Dr. S. Eichmann
Winter 2022/23

Solutions to exercise 3

Solution to exercise 3.1

Let $epi(F) := \{(t,x) \in \mathbb{R} \times B | \; F(x) \leq t\} \subseteq \mathbb{R} \times B$ the epigraph. We equip $\mathbb{R} \times B$ with the norm

$$\|(t,x)\|_{\mathbb{R}\times B} := |t| + \|x\|.$$

Therefore it is a Banachspace as well. we first show, that $epi(F)$ is weakly sequentially closed:
The epigraph is convex because take $(t_1,x_1),(t_2,x_2) \in epi(F)$ and $\lambda \in [0,1]$. Then the convexity of F yields

$$F(\lambda x_1 + (1-\lambda)x_2)) \leq \lambda F(x_1) + (1-\lambda)F(x_2) \leq \lambda t_1 + (1-\lambda)t_2.$$

Hence the definition of $epi(F)$ gives us

$$\lambda(t_1,x_1) + (1-\lambda)(t_2,x_2) = (\lambda t_1 + (1-\lambda)t_2, \lambda x_1 + (1-\lambda)x_2) \in epi(F)$$

and $epi(F)$ is convex.
Now $epi(F)$ is also closed: Let $(t_k,x_k) \to (t,x)$ w.r.t. to the norm with $(t_k,x_k) \in epi(F)$. Hence for all $k \in \mathbb{N}$ we have $t_k \geq F(x_k)$ and the lower semicontinuity yields

$$F(x) \leq \liminf_{k\to\infty} F(x_k) \leq \liminf_{k\to\infty} t_k = t,$$

hence $(t,x) \in epi(F)$.
By Theorem 2.18 $epi(F)$ is weakly sequentially closed.
Now let $x_k \in B$ with $x_k \to x$ weakly. After extracting a subsequence we can assume

$$\liminf_{k\to\infty} F(x_k) = \lim_{k\to\infty} F(x_k) =: t.$$

Let $L : \mathbb{R} \times B$ be linear and continuous. Then

$$L(F(x_k),x_k)) = L((F(x_k),0)) + L((0,x_k)) \to L((t,0)) + L((0,x)) = L((t,x)),$$

because $L|_{\mathbb{R}}$ and $L|_B$ are still continuous. Hence $(F(x_k),x_k) \to (t,x)$ weakly. Since

$$F(x_k) \leq F(x_k) \;\Rightarrow\; (F(x_k),x_k) \in epi(F),$$

we have $(t,x) \in epi(F)$. This yields

$$\liminf_{k\to\infty} F(x_k) = t \geq F(x)$$

and F is sequentially weakly lower semicontinuous.

Solution to exercise 3.2

By density we can assume $u \in C_0^\infty(\Omega)$. Let $x_0 \in \Omega$. Then

$$\int_\Omega |u|^p\,dx = \int_\Omega |u|^p \operatorname{div}(x - x_0)\frac{1}{n}\,dx = -\frac{1}{n}\int_\Omega \langle \nabla(|u|^p), x - x_0\rangle\,dx$$
$$= -\frac{1}{n}\int_\Omega p|u|^{p-1}\operatorname{sgn}(u)\langle \nabla u, x - x_0\rangle\,dx \le C(n,p)\int_\Omega |u|^{p-1}|\nabla u||x - x_0|\,dx$$
$$\le C(n,p,\operatorname{diam}(\Omega))\int_\Omega |u|^{p-1}|\nabla u|\,dx$$
$$\le C\left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}}\left(\int_\Omega \left(|u|^{p-1}\right)^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} = C\|\nabla u\|_{L^p(\Omega)}\left(\int_\Omega |u|^p\,dx\right)^{\frac{p-1}{p}}$$

Rearranging yields

$$\left(\int_\Omega |u|^p\,dx\right)^{1-\frac{p-1}{p}} \le C\|\nabla u\|_{L^p(\Omega)}.$$

Since $1 - \frac{p-1}{p} = \frac{1}{p}$, the result follows.

Solution to exercise 3.3

1. Let $v_1, v_2 \in L^1_{loc}(\Omega)$ be two possible weak derivatives of u. Then for every $\varphi \in C_0^\infty(\Omega)$ we have

   $$\int_\Omega v_1\varphi\,dx = -\int_\Omega uD^{(i)}\varphi\,dx = (-1)\cdot(-1)\int_\Omega v_2\varphi\,dx.$$

   Hence
   $$\int_\Omega (v_1 - v_2)\varphi\,dx = 0.$$

   By the fundamental lemma of variational calculus, we have

   $$v_1(x) = v_2(x) \text{ for } \mathcal{L}^n\text{-a.e } x \in \Omega.$$

2. Let $\varphi \in C_0^2(\Omega)$. Then by definition of the weak derivative and Schwarzes Theorem, we have

   $$\int_\Omega D^{(i,j)}u\varphi\,dx = \int_\Omega uD^{(i,j)}\varphi\,dx = \int_\Omega uD^{(j,i)}\varphi\,dx.$$

   Hence by Definition 3.1 the weak derivative $D^{(j,i)}u$ exists and by the first part of the exercise (i.e. uniqueness) we have

   $$D^{(i,j)}u = D^{(j,i)}u \;\mathcal{L}^n\text{-a.e.}.$$

Solution to exercise 3.4

We define the vector space

$$V_m := L^p(\Omega) \times \ldots \times L^p(\Omega)$$

with $m+1$-factors. A norm is given by

$$\|(u_0, \dots, u_m)\|_{V_m} := \sum_{j=0}^{m} \|u_j\|_{L^p(\Omega)}.$$

By Riesz-Fischer this space is complete. Furthermore by Example 2.13 V_m is reflexive, if $1 < p < \infty$. Now we define an operator $L : W^{k,p}(\Omega) \to V_m$ for a suitable $m \in \mathbb{N}$:

$$L(u) := (u, D^{(1,0,\dots,0)}u, D^{(0,1,0\dots,0)}u, \dots, D^{(k,\dots,k)}u).$$

Choosing m suitably, this map is well defined, linear, injective and by Remark 3.6 continuous (choosing the norms correctly, it becomes an isometry). Let $u_k \in L(W^{k,p}(\Omega))$ be a Cauchy sequence (i.e. $L^{-1}(u_k)$ is as well). Then there exists a limit $u \in V_m$. Now let $\varphi \in C_0^\infty(\Omega) \subseteq L^q(\Omega)$ ($q \in [1,\infty]$ arbitrary) and α be a multiindex with $|\alpha| \leq k$. Since norm convergence is stronger than weak convergence and by Example 2.4

$$\int_\Omega u_0 D^\alpha \varphi \, dx \leftarrow \int_\Omega (u_k)_0 D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha (u_k)_0 \varphi \, dx \to (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi \, dx.$$

Hence u_α (i.e. the α-component of $u \in V_m$) is the α-th weak derivative of the first component u_0. Hence $u \in L(W^{k,p}(\Omega))$. Therefore $L(W^{k,p}(\Omega))$ is a closed subspace of V_m. By Exercise 2.4 it is reflexive. Since $L : W^{k,p}(\Omega) \to L(W^{k,p}(\Omega))$ is a bijective isometry, the results follow.