "$C_{\rm unif}(G)$" and "$C_{unif}(G)_2$" to "$C_{\rm unif}(G)^2$".
- p. 323, reference Coh93: Change "Donald, L.C." to "Cohn, D.L.".
- p. 323, reference Fuehr: Change "Fuehr" to "Fü05".

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