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Exercises in variational calculus
Sheet 3

3.1 Exercise (Lower semicontinuity and convexity)

Let $(B, \|\cdot\|)$ be a Banachspace and $F : B \to \mathbb{R}$ be convex and lower semicontinuous. Show that F is also weakly lower semicontinuous, i.e. show that for every $x_k \in B$ weakly converging to an $x \in B$ that

$$F(x) \leq \liminf_{k\to\infty} F(x_k).$$

Hint: Examine the epigraph $epi(F) := \{(t,x) \in \mathbb{R} \times B |\ F(x) \leq t\}$ and use Theorem 2.15.

3.2 Exercise (Poincaré Inequality)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $1 \leq p < \infty$. Show that there exists a constant $C = C(\mathrm{diam}(\Omega), n, p) > 0$, such for all $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Hint: Density, partial integration and Hölders inequality.

3.3 Exercise (Basics about weak derivatives)

Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$. Show the following:

1. The weak derivative is unique, i.e. if two functions $v_1, v_2 \in L^1_{loc}(\Omega)$ exist, which are both a weak derivative of u w.r.t. the i-th component $(i = 1, \ldots, n)$, then

$$v_1 = v_2 \ \mathcal{L}^n\text{-a.e..}$$

2. Let $i, j \in \{1, \ldots, n\}$ and we assume $D^{(i)}u$, $D^{(j)}u$ and $D^{(i,j)}u$ to exist weakly. Show that $D^{(j,i)}u$ exists weakly as well and $\mathcal{L}^n$-a.e.

$$D^{(i,j)}u = D^{(j,i)}u.$$

Hint: Fundamental lemma of variational calculus.

3.4 Exercise (Sobolev spaces and Functional Analysis)

Let $\Omega \subseteq \mathbb{R}^n$ be open.

1. Show that $W^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ is complete.
2. Show that for $1 < p < \infty$ the Sobolev space $W^{k,p}(\Omega)$ is reflexive.

Hint: You may use without proof, that $(L^p(\Omega))^m$ for $m \in \mathbb{N}$ is reflexive if, $1 < p < \infty$.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.