Dr. S. Eichmann
Winter 2022/23

Solutions to exercise 1

Solution to exercise 1.1

Since $\Omega \subseteq \mathbb{R}^n$ is open, we have for every $x \in \Omega$ an $R > 0$ such that $B_R(x) \subset\subset \Omega$ (i.e. $\overline{B_R(x)} \subseteq \Omega$). If we show $f|_{B_R(x)} = 0$ a.e., we have already achieved our desired conclusion. Hence we can assume w.l.o.g. that $f \in L^1(\Omega)$ and Ω is bounded. We define

$$A := [f > 0] := \{x \in \Omega : \ f(x) > 0\}.$$

Therefore $\chi_A \in L^1(\Omega)$ and by the hint we find a sequence of functions $\varphi_k \in C_0^\infty(\Omega)$, such that

$$|\varphi_k| \leq 1, \ \|\varphi_k - \chi_A\|_{L^1(\Omega)} \to 0.$$

After extracting a subsequence (By an addendum of the Riesz-Fischer Theorem) and relabeling we can assume

$$\varphi_k \to \chi_A \text{ pointwise a.e..}$$

Hence $f \cdot \varphi_k \to f\chi_A$ pointwise a.e. and $|f \cdot \varphi_k| \leq |f| \in L^1(\Omega)$. The dominated convergence theorem yields

$$0 = \lim_{k\to\infty} \int_\Omega f \cdot \varphi_k \, dx = \int_\Omega f\chi_A \, dx = \int_{[f>0]} f \, dx = \int_{[f>0]} |f| \, dx.$$

Hence $f = 0$ on A a.e. By the same argument $\mathcal{L}^n([f < 0]) = 0$.
Addendum: The existence of such φ_k can be explicitly shown via a convolution with a smoothing kernel, see e.g. Gilbarg&Trudinger Lemma 7.2.

Solution to exercise 1.2

Since φ has compact support, we can extend it to $\mathbb{R}^n$ by zero and still have a smooth function. Furthermore let $R > 0$ be such that

$$\operatorname{spt}(\varphi) := \overline{\{x \in \Omega : \ \varphi(x) \neq 0\}} \subset \left[-\frac{R}{2}, \frac{R}{2}\right]^n$$

Then Fubini yields

$$\begin{aligned}
\int_\Omega \operatorname{div}(\varphi) \, dx &= \sum_{j=1}^n \int_{[-R,R]^n} \partial_j \varphi(x) \, dx \\
&= \sum_{j=1}^n \int_{-R}^R \cdots \int_{-R}^R \partial_j \varphi \, dx_j dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n. \\
&= \sum_{j=1}^n \int_{-R}^R \cdots \int_{-R}^R [\varphi]_{-R}^R \, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n. \\
&= \sum_{j=1}^n \int_{-R}^R \cdots \int_{-R}^R 0 \, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n = 0
\end{aligned}$$

with the help of the fundamental theorem of calculus.
For second part of the exercise we define a $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$ by

$$\varphi_j = fg\delta_{ij}.$$

By the first part we then have

$$0 = \int_\Omega \operatorname{div} \varphi \, dx = \int_\Omega \partial_i(fg) \, dx = \int_\Omega \partial_i f g + f \partial_i g \, dx$$

and the result follows.
If f or φ do not have compact support anymore, we have to add some assumptions: $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$, Ω bounded and with C^1 boundary. Then the divergence theorem yields

$$\int_\Omega \operatorname{div} \varphi \, dx = \int_{\partial\Omega} \langle \varphi, \nu \rangle \, darea_{\partial\Omega}$$

and $\nu : \partial\Omega \to \partial B_1(0)$ is the outer unit normal of Ω.
If $f, g \in C^1(\overline{\Omega})$ we can apply the same trick as above and obtain:

$$\int_\Omega \partial_i f g + f \partial_i g \, dx = \int_{\partial\Omega} \nu_i f g \, darea_{\partial\Omega}.$$

By Rademachers theorem one might lower the regularity assumptions on $\partial\Omega$, φ, f and g so that they only have to be Lipschitz.

Solution to exercise 1.3

Let $t \in I$ and $h_k \in \mathbb{R} \setminus \{0\}$ with $h_k \to 0$. We need to show

$$\lim_{k\to\infty} \frac{1}{h_k} \int f(t + h_k, x) \, d\mu(x) - \int f(t, x) \, d\mu(x) = \int \partial_t f(t, x) \, d\mu.$$

Hence we define a sequence of functions $g_k : X \to \mathbb{R}$ by

$$g_k(x) := \frac{f(t + h_k, x) - f(t, x)}{h_k}.$$

Since f is μ-measurable, so is g. Since $h_k \neq 0$ we then have $g_k \in L^1(\mu)$. Since f is differentiable w.r.t. t, we have for almost every x

$$g_k(x) \to \partial_t f(t, x) \text{ for } k \to \infty.$$

By the mean value theorem we have an $s_k \in [-|h_k|, |h_k|]$ such that

$$|g_k(x)| = |\frac{f(t + h_k, x) - f(t, x)}{h_k}| = |\frac{\partial_t f(t + s_k, x) h_k}{h_k}| \leq |g(x)|.$$

Since $g \in L^1(\mu)$, the dominated convergence theorem yields

$$\int g_k \, d\mu \to \int \partial_t f(t, x) \, d\mu,$$

which is the desired result.

Solution to exercise 1.4

Let $\varphi \in C_0^\infty((a,b),\mathbb{R})$ be arbitrary. Since $\varphi(a) = \varphi(b) = 0$, we have for all $s \in \mathbb{R}$

$$(t \mapsto x(t) + s \cdot \varphi(t)) \in L.$$

Since $[a,b] \subset \mathbb{R}$ is compact and $t \mapsto h(t, x(t), x'(t))$ is continuously differentiable, the function

$$s \mapsto F(x + s\varphi)$$

is continuously differentiable by Exercise 1.3. Furthermore the derivatives of h are bounded and hence integral and derivative can be exchanged.
Also $s \mapsto F(x + s\varphi)$ does possess a minimum in $s = 0$, hence

$$\begin{aligned}
0 =& \frac{d}{ds} F(x + s\varphi)|_{s=0} = \frac{d}{ds} \int_a^b h(t, x(t) + s\varphi(t), x'(t) + s\varphi'(t))\, dt|_{s=0} \\
=& \int_a^b \frac{\partial}{\partial s}(h(t, x(t) + s\varphi(t), x'(t) + s\varphi'(t)))\, dt|_{s=0} \\
=& \int_a^b \frac{\partial h}{\partial y}(t, x(t) + s\varphi(t), x'(t) + s\varphi'(t)) \cdot \varphi(t) \\
&+ \frac{\partial h}{\partial p}(t, x(t) + s\varphi(t), x'(t) + s\varphi'(t)) \cdot \varphi'(t)\, dt|_{s=0} \\
=& \int_a^b \frac{\partial h}{\partial y}(t, x(t), x'(t)) \cdot \varphi(t) + \frac{\partial h}{\partial p}(t, x(t), x'(t)) \cdot \varphi'(t)\, dt \\
=& \int_a^b \frac{\partial h}{\partial y}(t, x(t), x'(t)) \cdot \varphi(t)\, dt + \left[\frac{\partial h}{\partial p}(t, x(t), x'(t)) \cdot \varphi(t)\right]_{t_0}^{T} \\
&- \int_a^b \frac{d}{dt}\left(\frac{\partial h}{\partial p}(t, x(t), x'(t))\right) \cdot \varphi(t)\, dt \\
=& \int_a^b \left(\frac{\partial h}{\partial y}(t, x(t), x'(t)) - \frac{d}{dt}\left(\frac{\partial h}{\partial p}(t, x(t), x'(t))\right)\right) \cdot \varphi(t)\, dt.
\end{aligned}$$

Since $\varphi \in C_0^\infty((a,b),\mathbb{R})$ is arbitrary, exercise 1.1 yields the desired conclusion:

$$\frac{d}{dt}\left(\frac{\partial h}{\partial p}(t, x(t), x'(t))\right) - \frac{\partial h}{\partial y}(t, x(t), x'(t)) = 0.$$