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Exercises in variational calculus
Sheet 5

5.1 Exercise (p-Laplace)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Examine the following boundary value problem for the p-Laplace operator for $1 < p < \infty$:

$$\begin{cases} -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Find a suitable weak formulation of this boundary value problem and assumptions on f, such that you can show existence.
Hint: Try $f \in L^r(\Omega)$ and find a good r by Hölder inequality and Sobolev embeddings. If this fails, assume $f \in L^\infty(\Omega)$. Also examine $u \mapsto \frac{1}{p}\int_\Omega |\nabla u|^p \, dx$.
Further use Exercise 4.3.

5.2 Exercise (Poincaré inequality, Part II)

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and connected with C^1 boundary. For $u \in L^1(\Omega)$ we define the mean $\overline{u} \in \mathbb{R}$ by

$$\overline{u} := \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega u \, dx.$$

Show that there exists a constant $C = C(n, \Omega) > 0$, such that for all $u \in W^{1,2}(\Omega)$ we have

$$\int_\Omega |u - \overline{u}|^2 \, dx \leq C \int_\Omega |\nabla u|^2 \, dx.$$

Hint: Proceed by contradiction and use the compactness part of the Sobolev embedding theorem.

5.3 Exercise (Natural boundary conditions)

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, connected and with C^1-boundary. For $f \in L^2(\Omega)$ with $\int_\Omega f \, dx = 0$ we define the Energy $E : W^{1,2}(\Omega) \to \mathbb{R}$ by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx.$$

Do the following:

1. Show, that E does possess a minimiser in $W^{1,2}(\Omega)$.
2. What kind of boundary value problem does the minimiser from the previous point satisfy weakly?
3. Can we still expect to have a minimiser, if $\int f \, dx \neq 0$?

Hint: Modify a minimising sequence with the help of Exercise 5.2.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.