Dr. S. Eichmann 11. January 2023
Winter 2022/23

Exercises in variational calculus
Sheet 6

On this exercise sheet we will examine an example of a geometric energy defined for curves. We use the following definition of a curve:
We call $P \subseteq \mathbb{R}^2$ a $W^{2,2}$-regular curve, if there exists an open interval $I \subseteq \mathbb{R}$ and a function $c : I \to \mathbb{R}^2$, such that the coordinates satisfy $c_1, c_2 \in W^{2,2}(I)$, $|\dot{c}(t)| \neq 0$ for all $t \in I$ and

$$c(I) = P.$$

We call c a $W^{2,2}$-parametrisation of P.
Now we define two energies for a $W^{2,2}$-curve P by its parametrisation $c : I \to \mathbb{R}^2$:
The length of P:

$$L(P) := \int_I |c'(t)|\, dt.$$

The elastic energy of P

$$E(P) := \int_I \frac{1}{|c'|} \left| \frac{d}{dt} \left(\frac{c'}{|c'|} \right) \right|^2 dt.$$

6.1 Exercise (Inner Invariances)

Show that L and E are well defined, i.e. independent of a specific parametrisation. Hence let $c : [a,b] \to \mathbb{R}^2$ be smooth with $c'(t) \neq 0$ for all $t \in [a,b]$ and $\varphi : [d,e] \to [a,b]$ be a smooth diffeomorphism with $\varphi'(t) \neq 0$ for all t. Now show

$$\int_d^e |(c \circ \varphi)'|\, dt = \int_a^b |c'(t)|\, dt$$

and

$$\int_d^e \frac{1}{|(c \circ \varphi)'|} \left| \frac{d}{dt} \left(\frac{(c \circ \varphi)'}{|(c \circ \varphi)'|} \right) \right|^2 dt = \int_a^b \frac{1}{|c'|} \left| \frac{d}{dt} \left(\frac{c'}{|c'|} \right) \right|^2 dt.$$

6.2 Exercise (Arclength parametrisation)

Let $c \in W^{2,2}((a,b), \mathbb{R}^2)$ with $c'(t) \neq 0$ for all t and $P = c(a,b)$ the corresponding curve. Show that there exists a diffeomorphism $\varphi : [0,1] \to [a,b]$ with $\varphi' > 0$, such that

$$|(c \circ \varphi)'(t)| = L(P) \text{ for all } t \in (0,1).$$

Hint: Invert $s \mapsto \int_a^s |c'(t)|\, dt$.

6.3 Exercise (Existence of minimiser under Dirichlet data)

Let $a, b \in \mathbb{R}^2$, $v, w \in \partial B_1(0) \subseteq \mathbb{R}^2$. We define

$$M := \left\{ \begin{array}{c} P \subseteq \mathbb{R}^2 \ W^{2,2}\text{-curve with parametrisation } c : (d,e) \to \mathbb{R}^2 \text{ such that} \\ c(d) = a,\ c(e) = b,\ \frac{c'(d)}{|c'(d)|} = v,\ \frac{c'(e)}{|c'(e)|} = w \end{array} \right\}.$$

We define for $\lambda > 0$

$$W_\lambda : M \to \mathbb{R},\ W_\lambda(P) := E(P) + \lambda L(P).$$

Show that W_λ does possess a minimiser in M.
Hint: Use exercise 6.2 to control the inner invariances of the functional.

The exercises and other material can be found on the homepage of the lecture:
https://www.math.uni-tuebingen.de/user/eichmann/Lehre/Variationsrechnung_22_23/.