Dr. S. Eichmann 30. November 2022
Winter 2022/23

Exercises in variational calculus
Sheet 4

4.1 Exercise (Compact operators)

Let $(B, \|\cdot\|_B)$ and $(V, \|\cdot\|_V)$ be two Banachspaces and $T : B \to V$ be a linear compact operator. Show the following:

1. T is continuous.
2. For any weakly converging sequence $x_k \in B$ with $x_k \to x \in B$ weakly, there exists a subsequence x_{k_j}, such that
$$\lim_{j\to\infty} T(x_{k_j}) = T(x) \text{ w.r.t. } \|\cdot\|_V.$$

Hint: Exercise 2.1.

4.2 Exercise (nonlinear oscillation)

Let $a, b \in \mathbb{R}$. Examine the nonlinear oscillator equation:

$$\begin{cases} -u'' - \sin(u) = 0, & \text{in } (0,1) \\ u(0) = a, \ u(1) = b. \end{cases}$$

Give a suitable weak formulation of the above boundary value problem, so that you can show existence of a such a weak solution.

4.3 Exercise (basic estimates)

Show the following estimates:

1. Let $x_1, \dots, x_n \in \mathbb{R}$ and $1 < p < \infty$. Then
$$\left(\sum_{j=1}^n |x_j|\right)^p \le n^{p-1} \sum_{j=1}^n |x_j|^p.$$
2. Let $x, y \ge 0$, $\varepsilon > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Then
$$x \cdot y \le \varepsilon x^p + \frac{(p\varepsilon)^{1-q}}{q} y^q.$$

Hint: For 1: Use Hölders inequality. For 2: Show $xy \le \frac{1}{p}a^p + \frac{1}{q}b^q$ first with the help of the weighted inequality of arithmetic and geometric means and afterwards apply an appropriate scaling.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.