Dr. S. Eichmann 19. October 2022
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Exercises in variational calculus
Sheet 1

1.1 Exercise (Fundamental lemma of variational calculus)

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^1_{loc}(\Omega)$. Furthermore we assume, that for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega f\varphi \, d\mathcal{L}^n = 0.$$

Show $f(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \Omega$.
Hint: You may use, that any characteristic function $\chi_A \in L^1(\Omega)$ can be approximated by a sequence of functions $\varphi_k \in C_0^\infty(\Omega)$ in $L^1(\Omega)$ with $|\varphi_k(x)| \leq 1$ for all $x \in \Omega$.
If this fails, assume alternatively $f \in L^p(\Omega)$ for $p > 1$ and use that $C_0^\infty(\Omega) \subseteq L^p(\Omega)$ is dense.

1.2 Exercise (Partial integration)

Let $\Omega \subseteq \mathbb{R}^n$ be open. Show the following statements:

1. Let $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$. Then
$$\int_\Omega \operatorname{div} \varphi \, d\mathcal{L}^n = 0.$$
2. Let $f \in C_0^1(\Omega)$ and $g \in C^1(\Omega)$. Then we have for all $i = 1, \dots, n$:
$$\int_\Omega \partial_i f g \, d\mathcal{L}^n = -\int_\Omega f \partial_i g \, d\mathcal{L}^n.$$

What happens if φ or f do not have compact support anymore? Can these formulas be salvaged?
Hint: Fubini and/or divergence theorem.

1.3 Exercise (Exchanging integrals with derivatives)

Let μ be a measure on a set X, $I \subseteq \mathbb{R}$ an open Intervall and $f : I \times X \to \mathbb{R}$ such that for all $t \in I$ we have $f(t, \cdot) \in L^1(\mu)$ and the partial derivative w.r.t. t exists μ-a.e. Furthermore we assume the existence of a $g \in L^1(\mu)$ with

$$|\partial_t f(t, x)| \leq |g(x)| \text{ for all } t \in I \text{ and } \mu\text{-almost every } x \in X.$$

Show $t \mapsto \int f(t, x) \, d\mu(x)$ is differentiable and

$$\frac{d}{dt} \int f(t, x) \, d\mu = \int \partial_t f(t, x) \, d\mu$$

for all $t \in I$.
Hint: Dominated convergence theorem.

1.4 Exercise (Euler-Lagrange equation)

Let $(t, y, p) \mapsto h(t, y, p)$ be a function with $h \in C^\infty(\mathbb{R}^3)$. We define the energy $F : C^2([a, b]) \to \mathbb{R}$ by

$$F(x) = \int_a^b h(t, x(t), x'(t))\, dt.$$

We set

$$L := \{w \in C^2([a, b]) \text{ such that } w(a) = w(b) = 0\}$$

and assume in L exists a minimiser $x \in L$ of F, i.e. for all $w \in L$ we have

$$F(x) \leq F(w).$$

Show that x satisfies the following boundary value problem:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial h}{\partial p}(t, x(t), x'(t))\right) - \frac{\partial h}{\partial y}(t, x(t), x'(t)) = 0 \text{ in } (a, b), \\ x(a) = x(b) = 0. \end{cases}$$

Hint: The preceeding exercises.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.