Dr. S. Eichmann 02. November 2022
Winter 2022/23

Exercises in variational calculus
Sheet 2

2.1 Exercise (Fundamentals of linear maps)

Let $(B, \|\cdot\|_B)$ be a normed space. Show the following items:

1. The operatornorm $\|\cdot\|_{B^*}$ is a norm on B^*.
2. Let $L : B \to V$ be linear and $(V, \|\cdot\|_V)$ a normed space. Then L is continuous if and only if

$$\|L\|_{B,V} := \sup_{x \in B, \|x\|_B \leq 1} \|L(x)\|_V < \infty.$$

3. For all $L : B \to V$ linear we have

$$\|L\|_{B,V} = \sup_{x \in B \setminus \{0\}} \frac{\|Lx\|_V}{\|x\|_B}.$$

2.2 Exercise (Strong/weak convergence)

Let $(B, \|\cdot\|)$ be a Banach space, $x_k, x \in B$. Show:

1. If $x_k \to x$ w.r.t. to the norm $\|\cdot\|$, then $x_k \to x$ weakly.
2. The limit of a weakly converging sequence is unique.
3. If $B = \mathbb{R}^n$, weak convergence implies convergence in norm.

2.3 Exercise (Weak compactness in Hilbert spaces)

Without using Theorem 2.14 show that in a seperable Hilbert space any bounded sequence does possess a weakly converging subsequence.
Hint: Riesz representation theorem and Bolzano-Weierstrass.

2.4 Exercise (Reflexive spaces)

Let $(B, \|\cdot\|)$ be a reflexive Banach space and $X \subseteq B$ be a closed subspace. Show that X is reflexive as well.
Hint: Hahn-Banach seperation theorem.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.