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# Solutions to exercise 2

## Solution of exercise 2.1

1. - Positive definite: Let $L \in B^*$ with $\|L\|_{B^*} = 0$. Then for all $x \in B$ with $\|x\|_B \leq 1$ we have
$$|L(x)| \leq \sup_{y \in B,\ \|y\|_B \leq 1} |L(y)| = \|L\|_{B^*} = 0,$$
hence $L(x) = 0$. Now let $x \in B \setminus \{0\}$ be arbitrary. Then by linearity of $L$ we have
$$L(x) = \|x\|_B L\left(\frac{x}{\|x\|_B}\right) = \|x\|_B \cdot 0 = 0$$
and it follows, that $L = 0$.
   - Homogenity: Let $L \in B^*$ and $\alpha \in \mathbb{R}$. Then
$$\|\alpha L\|_{B^*} = \sup_{x \in B, \|x\|_B \leq 1} |\alpha L(x)| = |\alpha| \sup_{x \in B, \|x\|_B \leq 1} |L(x)| = |\alpha| \|L\|_{B^*}.$$
   - Triangle inequality: Let $L, G \in B^*$. Then
$$\|L + G\|_{B^*} = \sup_{x \in B, \|x\|_B \leq 1} |L(x) + G(x)| \leq \sup_{x \in B, \|x\|_B \leq 1} |L(x)| + |G(x)|$$
$$\leq \sup_{x \in B, \|x\|_B \leq 1} |L(x)| + \sup_{x \in B, \|x\|_B \leq 1} |G(x)| = \|L\|_{B^*} + \|G\|_{B^*}.$$

2. Let $L : B \to V$ be linear and continuous. Since $L$ is continuous in $x = 0$, we find a $\delta > 0$, such that for all $y \in B_\delta(0)$ we have
$$\|L(y)\|_V = \|L(0) - L(y)\|_V < 1.$$
Now let $y \in B$ with $\|y\|_B \leq 1$. Then $\|\frac{\delta}{2} y\|_B < \delta$ and therefore
$$\|L(\frac{\delta}{2} y)\|_V < 1.$$
Hence by linearity
$$\|L(y)\|_V < \frac{2}{\delta} < \infty.$$
Therefore the operatornorm is bounded.
Now let $\|L\|_{B,V} < \infty$. Furthermore let $x, y \in B$ with $x - y \neq 0$. Then
$$\left\| L\left(\frac{x - y}{\|x - y\|_B}\right) \right\|_V \leq \|L\|_{B,V}.$$
Since $L$ is linear, we then obtain
$$\|L(x) - L(y)\|_V \leq \|L\|_{B,V} \|x - y\|_B.$$
Therefore $L$ is Lipschitz and also continuous.

3. Let $x \in B \setminus \{0\}$. Then

$$\frac{\|L(x)\|_V}{\|x\|_B} = \|L\left(\frac{x}{\|x\|_B}\right)\|_V \leq \sup_{y \in B, \|y\|_B \leq 1} \|L(y)\|_V = \|L\|_{B,V}.$$

On the other hand let $x \in B \setminus \{0\}$ with $\|x\|_B \leq 1$. Then

$$\|L(x)\|_V \leq \frac{\|L(x)\|_V}{\|x\|_B} \leq \sup_{y \in B \setminus \{0\}} \frac{\|L(y)\|_V}{\|y\|_B}.$$

## Solution of exercise 2.2

1. Let $L \in B^*$ be arbitrary. Since $L$ is Lipschitz by Ex. 2.1 we have

$$|L(x_k) - L(x)| = |L(x_k - x)| \leq \|L\|_{B^*} \|x_k - x\| \to 0 \text{ for } k \to \infty.$$

2. Let $x_k$ weakly converge to $x \in B$ and $y \in B$. We proceed by contradiction and assume $x \neq y$. By Hahn-Banach 2.5 3) we find an $L \in B^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\lim_{k \to \infty} L(x_k) = L(x) < \gamma_1 < \gamma_2 < L(y) = \lim_{k \to \infty} L(x_k),$$

a contradiction.

3. Let $x_k \in \mathbb{R}^n$ weakly converge to $x \in \mathbb{R}^n$. We define $L_j : \mathbb{R}^n \to \mathbb{R}$ by $L_j(x) = \langle e_j, x \rangle$. Then the $j$-th component converges, i.e.

$$x_k^j = \langle e_j, x \rangle = L_j(x_k) \to L_j(x) = x^j.$$

Since $j = 1, \ldots, n$ is finite, all components converge and we have $\|x_k - x\| \to 0$ for $k \to \infty$. Please note that since we are in a finite dimensional normed space, all norms are equivalent and therefore the choice of norm does not matter.

## Solution of exercise 2.3

Let $H$ be a seperable Hilbertspace and $p_n \in H$ a sequence, such that there exists a $C > 0$ with

$$\|p_n\| \leq C < \infty \text{ for all } n \in \mathbb{N}.$$

Let $\{q_\ell, \ell \in \mathbb{N}\} \subset H$ be dense. Via Cauchy-Schwartzes inequality we have

$$|\langle p_n, q_\ell \rangle| \leq C \|q_\ell\|.$$

With Bolzano-Weierstraßwe can successively extract subsequences and we obtain

$$\begin{array}{llll}
\langle p_{1_1}, q_1 \rangle, & \langle p_{1_2}, q_1 \rangle, & \langle p_{1_3}, q_1 \rangle, & \ldots \to f(q_1) \\
\langle p_{2_1}, q_2 \rangle, & \langle p_{2_2}, q_2 \rangle, & \langle p_{2_3}, q_1 \rangle, & \ldots \to f(q_2) \\
\langle p_{3_1}, q_3 \rangle, & \langle p_{3_2}, q_3 \rangle, & \langle p_{3_3}, q_3 \rangle, & \ldots \to f(q_3) \\
\vdots & \ddots & &
\end{array}$$

Here $n_j$ is a subsequence of $k_j$ if $n \geq k$. Choosing the diagonal as a subsequence, i.e.

$$p_n := p_{n_n}$$

we have

$$\langle p_n, q_\ell \rangle \to f(q_\ell) \text{ for all } \ell \in \mathbb{N}.$$

This defines a function $f : \{q_\ell, \ell \in \mathbb{N}\} \to \mathbb{R}$. In the next step we use the density of the $q_\ell$ to extend $f$ continuously to the whole of $H$:
Let $q \in H$ be arbitrary, $\varepsilon > 0$ and $q_\ell$ such that $\|q - q_\ell\| < \varepsilon$. Then

$$\begin{aligned}\limsup_{n,j\to\infty} |\langle p_n, q\rangle - \langle p_j, q\rangle| \leq \limsup_{n,j\to\infty} &|\langle p_n, q - q_\ell\rangle| + |\langle p_n - p_j, q_\ell\rangle| \\ &+ |\langle p_j, q_\ell - q\rangle| \leq 2C\varepsilon\end{aligned}$$

Hence $\langle p_n, q\rangle$ is a Cauchy sequence and since $H$ is complete, it converges to a limit $f(q)$. Therefore

$$\forall q \in H: \ \langle p_n, q\rangle \to f(q), \ n \to \infty.$$

Since the scalarproduct is linear, $f$ is as well. We also have

$$|f(q)| = \lim_{n\to\infty} |\langle p_n, q\rangle \leq \lim_{n\to\infty} \|p_n\|\|q\| \leq C\|q\|,$$

hence $f$ is bounded and by Ex. 2.1 it is continuous. By the Riesz representation theorem we therefore find a unique $p \in H$, such that for all $q \in H$ we have

$$\lim_{n\to\infty} \langle p_n, q\rangle = f(q) = \langle p, q\rangle.$$

Again by the Riesz representation theorem we can represent any linear $L \in h^*$ by such a $q$. Hence $p_n \to p$ weakly.

## Solution of exercise 2.4

Let $x^{**} \in X^{**}$ be arbitrary. We define

$$\tilde{x}^{**} \in B^{**} \text{ by } \tilde{x}^{**}(G) = x^{**}(G|_X).$$

Since $B$ is reflexive, we find an $x \in B$ such that

$$i_B(x) = \tilde{x}^{**}.$$

We proceed by contradiction and assume $x \notin X$. Then the Hahn-Banach seperation theorem 2.5 3) yields an $L \in B^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$L(x) < \gamma_1 < \gamma_2 < L(y) \text{ for all } y \in X.$$

Since $X$ is a subspace, we could scale the right hand of the inequality as much as we like, if $L|_X \neq 0$. Hence $L|_X = 0$. This yields

$$0 > L(x) = i_B(x)(L) = \tilde{x}^{**}(L) = x^{**}(L|_X) = x^{**}(0) = 0,$$

a contradiction. Hence $x \in X$.
Now let $G \in X^*$. Since $X$ is closed we can find by Hahn-Banach 2.5 2) a $\tilde{G} \in B^*$, such that $\tilde{G}|_X = G$. Hence

$$x^{**}(G) = \tilde{x}^{**}(\tilde{G}) = i_B(x)(\tilde{G}) = \tilde{G}(x) \overset{x\in X}{=} G(x) = i_X(x)(G).$$

Hence $i_X(x) = x^{**}$ and $X$ is reflexive.